A Two-Factor Model for the Electricity Forward Market

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Abstract

This paper aims to describe and calibrate a two-factor model for electricity futures, which captures the main features of the market, in particular seasonality, and fits the term structure of volatility. Additionally, options on futures will be priced within this context and we will especially take care of the existence of delivery periods in the underlying future. The approach is motivated by the one-factor-model of Clewlow and Strickland applied to the electricity market and extends it to a two-factor-model.

Keywords: Energy derivatives, Futures, Option, Two-Factor Model, Volatility term Structure
1 Introduction

Since the deregulation of electricity markets in the end of the 1990s, power can be traded at exchanges like the Nordpool or the European Energy Exchange (EEX). All exchanges have established spot and futures markets.

The spot market usually is organised as an auction, which manages the distribution of power in the near future, i.e. one day ahead. Empirical studies, such as [KR01] using hourly prices in the California power market, show that spot prices exhibit seasonalities on different time scales, a strong mean-reversion and are very volatile and spiky in nature. Because of inherent properties of electricity as an almost non-storable commodity such a price behaviour has to be expected, see [GE99].

Due to the volatile behaviour of the spot market and to ensure that power plants can be deployed optimally, power forwards and futures are traded. Power exchanges established the trade of forwards and futures early on and by now large volumes are traded. A power forward contract is characterized by a fixed delivery price per MWh, a delivery period and the total amount of energy to deliver. Especially the length of the delivery period and the exact time of delivery determine the value and statistical characteristics of the contract vitally. One can observe, that contracts with a long delivery period show less volatile prices than those with short delivery. These facts give rise to a term structure of volatility in most power forward markets, which has to be modelled accurately in order to be able to price options on futures. Figure 1 gives an example of such a term structure for futures traded at the EEX. Additionally, seasonalities can be observed in the forward curve within a year. Monthly contracts during winter months show higher prices than comparable contracts during the summer (cp. Figure 2).

Aside from spot and forward markets, valuing options is an issue for market participants. While some research has been done on the valuation of options on spot power, hardly any results can be found on options on forwards and futures. Both types impose different problems for the valuation.

Spot options fail most of the arbitrage and replication arguments, since power is almost non-storable. Some authors take the position to find a realistic model to describe the prices of spot prices and then value options via risk-neutral expectations (cp. [dJH02], [BDK04], [BKMS04]). Other ideas explicitly take care of the special situation in the electricity production and use power plants to replicate certain contingent claims (cp. [GE99]).

Forward and futures options are heavily influenced by the length of their delivery period and their time to maturity. In [CS99], for example, a one-factor model is presented, that tries to fit the term structure of volatility, but that does not incorporate a delivery period, since it is constructed for oil and gas markets.
Figure 1: Implied volatilities of futures with different maturities and delivery periods, Sep. 14

Figure 2: Forward prices of futures with different maturities and delivery periods, Feb. 18
Our approach is to describe and calibrate a two-factor-model for electricity futures, which captures the main features of the market, in particular seasonality, and fits the term structure of volatility. Additionally, options on futures will be priced within this context and we will especially take care of the existence of delivery periods in the underlying future. The approach is motivated by the one-factor-model of Clewlow and Strickland ([CS99]) applied to the electricity market and extends it to a two-factor-model. (Compare also the two-factor model proposed by Schwartz and Smith ([SS00]), which can be used for commodities in general, but fails to address specifics of the electricity market.)

2 Description of the Model and Option Pricing

2.1 General Model Formulation

In energy markets we observe futures with a certain delivery period. In the following, energy futures with 1-month delivery are the building blocks of our model. Note, that futures with other delivery period are derivatives now, i.e. a future with delivery of a year is a portfolio of 12 appropriate month-futures.

Let $F(t, T)$ denote the time $t$ forward price of 1MWh electricity to be delivered constantly over a 1-month-period starting at $T$. Then, assuming a deterministic and flat rate of interest $r$, the time $t$ value of this futures contract with delivery price $D$ is given by

$$V^{\text{future}}(t, T) = e^{-r(T-t)}(F(t, T) - D).$$

Assuming the existence of a risk-neutral measure, discounted value processes have to be martingales under this measure, which in this case is equivalent to forward prices being martingales.

Thus, in the spirit of LIBOR market models, we model the observable forward prices directly under a risk-neutral measure as martingales via the stochastic differential equation

$$dF(t, T) = \sigma(t, T)F(t, T)dW(t),$$

where $\sigma(t, T)$ is an adapted $d$-dimensional deterministic function and $W(t)$ a $d$-dimensional Brownian motion. The initial value of this SDE is given by the condition to fit the initial forward curve observed at the market.

The solution of the SDE is given by

$$F(t, T) = F(0, T) \exp\left(\int_0^t \sigma(s, T)dW(s) - \frac{1}{2} \int_0^t ||\sigma(s, T)||^2 ds\right)$$

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2.2 Option Pricing

A European call option on $F(t, T)$ with maturity $T_0$ and strike $K$ can be easily evaluated by the Black-formula

$$V_{\text{option}}(t) = e^{-r(T_0-t)} (F(0, T)N(d_1) - K N(d_2)),$$

where $N$ denotes the normal distribution and

$$d_1 = \frac{\log \frac{F(0, T)}{K} + \frac{1}{2} \text{Var}(\log F(T_0, T))}{\sqrt{\text{Var}(\log F(T_0, T))}}$$

$$d_2 = d_1 - \sqrt{\text{Var}(\log F(T_0, T))}$$

Options on month-futures are not the only derivatives in the market. What is more, options on year-futures are the most heavily traded products in the option market. In this setup, year-futures (i.e., futures with delivery period 1 year) are a portfolio of the building blocks, the month-futures. Thus, the pricing of an option on such a portfolio is not straightforward and a closed form formula is not known in general, but in some special cases. Clewlow and Strickland for example derive an option formula in a one-factor model ($d = 1$) in [CS99]. The issue is closely related to the pricing of swaptions in the context of LIBOR market models and is discussed in [BM01].

The value of a portfolio of month-futures (e.g., a year-future) with delivery starts at $T_i, i = 1, \ldots, n$ normalized to the delivery of 1 MWh and delivery price $D$ is given by

$$V_{\text{portfolio}}(t, T_1, \ldots, T_n) = \frac{1}{n} \sum_{i=1}^{n} e^{-r(T_i-t)} (F(t, T_i) - D).$$

In the context of interest rate swaps, the value of a swap is expressed in terms of a swap rate $Y$, which is here:

$$V_{\text{portfolio}}(t, T_1, \ldots, T_n) = \frac{1}{n} \sum_{i=1}^{n} e^{-r(T_i-t)} (Y_{T_i, \ldots, T_n}(t) - D),$$

where

$$Y_{T_1, \ldots, T_n}(t) = \frac{\sum_{i=1}^{n} e^{-r(T_i-t)} F(t, T_i)}{\sum_{i=1}^{n} e^{-r(T_i-t)}}.$$

In case the portfolio represents a 1-year-future, the swap rate is the forward price of the 1-year-future, which can be also observed in the market.

Evaluating an option on this 1-year-forward price (i.e., on the swap par rate) poses the problem of computing the expectation in

$$e^{-rT_0} \mathbb{E} [(Y(T_0) - K)^+] ,$$
where the distribution of $Y$ as a sum of lognormals is unknown. Often, an approximation is suggested (e.g. [BM01]), which assumes $Y$ to be lognormal. Formally, we can approximate $Y$ by a random variable $\hat{Y}$, which is lognormal and coincides with $Y$ in mean and variance. Then,

$$\log \hat{Y} \sim \mathcal{N}(m, s)$$

with $s^2$ depending on the choice of the volatility functions $\sigma(t, T_i)$. An analysis of the goodness of the approximation can be found in [BL03]. Using this approximation, it is possible to apply a Black-option formula again to obtain the option value as

$$V_{\text{option}} = e^{-rT_0}E[(Y(T_0) - K)^+]$$

$$\approx e^{-rT_0}E[(\hat{Y}(T_0) - K)^+]$$

$$= e^{-rT_0} (Y(0)\mathcal{N}(d_1) - K\mathcal{N}(d_2))$$

with

$$d_1 = \frac{\log \frac{Y(0)}{K} + \frac{1}{2} s^2}{s}$$

$$d_2 = d_1 - s$$

3 The Special Case of a Two-Factor-Model

3.1 Model Formulation and Option Pricing

As motivated in the introductory part, a special choice of the volatility function is needed to resemble market observations of the term structure of volatility in the futures contracts. In this section we will use a two-factor model given by the SDE

$$\frac{dF(t, T)}{F(t, T)} = e^{-\kappa(T-t)}\sigma_1 dW_1^1 + \sigma_2 dW_2^1,$$  \hspace{1cm} (3)

for a fixed $T$. The Brownian motions are assumed to be uncorrelated.

This is a special case of the general setup in the previous section with

$$\sigma(t, T) = (e^{-\kappa(T-t)}\sigma_1, \sigma_2)$$

and $W(t)$ a 2-dimensional Brownian motion.

For ease of notation assume in the following that today’s time $t = 0$.

In this case the variance of the logarithm of the future contract at some future time $T_0$ can be shown to be (see Appendix 7.1)

$$\text{Var}(\log F(T_0, T)) = \frac{\sigma_1^2}{2\kappa} (e^{-2\kappa(T-T_0)} - e^{-2\kappa T}) + \sigma_2^2 T_0$$  \hspace{1cm} (4)
This quantity has to be used to price an option on month-futures with maturity \( T_0 \) with the option formula (1).

In the case of options on quarter- or year-futures, it is necessary to compute the quantity \( s^2 \) of the lognormal approximation in equation (2). The derivation of \( s^2 \) in this two-factor model can be found in Appendix 7.2 and is given by

\[
\exp(s^2) = \frac{\sum_{i,j} e^{-r(T_i+T_j)} F(0, T_i) F(0, T_j) \cdot \exp(Cov_{ij})}{(\sum e^{-rT_i} F(0, T_i))^2}
\]

(5)

\[
Cov_{ij} = \text{Cov}(\log F(T_0, T_i), \log F(T_0, T_j))
\]

\[
= e^{-\kappa(T_i+T_j-2T_0)} \frac{\sigma_1^2}{2\kappa} (1 - e^{-2\kappa T_0}) + \sigma_2^2 T_0
\]

4 Calibration to Option Prices

In the following, we will calibrate the two-factor model to market data, i. e. we will estimate the parameters \( \phi = (\sigma_1, \sigma_2, \kappa) \) such that the model resembles the market behaviour as good as possible. Since we have modelled under a risk-neutral measure, we need to find risk-neutral parameters, which can be observed using option-implied parameters.

Given the market price of a futures-option (month-, quarter- or year-futures), we can observe its implied variance \( \text{Var}(\log F(T_0, T)) \) for month-future or \( s^2 \) for quarter- or year-futures.

Furthermore, we can compute the corresponding model implied quantities, which depend on the choice of the parameter set \( \phi = (\sigma_1, \sigma_2, \kappa) \) as described in the previous section. We will estimate the model parameters such that market and model implied quantities coincide as good as possible in the mean square sense, i. e.

\[
\sum_i \left( \text{Var}_{\text{market}}(\log Y_{T_1, \ldots, T_n}(T_0_i)) - \text{Var}_{\text{model}}(\log Y_{T_1, \ldots, T_n}(T_0_i)) \right)^2 \rightarrow \min_{\phi}
\]

(6)

where \( i \) represents an option with maturity \( T_0_i \) and delivery covering the months \( T_1, \ldots, T_n \). Depending on the delivery period, which of course may be longer than one month, the model variance is either the true model implied variance according to equation (4) or the approximated variance according to equation (5). The minimum is taken over all admissible choices of \( \phi = (\sigma_1, \sigma_2, \kappa) \), that means \( \sigma_1, \sigma_2, \kappa > 0 \).

Since our model is not capable of capturing volatility smiles, which can be observed in option prices very often, we will use at-the-money options only.

The minimization can be done with standard programming languages and their implemented optimizers. The objective function (6) is given to the optimizer, which has to
<table>
<thead>
<tr>
<th>Product</th>
<th>Delivery Start</th>
<th>Strike</th>
<th>Forward</th>
<th>Market Price</th>
<th>Implied Vola</th>
</tr>
</thead>
<tbody>
<tr>
<td>Month</td>
<td>October 05</td>
<td>48</td>
<td>48.90</td>
<td>2.023</td>
<td>43.80%</td>
</tr>
<tr>
<td>Month</td>
<td>November 05</td>
<td>49</td>
<td>50.00</td>
<td>3.064</td>
<td>37.66%</td>
</tr>
<tr>
<td>Month</td>
<td>December 05</td>
<td>49</td>
<td>49.45</td>
<td>3.244</td>
<td>34.72%</td>
</tr>
<tr>
<td>Quarter</td>
<td>October 05</td>
<td>48</td>
<td>49.44</td>
<td>2.086</td>
<td>35.15%</td>
</tr>
<tr>
<td>Quarter</td>
<td>January 06</td>
<td>47</td>
<td>48.59</td>
<td>3.637</td>
<td>28.43%</td>
</tr>
<tr>
<td>Quarter</td>
<td>April 06</td>
<td>40</td>
<td>40.71</td>
<td>3.421</td>
<td>26.84%</td>
</tr>
<tr>
<td>Quarter</td>
<td>July 06</td>
<td>42</td>
<td>41.80</td>
<td>3.758</td>
<td>27.19%</td>
</tr>
<tr>
<td>Quarter</td>
<td>October 06</td>
<td>43</td>
<td>43.71</td>
<td>4.566</td>
<td>25.35%</td>
</tr>
<tr>
<td>Year</td>
<td>January 06</td>
<td>44</td>
<td>43.68</td>
<td>1.521</td>
<td>20.19%</td>
</tr>
<tr>
<td>Year</td>
<td>January 07</td>
<td>43</td>
<td>42.62</td>
<td>3.228</td>
<td>19.14%</td>
</tr>
<tr>
<td>Year</td>
<td>January 08</td>
<td>42</td>
<td>42.70</td>
<td>4.286</td>
<td>17.46%</td>
</tr>
</tbody>
</table>

Table 1: ATM calls and implied Black-volatility, Sep 14

compute the model implied variances of all options for different parameters. The worst case (the computation of the variance of a year-contract) involves the evaluation of all covariances $C_{ij}$ between the underlying month-futures in equation (5), which is a 12 by 12 matrix, thus computationally not too expensive. As there are usually not more than 15 at-the-money options available (e. g. at the EEX), the optimization can be done within a few minutes.

Additionally, it is possible to use the gradient of the objective function for the optimization. The gradient can be computed explicitly, which makes the numerical evaluation of the gradient in the optimizers unnecessary. Usually, there is a smaller number of function calls necessary to reach the optimal point within a given accuracy using the gradient than using a numerical approximation. But, the explicit calculation again involves matrices up to size 12 by 12. We found, that the time saved by less function calls is eaten up by the increased complexity of the problem. Both methods end up with about the same optimization time, though the gradient method finds minima, which usually give slightly smaller optimal values than methods without gradient.

5 Calibration Result

In the following we will apply the two-factor model introduced in Section 3.1 to the German market, i. e. we will calibrate it to EEX prices. We used data of different days, but the results will be discussed as of September 14. Properties of the term structure on this day have been discussed in the introductory part. The data set is shown in Table 1.

The column implied volatility in Table 1 shows the implied volatility by the Black76-
The optimizers converge in less than a minute and optimizing with and without gradient delivers the same results up to two decimal places. Even not restricting the parameters does not change the estimates (see Table 2).

The calibration leads to parameter estimates $\sigma_1 = 0.37$, $\sigma_2 = 0.15$, and $\kappa = 1.40$. This implies, that options, which are far away from maturity, will have a volatility of about 15%, which can add up to more than 50%, when time to maturity decreases. A $\kappa$ value of 1.40 indicates, that disturbances in the futures market halve in $\frac{1}{\kappa} \cdot \log 2 \approx 0.69$ years.

**Table 2**: Parameter estimates with different optimizers, market data as of Sep 14

<table>
<thead>
<tr>
<th>Method</th>
<th>Constraints</th>
<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
<th>$\kappa$</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Function calls and numerical gradient</td>
<td>yes</td>
<td>0.37</td>
<td>0.15</td>
<td>1.40</td>
<td>&lt;1min</td>
</tr>
<tr>
<td>Least Square Algorithm</td>
<td>no</td>
<td>0.37</td>
<td>0.15</td>
<td>1.41</td>
<td>&lt;1min</td>
</tr>
</tbody>
</table>

**Figure 3**: Market-implied volatilities of futures with model-implied volatilities, Sep 14

formula. After filtering out data points, where the option price is only the inner value of the option, these 11 options are left out of 15 observable in the EEX data set. Now, one can observe a strong decreasing term structure with increasing time to maturity (there are two outliers, which do not confirm the hypothesis) and a decreasing volatility level with increasing delivery period.
<table>
<thead>
<tr>
<th>Delivery Start</th>
<th>Market Price</th>
<th>Model Price</th>
<th>Market Volatility</th>
<th>Model Volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td>M-October 05</td>
<td>2.023</td>
<td>1.844</td>
<td>43.80%</td>
<td>38.52%</td>
</tr>
<tr>
<td>M-November 05</td>
<td>3.064</td>
<td>3.000</td>
<td>37.66%</td>
<td>36.70%</td>
</tr>
<tr>
<td>M-December 05</td>
<td>3.244</td>
<td>3.279</td>
<td>34.72%</td>
<td>35.13%</td>
</tr>
<tr>
<td>Q-October 05</td>
<td>2.086</td>
<td>2.089</td>
<td>35.15%</td>
<td>35.25%</td>
</tr>
<tr>
<td>Q-January 06</td>
<td>3.637</td>
<td>3.865</td>
<td>28.43%</td>
<td>30.82%</td>
</tr>
<tr>
<td>Q-April 05</td>
<td>3.421</td>
<td>3.539</td>
<td>26.84%</td>
<td>27.88%</td>
</tr>
<tr>
<td>Q-July 06</td>
<td>3.758</td>
<td>3.520</td>
<td>27.19%</td>
<td>25.51%</td>
</tr>
<tr>
<td>Q-October 06</td>
<td>4.566</td>
<td>4.315</td>
<td>25.35%</td>
<td>23.83%</td>
</tr>
<tr>
<td>Y-January 06</td>
<td>1.521</td>
<td>1.746</td>
<td>20.19%</td>
<td>22.92%</td>
</tr>
<tr>
<td>Y-January 07</td>
<td>3.228</td>
<td>3.074</td>
<td>19.14%</td>
<td>18.28%</td>
</tr>
<tr>
<td>Y-January 08</td>
<td>4.286</td>
<td>4.131</td>
<td>17.62%</td>
<td>16.93%</td>
</tr>
</tbody>
</table>

Table 3: Comparison between market and model quantities, Sep 14

The model-implied volatility term structure is shown in Figure 3 together with the observed market values. One can see, that, qualitatively, most of the desired properties are described by the model. Futures with a long delivery period show a lower level of volatility compared to those with short delivery. The volatility term structure is decreasing as the time to maturity increases, but it does not go down to zero. Yet, quantitatively, there are some drawbacks. Especially the month-futures show a volatility term structure, that has a much steeper descent than the model implies. Also, the level of volatility is mostly higher than observed in the market. Yet, the model implies reasonable values for all contracts. Absolute values in terms of volatilities and option prices can be taken from Table 3.

In order to assess the time stability of the parameter estimation, the calibration has been repeated during a four week period in July. Figure 4 shows the percentage change in the estimates from day to day.

One can see, that the estimates are rather stable over time. Yet, there is a day, which is totally out of the regular values. This might be due to several issues. First of all, the market data is not very reliable up to this date, since the traded volume is small. It is not clear, if at some given day the quoted price includes all information. Besides, the number of contracts available for calibration is rather small so that each option has big influence on the estimate. Finally, the objective function seems to show several local minima, which leads to fluctuations in the parameter.
Conclusion

We have presented a two-factor-model for the electricity futures market. It is embedded in a bigger class of market models, which are very popular in interest rate forward markets. We have developed pricing formulas for relevant products in the market and shown a procedure to fit the market data. A first visual analysis shows, that the model resembles the main characteristics of the futures market (i.e. level of volatility depending on time to maturity, length of delivery period) reasonably well and is, thus, appropriate for pricing similar claims.
7 Appendix

All following results will be derived under the assumption of correlated Brownian motions, i.e. \( dW_t^{(1)}dW_t^{(2)} = \rho dt \) (in contrast to the independence assumption in the article). The results of the article will be obtained by setting \( \rho = 0 \).

7.1 Derivation of the Variance of a Month-Futures Contract

We will derive equation (4), i.e. \( \text{Var}(\log F(T_0, T)) \). The SDE describing the futures dynamics in equation (3) can be solved by

\[
F(t, T) = F(0, T) \exp \left\{ -\frac{1}{2} \int_0^t \tilde{\sigma}^2(s, T) ds + \int_0^t e^{-\kappa(T-s)} \sigma_1 dW_s^{(1)} + \int_0^t \sigma_2 dW_s^{(2)} \right\}
\]

with

\[
\tilde{\sigma}^2(s, t) = \sigma_1^2 e^{-2\kappa(t-s)} + 2\rho \sigma_1 \sigma_2 e^{-\kappa(t-s)} + \sigma_2^2.
\]

Now

\[
\text{Var}(\log F(T_0, T)) = \int_0^{T_0} \tilde{\sigma}^2(s, T) ds
\]

\[
= \frac{\sigma_1^2}{2\kappa} (e^{-2\kappa(T-T_0)} - e^{-2\kappa T}) + \sigma_2^2 T_0 + 2\frac{\rho \sigma_1 \sigma_2}{\kappa} (e^{-\kappa(T-T_0)} - e^{-\kappa T})
\]

7.2 Derivation of the Variance of Quarter- and Year-Futures Contracts

We will derive equation (5), i.e. \( s^2 \) at time \( T_0 \). We have

\[
\mathbb{E}(Y) = \mathbb{E}(\tilde{Y})
\]

\[
\text{Var}(Y) = \text{Var}(\tilde{Y})
\]

\[
\log \tilde{Y} \sim \mathcal{N}(m, s^2)
\]

Moments of normal and lognormal distributions are related via

\[
\mathbb{E}(Y) = \exp(m + \frac{1}{2}s^2)
\]

\[
\text{Var}(Y) = \exp(2m + 2s^2) - \exp(2m + s^2)
\]
Solving this system, we get

\[
\exp(s^2) = \frac{\text{Var}(Y)}{(\mathbb{E}(Y))^2} + 1 \\
= \frac{\mathbb{E}(Y^2)}{\mathbb{E}(Y)^2}
\]

It can be seen easily that

\[
\mathbb{E}(F_{T_0,T_i}) = F_{0,T_i} \\
\mathbb{E}(Y_{T_1,\ldots,T_n}(T_0)) = \frac{\sum e^{-r(T_i-T_0)} F_{0,T_i}}{\sum e^{-r(T_i-T_0)}}
\]

Further

\[
\mathbb{E}(Y_{T_1,\ldots,T_n}(T_0)^2) = \frac{1}{(\sum e^{-r(T_i-T_0)})^2} \cdot \ldots \\
\sum_{i,j} e^{-r(T_i+T_j-2T_0)} F_{0,T_i} F_{0,T_j} \cdot \exp \text{Cov}_{ij} \\
\text{Cov}_{ij} = \text{Cov}(\log F(T_0, T_i), \log F(T_0, T_j))
\]

The covariance can be computed directly from the explicit solution of the SDE

\[
\text{Cov}(\log F(T_0, T_i), \log F(T_0, T_j)) = e^{-\kappa(T_i+T_j-2T_0)} \frac{\sigma_1^2}{2\kappa} (1 - e^{-2\kappa T_0}) + \sigma_2^2 T_0 + \ldots \\
+ \frac{\rho \sigma_1 \sigma_2}{\kappa} (1 - e^{-\kappa T_0}) (e^{-\kappa(T_i-T_0)} + e^{-\kappa(T_j-T_0)})
\]
References


