A Kalman Filter Approach for Structural Firm Value Models

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Abstract

This paper proposes a direct empirical implementation of the Corporate Securities Framework. To date, only time series of equity prices have been used to estimate parameters of structural firm value models. We develop a Kalman filter that includes time series of bond prices which usually convey important information about a firm’s economic condition, and measurement errors of security prices. We suggest that the use of time series of all traded securities in an empirical study will improve the quality of the estimators of the latent EBIT-process because we can identify not only the EBIT-volatility but also the EBIT risk-neutral drift which used to be fixed by unreasonable assumptions in applications so far. The estimators are unbiased even in small samples and robust to specification errors.

JEL-Classification: G12, G13, G33, C13, C32
Keywords: firm value models, empirical implementation, Kalman filter

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A Kalman Filter Approach for Testing of Structural Firm Value Models

Abstract

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1 Introduction

This paper proposes a direct empirical implementation of the class of EBIT-based structural firm value models as discussed in Genser (2005a). To date, only a mixture of accounting data and time series of equity prices have been used to estimate parameters of structural firm value models. We develop a Kalman filter that incorporates time series of bond prices which usually convey important information about a firm’s economic condition. We suggest that the use of time series of all traded securities in an empirical study will improve the quality of the estimators of the latent EBIT-process.

In Section 2, we give a brief overview of empirical studies which are based on firm value models. Section 3 discusses estimation procedures used in the literature and proposes a suitable Kalman filter for the Corporate Securities Framework. Section 4 raises practical issues for the actual estimation. Finally, Section 5 collects the results of the simulation study. A brief summary is given in Section 6.

2 Existing Literature and Shortcomings

Tests of reduced-form asset pricing models can be designed and performed relatively easily. In contrast, direct implementations of structural credit risk models have been rare due to two reasons:
Theoretical firm value models have been too restrictive to be applied to corporate data. To fit real world firms into e.g. the model of Merton (1974) or Black and Cox (1976), several ad-hoc adjustments to the firm’s debt structure are needed. In most cases the debt structure is artificially reduced to a single debt issue by constructing a zero or coupon bond with a notional of the total debt outstanding and a maturity equal to the duration of the entire debt structure. Empirical results may therefore be influenced by the adjustments. Only the proposed EBIT-based Corporate Securities Framework of Genser (2005b) and Ericsson and Reneby (1998) represent more general frameworks for the valuation of corporate securities which seem appropriate for direct estimation. These models allow to exploit trading data of corporate securities, i.e. time series of equity and corporate bond prices, that are abundantly available, and need not rely solely on accounting data.

Firm value models need more advanced econometric methods. Most firm value models operate with a latent variable. The dependence of corporate securities on the latent variable implies non-stationary parameters of the processes of corporate securities.

As a result, most of the empirical literature on firm value models has been reduced to a test of stylized model behavior on aggregate levels such as industry, rating classes, or firm sizes. Predictions of firm characteristics such as leverage ratios, corporate bond coupons, corporate bond issue discounts, credit spreads were regressed on time series of equity and bond prices.

Fischer, Heinkel and Zechner (1989a) find that observed leverage ratio ranges can be explained by a firm value model where dynamic recapitalization is costly. Importantly, they point out that the current leverage ratio of a firm might stem from an optimal decisions made several periods ago. As a result, the observed leverage ratio might not be optimal but lie in a range of inertia where high recapitalization cost prevent the firm from adjusting the leverage.

Another aspect of corporate debt is the existence of call premia and issue discounts. In a similar framework as in their first paper, Fischer, Heinkel and Zechner (1989b) find empirical support that debt contracts are designed to mitigate agency conflicts between debt and equity holders. Call premia and issue discounts are related to firm risk, which the model predicts.

\[1\text{See e.g. Jones, Mason and Rosenfeld (1984).}\]

\[2\text{See Section 3. Also Ericsson and Reneby (2002) for a discussion of estimator design. Duan (1994) describes a more general setting of latent variable estimation.}\]
In a model with a constant capital structure, Longstaff and Schwartz (1995) extend the pricing model for corporate debt to include stochastic interest rates which are correlated with the firm value. Their empirical section strongly supports the incorporation of stochastic interest rates in firm value models. However, Delianedis and Geske (2001) and Huang and Huang (2003) report contradictory evidence.

One of the primary issues of the option pricing literature is the source of implied Black and Scholes (1973) volatility smirks as observed in option prices. Toft and Prucyk (1997) relate option prices to the leverage ratio of firms in the Leland (1994) model. Despite the simple capital structure, their regression results support a strong dependence of the pricing bias of equity options on the leverage of the firm. Interpreting short term debt as a kind of debt covenant they can even relate steeper smirk functions to firms which are mainly financed by short term debt.

Recently, Brockman and Turtle (2003) contrast the performance of a barrier option approach of pricing corporate debt which is used in firm value models such as the framework of Genser (2005b) and Ericsson and Reneby (1998) with the Merton (1974)-like approach where bankruptcy can only occur path-independently at the maturity of a debt issue. The cross section of a large sample of US firms indicates that the barrier option feature of corporate securities is statistically significant and outperforms the traditional static approach.

Summarizing, most tests of stylized facts support firm value models and attest predictive power with respect to distinct features.

Parameter estimation of firm value models until so far was restricted to the simple Merton (1974)-model and with unsatisfactory empirical methods. Jones et al. (1984) were the first to parameterize the Merton (1974)-model by using accounting data and equity times series. They are not able to reproduce observed bond prices. Apart from the criticism of their parameter estimation methods, they have bond types in their sample that clearly contradict the stringent assumptions of the Merton (1974)-model, i.e. only a single zero-coupon bond is outstanding. Subsequent studies tried to overcome some of the problems in Jones et al. (1984). Delianedis and Geske (1999), Delianedis and Geske (2001), Eom, Helwege

\[ \text{Longstaff and Schwartz (1995) approximate the hitting probabilities and the hitting prices by an algorithm that extends Buonocore, Nobile and Ricciardi (1987)’s one dimensional version to a two-dimensional stochastic process. As Collin-Dufresne and Goldstein (2001) show, Longstaff and Schwartz (1995)’s approximations are incorrect which has an impact on the credit spreads shown in their paper. However, Collin-Dufresne and Goldstein (2001)’s implementation of their analytically correct version bears an error which changes some of their conclusions. So, all of these results have to be taken with care.} \]

\[ \text{See also the critical remarks on the Toft and Prucyk (1997) in Genser (2005b).} \]

\[ \text{See Section 3 for details.} \]
and Huang (2004), and Huang and Huang (2003) resort to a calibration method in order to extract the time series of asset values. Their primary focus was to explain why structural models fail to explain the whole credit spread. Other factors such as stochastic interest rates, stochastic recovery rates, differential tax treatments, illiquidity premia, and market risk factors were tried as explanatory variables additional to default risk. However, none of the studies was able to find a consistent and reasonable explanation.\footnote{Similar results are reported in Collin-Dufresne, Goldstein and Martin (2001). In a regression study they try to explain credit spreads by several economic and market wide factors, but conclude that the most important explanatory factor as of a principal component analysis is not among the variables tried out. In contrast to this evidence, Longstaff, Mithal and Neis (2004) can attribute the majority of the observed credit spread to default risk. However, they use credit default swap data and a reduced form approach in their study.}

A very promising example of direct estimation of structural credit risk models is Ericsson and Reneby (2004) who are able to parameterize a structural model with a simple capital structure with Duan (1994)’s latent variable approach. Their estimates for individual firm’s corporate bonds are remarkably well. However, their estimation method only uses time series of equity prices.

The Kalman filter suggested here is based on the idea to use as much time series information as possible to estimate the model parameters.\footnote{Bruche (2004) suggests a more computationally demanding simulation approach which also can incorporate all kind of corporate securities.} We will argue that our approach is simple but more accurate than any estimation method suggested so far.\footnote{In financial applications, Kalman filters have primarily been used to estimate interest rate processes where the short rate is treated as a latent variable. See e.g. Geyer and Pichler (1999), Babbs and Nowman (1999).}

\section{Estimation of Parameters of the Corporate Securities Framework}

\subsection{The Corporate Securities Framework}

Recall from Ammann and Genser (2005)’s specification of the Corporate Securities Framework that the state of the firm is assumed to be described by the firm’s EBIT which is assumed to follow an arithmetic Brownian motion:

\begin{equation}
d\eta = \mu dt + \sigma \eta dz^{\mathbb{Q}},
\end{equation}
where $\mu$ is the risk-neutral drift of the EBIT-process under the equivalent martingale measure $Q$, $\sigma_\eta$ is the volatility of EBIT and $dz^{Q}$ describes a standard Brownian motion.

The firm is assumed to go bankrupt if EBIT hits a lower barrier $\eta_B$, where a fraction $\alpha$ of the then available firm value $V_B$ is lost.

The firm and investors are exposed to a tax system with three different kinds of taxes. Debt holders’ coupon payments are taxed at a tax rate $\tau^d$. A corporate tax rate $\tau^c$ is applied to corporate earnings, i.e. EBIT less coupon payments. Corporate earnings after tax are paid out as a dividend, which in turn is taxed at the personal tax rate of equity owners $\tau^e$.

The capital structure of the firm consists of $J$ debt issues with model prices $D_{C_j,T_j}$, $j = 1 \ldots J$, where $C_j$ denotes the coupon level and $T_j$ the time of maturity of debt issue $j$, and equity with a model price of $E$. Model prices of debt and equity are functions of a parameter vector $\Theta = \{\mu, \sigma_\eta, \alpha, \eta_B, \tau^c, \tau^d, \tau^e\}$, the risk-less interest rate $r$, and the current EBIT $\eta_t$.

Under the assumption that EBIT follows a geometric Brownian motion, only the process parameters $\bar{\mu}$ and $\bar{\sigma}_\eta$ are exchanged for $\mu$, $\sigma_\eta$, to arrive at the parameter vector $\bar{\Theta}$. Without loss of generality, we restrict the exposition to the case of arithmetic Brownian motion.

The closed form solutions to the case when EBIT follows a geometric Brownian motion is derived in Genser (2005b), Section 3.3.

### 3.2 Estimation Approaches Using Accounting Data

A very intuitive way of implementing structural credit risk models was first proposed by Jones et al. (1984). Jones et al. (1984) estimated parameter values of the classical Merton (1974)-model by using a mixture of accounting and market, i.e. equity time series, data. In the Merton (1974)-model, a time series of firm values $\hat{V}_n$, an estimate of the asset volatility $\hat{\sigma}_V$, and the interest rates $\hat{r}_n$, are needed. Jones et al. (1984) propose the following estimation procedures:

- The firm value time-series:
  Given the total liabilities of a firm as of quarterly reports, its firm value can be estimated as the sum of the market value of traded debt and equity. The remaining market value of non-traded debt is assumed to be valued proportionally to the value of traded debt. Alternatively, book values of non-traded assets could be employed with some netting of short-term liabilities with short-term assets.$^9$

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• Asset volatility:
  As a first guess, the standard deviation of the firm’s asset can be derived from the constructed firm value changes directly. Alternatively, the theoretical relation
\[
\hat{\sigma}_V = \hat{\sigma}_E \frac{E}{\partial E \hat{V}}
\]  

(1)
can be used given an estimate of the equity volatility \(\hat{\sigma}_E\) and the estimate of the firm value time series \(\hat{V}\). Equity volatility is estimated from time series of stock prices.

• Interest Rates:
  Forward par yields were extracted from government bond dirty prices. The estimation procedure has been criticized to have several drawbacks. First, the construction of the firm value time series depends crucially on the assumption on non-traded debt values. Any of the proposed approximations can lead to biases of the general conclusion. Second, a standard argument against estimating the firm value volatility directly from the constructed firm value time series is that the resulting volatility estimate inherits the deficiencies of the construction process and that it is strictly backward looking. The estimate of the firm value volatility as of equation (1) is conceptually inconsistent because the constant asset volatility \(\hat{\sigma}_V\) is calculated by assuming that \(\hat{\sigma}_E\) is constant.\(^{10}\) However, \(\hat{\sigma}_E\) changes if \(\hat{V}\) and consecutively \(E\) and \(\frac{\partial E}{\partial V}\) change.

3.3 Calibration Approach

A widely accepted procedure to come up with estimates of the time series of the firm value and the asset volatility calibrates both parameters to data simultaneously. Instead of constructing a time series of firm values from balance sheet data, a second equation together with equation (1) is used, i.e. the theoretical equity value, to solve for the two unknowns \(\hat{V}_n\) and \(\hat{\sigma}_{V,n}\) at each observation time.\(^{11}\) As Ericsson and Reneby (2002) point out, the system of two equations suffers the same criticism as the approach using accounting data because equation (1) is still applied as if \(\hat{\sigma}_E\) were constant.

\(^{10}\)See Ericsson and Reneby (2002) or Bruche (2004).
\(^{11}\)Delianedis and Geske (1999), Delianedis and Geske (2001), Eom et al. (2004), and Huang and Huang (2003) apply the calibration method in an academic context. Moodys/KMV is an example of commercial application, see Crosbie and Bohn (2003).
3.4 Duan’s Latent Variable Approach

In a simulation study Ericsson and Reneby (2002) find that a maximum-likelihood estimator treating the firm value as a latent variable outperforms a least squares estimator used traditionally within the calibration approach for the corporate security models suggested by Merton (1974), Briys and de Varenne (1997), Leland and Toft (1996), and Ericsson and Reneby (1998). Although the Ericsson and Reneby (1998) framework is comparable to the Corporate Securities Framework, a direct implementation as suggested by Ericsson and Reneby (2002) seems problematic if more than the time series of equity prices is used to estimate the latent variable process.\(^\text{12}\)

Ericsson and Reneby (2004) estimate the Ericsson and Reneby (2002) model and propose an estimator for an equity time series only. Denote the \(n\)-th observed market price of equity by \(e_n\) with \(n = 1, \ldots, N\). The log-likelihood function for the observed market prices is then

\[
L(e; \Theta) = \sum_{n=2}^{N} \ln f(e_n|e_{n-1}; \eta_n, \Theta). \quad (2)
\]

The \(N\)-dimensional vector \(e\) denotes the time-series of observed equity prices. \(f(\cdot|\cdot)\) is defined as the conditional density of the price observations.

Note that \(\eta_n\) is not directly observable, it must be calculated for \(n = 1, \ldots, N\) using the inverse function of equity prices given the parameter vector \(\Theta\). Denote this inverse transformation function by \(\eta_n = E^{-1}(e_n, \Theta)\). Since we model security prices in terms of a latent EBIT-variable \(\eta\), we need to change the variable of the conditional density from the observed equity price to the unobserved EBIT-levels.\(^\text{13}\)

In our case of arithmetic Brownian motion, EBIT is normally distributed: \(\Delta \eta = \eta_s - \eta_t \sim N(\mu \Delta t, \sigma^2 \eta \Delta t)\) where \(\Delta t = s - t\) is the time between two observations. Therefore, the conditional density can be transformed to

\[
f(e_n|e_{n-1}, d_{n-1}; \eta, \Theta) = n(\eta_{n-1} + \mu \Delta t, \sigma^2 \eta \Delta t) \big|_{\eta_n} \cdot \frac{dE}{dn} \big|_{\eta_n} , \quad (3)
\]

where \(d_{n-1}\) collects the time series of debt prices until \(t_{n-1}\) and \(n(\mu, \sigma^2)\) denotes the normal density with parameters \(\mu\) and \(\sigma\). Maximizing equation (2) with the specification made in equation (3) results in an estimator for the time series of \(\eta_n\) and the parameter vector \(\Theta\).

\(^{12}\)Additionally, results from a simulation study might not directly apply to pricing data where market forces such as trading strategies and liquidity issues may reduce data quality.

\(^{13}\)The method was proposed by Duan (1994) in the context of deposit insurance. The function \(E^{-1}(e_n, \Theta)\) represents Duan (1994)’s inverse transformation function necessary to calculate the transformed maximum-likelihood function.
3.5 A Kalman Filter Approach

If more than one time series of corporate security prices is available for estimation, the one-dimensional maximum-likelihood estimator of equation (2) must be extended to allow for multi-dimensionality which cannot be accomplished easily. The inverse transformation function used in equation (3) needs a more general concept of optimal \( \eta_n(e_n, d_n, \Theta) \) with respect to equity price \( e_n \), debt prices \( d_n \) of the period \( n \) and the parameter vector \( \Theta \). It is not obvious how to specify this function, so that the maximum-likelihood estimator keeps its statistical properties.

Therefore, the use of a Kalman filter approach\(^\text{14}\) seems more appropriate. A state representation seems especially suitable for EBIT-based structural credit risk models. EBIT cannot be observed directly. However, observations of time series of equity and debt prices are available which are in turn non-linear functions of the EBIT. The Kalman filter extracts for each period the best estimate of the latent state variable given the observed security prices of the period. In order to be applicable, the non-linear observation function must be linearized either by a Taylor series expansion as proposed in the literature on extended Kalman filters\(^\text{15}\) or a derivative free approximation algorithm\(^\text{16}\). Both approaches are discussed next, starting with the Taylor series expansion.

As before, consider \( N \) discrete observations of equity and corporate bond prices. The Kalman updating scheme has a simple one-dimensional state vector \( \eta_n \) with a disturbance term \( \Delta z_n^Q \sim N(0, \Delta t) \), \( n = 1, \ldots, N \).

\[
\eta_n = \eta_{n-1} + \mu \Delta t + \sigma \Delta z_n^Q \quad (4)
\]

Given an estimate of last period’s estimated EBIT \( \hat{\eta}_{n-1} \), the best prediction of this period’s EBIT, i.e. the conditional expectation of the \( n \)-th period’s EBIT, is\(^\text{17}\)

\[
\bar{\eta}_{n|n-1} = E_{n|n-1}(\eta_n) = \hat{\eta}_{n-1} + \mu \Delta t. \quad (5)
\]

Denote by \( \hat{\Sigma}_{\eta}(n-1) = E_{n-1}[(\eta_{n-1} - \hat{\eta}_{n-1})^2] \) the variance of last period’s estimation error. Then, the variance of the prediction error of the \( n \)-th

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\(^{14}\)See e.g. Maybeck (1979).

\(^{15}\)See e.g. Haykin (2002).

\(^{16}\)See Norgaard, Poulsen and Ravn (2000).

\(^{17}\)Note that in this subsection the firm’s equity \( E \) is denoted by the same letter as the expectation operator. To distinguish between the two symbols and to be precise on the information when taking expectations, \( E_n \) is reserved for the unconditional expectation in period \( n \) and \( E_{n|n-1} \) denotes the expectation for period \( n \) conditional on information up to period \( n - 1 \). The single letter \( E \) without subscript refers to equity.
period becomes
\[
\Sigma_\eta(n|n-1) = E_{n|n-1} \left[ (\eta_n - \bar{\eta}_{n|n-1})^2 \right] \\
= \hat{\Sigma}_\eta(n-1) + \sigma_\eta^2 \Delta t,
\] (6)

where the fact was used that \( \Delta z_{n-1}^Q \) is uncorrelated with the prediction error. Assume next that at each observation time \( t_n \), a stock price \( e_n \) and \( J \) bond prices \( d_n = \{d_{j,n}\} \) for \( j = 1, \ldots, J \), are observed with observation errors \( \varepsilon_{e,n} \) and \( \varepsilon_{d,n} = \{\varepsilon_{j,d,n}\} \), respectively. Collect the prices and observation errors in \( J + 1 \times 1 \) vectors \( y_n = (e_n, d_n') \) and \( \varepsilon_n = (\varepsilon_{e,n}, \varepsilon_{d,n}') \sim N(0, R) \). Since the equity and debt pricing functions depend non-linearly on EBIT, a Taylor series expansion around the predicted EBIT \( \bar{\eta}_{n|n-1} \) is needed to linearize both functions for use in a Kalman filter.

\[
y_n(\eta_n) = Y(\eta_n, \bar{\eta}_{n|n-1}, \varepsilon_n) \\
= \begin{pmatrix} E(\eta_n, \Theta, r_n, t_n) \\ D(\eta_n, \Theta, r_n, t_n) \end{pmatrix} + \begin{pmatrix} \varepsilon_{e,n} \\ \varepsilon_{d,n} \end{pmatrix} \\
\approx \begin{pmatrix} E(\bar{\eta}_{n|n-1}, \Theta, r_n, t_n) \left[ \frac{\partial Y}{\partial \eta} \right]_{\eta=\bar{\eta}_{n|n-1}} (\eta_n - \bar{\eta}_{n|n-1}) + \varepsilon_{e,n} \\ D(\bar{\eta}_{n|n-1}, \Theta, r_n, t_n) \left[ \frac{\partial Y}{\partial \eta} \right]_{\eta=\bar{\eta}_{n|n-1}} (\eta_n - \bar{\eta}_{n|n-1}) + \varepsilon_{d,n} \end{pmatrix}
\] (7)

In equation (7), the linearization of the observation error is additive because the derivative \( \partial Y/\partial \varepsilon = I \), have mean zero, and do not aggregate over time.\(^{18}\) The interest rate \( r_n \) and time \( t_n \) change deterministically while iterating through the filter implying that there is no correlation between interest rate and EBIT-changes, i.e. time and interest rates have no stochastic effect on EBIT. Given the linearized observation equation, the n-th period’s security prices would be predicted to be

\[
E_{n|n-1}(Y_n) = \bar{Y}_{n|n-1}(\bar{\eta}_{n|n-1}, \bar{\eta}_{n|n-1}, 0)
\] (8)

with an observation prediction error of

\[
v_n = y_n - \bar{Y}_{n|n-1}(\bar{\eta}_{n|n-1}, \bar{\eta}_{n|n-1}, 0) = G_{\eta}(\eta_n - \bar{\eta}_{n|n-1}) + \varepsilon_n.
\] (9)

\(^{18}\)Note that more complicated observation error models can be implemented without difficulty. If observation errors have non-linear impact on prices and interrelate with other observation errors, or have non-zero means, the partial derivative would no longer be equal to the identity matrix and the observation error is linearized the same way as the latent state variable.
In equation (9), the $J + 1 \times 1$-vector $G_\eta$ collects the derivatives of the observation vector $Y$ with respect to EBIT $\eta$. The observation prediction error has a variance of

\[
\tilde{\Sigma}_Y(n|n-1) = E_{n|n-1} \left[ (y_n - \bar{Y}_{n|n-1}(\tilde{\eta}_{n|n-1}, \bar{\eta}_{n|n-1}, 0)) \right] \\
= G_\eta \tilde{\Sigma}_\eta(n|n-1)G_\eta' + R
\]

(10)

and a covariance with the EBIT-prediction error of

\[
\tilde{\Sigma}_{Y\eta}(n|n-1) = E_{n|n-1} \left[ (\eta_n - \bar{\eta}_{n|n-1}) (y_n - \bar{Y}_{n|n-1}(\tilde{\eta}_{n|n-1}, \bar{\eta}_{n|n-1}, 0)) \right]' \\
= \tilde{\Sigma}_\eta(n|n-1)G_\eta.
\]

(11)

Having observed the new observation vector $y_n$, the predictions of equations (5) and (8) can be updated. The best estimator of the current EBIT becomes

\[
\hat{\eta}_n = E_n(\eta_n|y_n) = \bar{\eta}_{n|n-1} + K_n v_n
\]

(12)

with the Kalman gain vector $K_n$ defined by

\[
K_n = \tilde{\Sigma}_{Y\eta}(n|n-1)\tilde{\Sigma}_Y(n|n-1)^{-1}.
\]

(13)

The variance of the EBIT-estimator is

\[
\hat{\Sigma}_\eta(n) = V_{\eta}(\eta_n|y_n) = \tilde{\Sigma}_\eta(n|n-1) - K_n'\tilde{\Sigma}_{Y\eta}(n|n-1).
\]

(14)

The Kalman filter recursion begins with

(i) an initial estimate of the state variable $\hat{\eta}_0$ and its variance $\hat{\Sigma}_\eta(0)$.

(ii) Next, the predictions $\hat{\eta}_1$, $\hat{Y}_1$, $\tilde{\Sigma}_\eta(1|0)$, $\tilde{\Sigma}_Y(1|0)$, and $\tilde{\Sigma}_{Y\eta}(1|0)$ are formed.

(iii) With the observation $y_1$, the predictions are updated to $\hat{\eta}_1$, and $\hat{\Sigma}_\eta(1|0)$. The observation estimates can then be calculated by

\[
\hat{Y}_n = \begin{pmatrix} E(\hat{\eta}_n, \Theta, r_n, t_n) \\ D(\hat{\eta}_n, \Theta, r_n, t_n) \end{pmatrix}.
\]

(15)

(iv) Steps (ii) and (iii) are repeated until $n = N$.  

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The extended Kalman filter depends on the linearization of the observation equation (7) by a first order Taylor series expansion. Norgaard et al. (2000) criticize that the extended Kalman filter’s Taylor series expansion is only accurate near the expansion point and that optimizing the parameter vector Θ might suffer numerical difficulties. In particular, approximation errors accumulate because in many applications analytical derivatives for Taylor series expansions are not available and numerical methods are employed. Moreover, a Taylor series expansion cannot capture the stochasticity of the underlying functions. Therefore, they propose a polynomial approximation of second order which offers advantages for its implementation because no explicit analytical derivative is needed but only function evaluations near but not too close to the expansion point. As a result, the differences taken never get so small as to cause numerical difficulties. This divided difference approach eases the implementation of filtering schemes and improves accuracy.

In particular, equation (7) is replaced by a second order Stirling interpolation which yields for the equity value

\[ e_n = E(\bar{\eta}_{n|n-1}, \cdot) + \frac{1}{2h} [E(\bar{\eta}_{n|n-1} + h, \cdot) - E(\bar{\eta}_{n|n-1} - h, \cdot)] \Delta \eta + \frac{1}{h^2} \left[ E \left( \eta_{n|n-1} + \frac{h}{2}, \cdot \right) - E \left( \eta_{n|n-1} - \frac{h}{2}, \cdot \right) \right] (\Delta \eta)^2 + \varepsilon_{e,n} \tag{16} \]

where \( h \) is the step size of the approximation and \( \Delta \eta = (\eta_n - \bar{\eta}_{n|n-1}) \).\(^{19}\)

Alternatively, we could have used the linearly transformed variable

\[ Z = \frac{\eta}{s} \tag{17} \]

with \( s \) being some constant for an approximation of the function

\[ \bar{E}(Z, \cdot) \equiv E(sZ, \cdot) = E(\eta, \cdot). \tag{18} \]

This transformation changes the power series approximation of the original function to

\[ e_n = E(\bar{\eta}_{n|n-1}, \cdot) + \frac{1}{2h} [E(\bar{\eta}_{n|n-1} + hs, \cdot) - E(\bar{\eta}_{n|n-1} - hs, \cdot)] \Delta \eta + \frac{1}{h^2} \left[ E \left( \eta_{n|n-1} + \frac{h}{2}s, \cdot \right) - E \left( \eta_{n|n-1} - \frac{h}{2}s, \cdot \right) \right] (\Delta \eta)^2 + \varepsilon_{e,n} \tag{19} \]

which is strictly different to equation (16).\(^{20}\) By picking \( s \) to be the square root of \( \bar{\Sigma}_\eta \) and \( h = 3 \) as the kurtosis of the underlying distribution, the

\(^{19}\)For more complicated observation error models, \( \varepsilon_{e,n} \) would be replaced by a sum of approximation terms involving \( h \) around the expected observation error.

\(^{20}\)Note that due to equation (18), the Taylor series expansion around \( \eta_{n|n-1} \) is invariant to linear transformations.
transformation optimally approximates the stochastic function \( E(\eta_n, \cdot) \).\(^{21}\)

Similarly, all model bond prices \( D_{C_j T_j} \) can be approximated. In Subsection 5 evidence is presented that the Kalman filter outperforms the estimation procedures proposed by Ericsson and Reneby (2002) because all observable variables are incorporated in the measurement of the state variable whereas Ericsson and Reneby (2002) only use equity prices.\(^{22}\)

3.6 Parameter Estimation and Inference

The Kalman filter framework of the last subsection is convenient because it provides the necessary information for a maximum-likelihood estimator and inference. From equations (8) and (10), the observation error \( v_n \), \( n = 1, \ldots, N \) is normally distributed with

\[
v_n \sim N(0, \bar{\Sigma}_Y(n|n-1)). \tag{20}
\]

Therefore, the density of the estimated observation given the true observation is

\[
f_{\hat{Y}_{n|n-1}}(y_n|\hat{Y}_{n-1}, \hat{\eta}_{n-1}, \Theta) = \frac{1}{\sqrt{2\pi} \bar{\Sigma}_Y(n|n-1)^{\frac{1}{2}}} \exp \left\{ -\frac{1}{2} v_n^{\prime} \bar{\Sigma}_Y(n|n-1)^{-1} v_n \right\}. \tag{21}
\]

and the log likelihood\(^{23}\) of the whole filter becomes

\[
L(\Theta) = \sum_{n=1}^{N} \log \left[ f_{\hat{Y}_{n|n-1}}(y_n|\hat{Y}_{n-1}, \hat{\eta}_{n-1}, \Theta) \right]. \tag{22}
\]

All functions of equation (21) depend on the parameter vector \( \Theta \) which can be estimated by maximizing the likelihood function of equation (22).

\[
\hat{\Theta}_{ML} = \sup_{\Theta} L(\Theta). \tag{23}
\]

As an alternative to equation (23), another reasonable objective function can be used that has properties as to converge to a solution quicker and more robustly, i.e. the minimum of the sum of squared or absolute pricing

\(^{21}\)See Norgaard et al. (2000) for the argument of optimally choosing \( h \) and \( s \) in the sense that the approximation error is minimized.

\(^{22}\)See also Ericsson and Reneby (2004).

\(^{23}\)See e.g. Hamilton (1994), Chapter 13 for the details.
errors

\[ \hat{\Theta}_{SSQE} = \inf_{\Theta} \sum_{n=1}^{N} v_n' v_n, \]  
\[ \hat{\Theta}_{SAE} = \inf_{\Theta} \sum_{n=1}^{N} |v_n| \]  

(24) (25)

Since all three estimators belong to the class of extremum estimators and as such to general method of moment estimators, they converge to the true parameter values \( \Theta_0 \) as \( N \to \infty \). Therefore, irrespective of the objective function, the maximum-likelihood function of equation (22) can be used for inference. By the fact that observation error \( v_n \) are Gaussian, the estimated parameter \( \hat{\Theta} \) is

\[ \sqrt{N}(\hat{\Theta} - \Theta_0) \sim N(0, \hat{H}^{-1}) \]  

(26)

where the estimated information matrix

\[ \hat{H} = -\sum_{n=1}^{N} \frac{\partial^2 \log f_{\tilde{y}_n|\hat{\tilde{y}}_{n-1},\hat{\eta}_{n-1},\Theta}}{\partial \Theta \partial \Theta'} \bigg|_{\Theta = \hat{\Theta}} \]

\[ = -\sum_{n=1}^{N} \hat{H}_n \]

converges to its true value \( H_0 \) in probability for \( N \to \infty \). For stock and bond prices, similar asymptotic distributions can be derived by the delta-method

\[ \sqrt{N}(\hat{E}(\eta_n, \cdot) - E(\eta_n, \cdot)) \sim N(0, \hat{H}_{E,n}) \]  

(27)

with

\[ \hat{H}_{E,n} = -\frac{\partial E(\eta_n, \cdot)}{\partial \Theta'} \bigg|_{\Theta = \hat{\Theta}} \hat{H}_n^{-1} \frac{\partial E(\eta_n, \cdot)}{\partial \Theta} \bigg|_{\Theta = \hat{\Theta}} \]

and

\[ \sqrt{N}(\hat{D}(\eta_n, \cdot) - D(\eta_n, \cdot)) \sim N(0, \hat{H}_{D,n}) \]  

(28)

with

\[ \hat{H}_{D,n} = -\frac{\partial D(\eta_n, \cdot)}{\partial \Theta'} \bigg|_{\Theta = \hat{\Theta}} \hat{H}_n^{-1} \frac{\partial D(\eta_n, \cdot)}{\partial \Theta} \bigg|_{\Theta = \hat{\Theta}}. \]

---

24 See Mittelhammer, Judge and Miller (2000), Part III and Chapter 11.
26 See e.g. Campbell, Lo and MacKinlay (1997), Appendix A4.
4 Implementing the Corporate Securities Framework

The Corporate Securities Framework of Genser (2005b) is structured modular where the firm value is distributed among all claimants of the firm’s EBIT. To be fully specified the whole financing structure is needed to calculate security prices because the bankruptcy event interrelates all debt claims to one another. However, the value of all claims is not observed on a regular basis. Therefore, the model has to be restricted so that observations of only part of the issued securities suffices to estimate parameter values. Additional assumptions are needed.

Consider first the assumption that the total amount of debt outstanding is left unchanged, i.e. 
\[
\bar{P} = \sum_{j=1}^{J} P_j
\]
for all outstanding debt contracts \(j = 1, \ldots, J\) and all future points in time. The constant total debt volume makes the recovery values of individual debt contracts independent of the bankruptcy time since each outstanding debt issue recovers a fraction of the firm value after bankruptcy losses and taxes, that is proportional to total debt outstanding. So, the recovery fraction depends only on the initial debt structure. Therefore, maturing debt contracts are refinanced by issuing new debt. As shown in Genser (2005b), future debt issues flatten the optimal bankruptcy barrier. It is safe to assume \(\eta_B\) to be time invariant which eases the calculation of bankruptcy probabilities and prices considerably.

The value of a debt contract simplifies to
\[
D_{C_k, T_k} = e^{-r(T_k-t_0)} \left[ P_k - (1 - \tau^d) \frac{C_k}{r} \right] \left[ 1 - \Phi(t_0, T_k, \eta_{t_0}, \eta_B) \right] + (1 - \tau^d) \frac{C_k}{r} \left[ 1 - p_B(t_0, T_k, \eta_{t_0}, \eta_B) \right] + D_{C_k, T_k}^-.
\]

Recall from Genser (2005b) that \(\Phi(t, T, \eta_t, \eta_B)\) and \(p_B(t, T, \eta_t, \eta_B)\) denote the probability of going bankrupt in the period \([t, T]\) when EBIT starts at \(\eta_t\) facing the constant barrier \(\eta_B\) and the Arrow-Debreu price of a claim that pays one currency unit in bankruptcy, respectively. The time \(t_0\)-value of the funds that bond holders are able to recover in bankruptcy are
\[
D_{C_k, T_k}^- = (1 - \tau^{eff}) \min \left[ (1 - \alpha)V_B; \bar{P} \right] w_k p_B(t_0, T_k, \eta_{t_0}, \eta_B).
\]

where \(w_k = P_k/\bar{P}\) represents the fraction of bond \(k\) with respect to total debt \(\bar{P}^2\).

The original framework only allows for a single, constant risk-free interest rate. Even if we abstract from the stochasticity of risk-free interest

---

27See Genser (2005b) for a detailed derivation of the formulas and the exact definition of variables.
rates, the daily change of interest rates should be considered in the estimation process and a suitable maturity within the term structure has to be chosen. Denote by $B_{C_k,T_k}$ the value of a risk-free bond with the same features and tax treatment as the corporate bond $D_{C_k,T_k}$. The corporate bond price can be restated as

$$D_{C_k,T_k} = B_{C_k,T_k} [1 - \Phi(t_0, T_k, \eta_{t_0}, \eta_B)] - B_{C_k,\infty} [p_B(t_0, T_k, \eta_{t_0}, \eta_B)] - \Phi(t_0, T_k, \eta_{t_0}, \eta_B)] + D_{C_k,T_k}.$$  

(29)

If we allow for future starting bond issues, its value would be

$$D_{C_k,S_k,T_k} = B_{C_k,S_k,T_k} [1 - \Phi(t_0, T_k, \eta_{t_0}, \eta_B)] - B_{C_k,\infty} [p_B(S_k, T_k, \eta_{t_0}, \eta_B)] - \Phi(S_k, T_k, \eta_{t_0}, \eta_B)]B_{0,S_k} - \Phi(S_k, T_k, \eta_{t_0}, \eta_B)]B_{0,S_k} P_k I_{\{S_k > t\}} + D_{C_k,T_k}.$$  

(30)

where $S_k$ denotes the bonds issue time. The default probability during the bond’s life is

$$\Phi(S_k, T_k, \eta_{t_0}, \eta_B) = \Phi(t_0, T_k, \eta_{t_0}, \eta_B) - \Phi(t_0, S_k, \eta_{t_0}, \eta_B),$$

and the Arrow-Debreu price of bankruptcy for the subperiod $[S_k, T_k]$ is

$$p_B(S_k, T_k, \eta_{t_0}, \eta_B) = p_B(t_0, T_k, \eta_{t_0}, \eta_B) - p_B(t_0, S_k, \eta_{t_0}, \eta_B).$$

The dependence of the corporate bond price in equations (29) and (30) on the choice of the term of the risk-free interest rate is reduced considerably. Since the whole term-structure of risk-free interest rates is known at each point in time, the price of the risk-free equivalent bonds can be calculated easily. However, the risk-free interest rate $r$ still remains explicit when determining the equity value and might be set to a term between five and ten years which should approximately equal the average payback time of a firm’s investments.

A firm’s total liabilities $P$ can be set to the reported amount as of the last available annual report. Note that this value usually represents the notional amount of debt outstanding and not its current value as of the balance sheet date. We might also consider $P$ to be a constant or deterministically changing parameter determined within the estimation process.
5 The Simulation Study

5.1 Experiment Design

To evaluate whether the Kalman filter proposed in Section 3.5 is able to detect the parameters of the EBIT-process correctly, a simulation study was conducted for the ABM- and the GBM base case firm introduced in Section 3.1. With respect to the discussion in Subsection 4, where it was proposed to allow for future debt issues to simplify and speed up the Kalman filter evaluations, it is assumed that the firm refines each maturing debt issues by a 6 % coupon bond with an identical notional and infinite maturity.\textsuperscript{28} The model parameters are summarized in Table 1.

<table>
<thead>
<tr>
<th>Panel A: Model Parameters</th>
<th>Panel B: Initial Financing Structure</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Economic Variables</strong></td>
<td><strong>Firm Specific Variables</strong></td>
</tr>
<tr>
<td>$r$</td>
<td>$\eta_0$</td>
</tr>
<tr>
<td>5 %</td>
<td>100</td>
</tr>
<tr>
<td>$\tau^c$</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>35 %</td>
<td>50 %</td>
</tr>
<tr>
<td>$\tau^d$</td>
<td>$V_B$</td>
</tr>
<tr>
<td>10 %</td>
<td>3'050</td>
</tr>
<tr>
<td>$\tau^e$</td>
<td>$\mu$</td>
</tr>
<tr>
<td>10 %</td>
<td>10</td>
</tr>
<tr>
<td>$\varepsilon_{e,n}$</td>
<td>$\sigma_{\eta}$</td>
</tr>
<tr>
<td>$N(0, 0.005)$</td>
<td>40</td>
</tr>
<tr>
<td>$\varepsilon_{d,j,n}$</td>
<td>18 %</td>
</tr>
<tr>
<td>$N(0, 0.0025)$</td>
<td></td>
</tr>
<tr>
<td>$\forall n, j$</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Model parameters of the ABM- and GBM-firm in the Kalman filter simulation study

\textsuperscript{28}As illustrated in Ammann and Genser (2005), the refinancing bonds have a current value of approximately zero.
in Table 1, for both the ABM- and the GBM-firm, 500 EBIT-paths of 50 daily observations have been simulated. For each EBIT-realization, the security values have been calculated according to pricing equations as of Genser (2005a) and Ammann and Genser (2005). The security values have then been translated into price quotes, assuming that the firm issued 200 stocks in total. All price quotes have been perturbed by an observation error drawn from the distribution of $\varepsilon_{e,n}$ and $\varepsilon_{d,j,n}$.

The stock price and the dirty prices of the currently issued bonds, i.e. bonds $j = 1, \ldots, 4$, are observable. The parameter vector $\Theta$ was reduced to contain the process parameters $\mu$ and $\sigma_\eta$ only.

In principle, we could have included the bankruptcy barrier $V_B$ and/or the refinancing coupons $C_j$, $j > 4$ in the parameter vector, as well. If the bankruptcy barrier were included, we might have allowed equity holders to pick the bankruptcy barrier optimally assuming that the firm is already in the infinite maturity state. This appears viable because it was shown in Ammann and Genser (2005) that the inclusion of future debt issues flattens the optimal bankruptcy barrier. However, for each pair of $\mu$ and $\sigma_\eta$, the flat optimal bankruptcy barrier is equivalent to a deterministically set constant bankruptcy barrier without the additional burden to calculate the optimal bankruptcy barrier. Therefore, for estimating the Kalman filter, the optimization of the bankruptcy barrier would only add complexity without making the simulation study more accurate. A similar argument is true for the refinancing coupons. If included in the parameter vector $\Theta$, refinancing coupons would have to be determined by finding the par forward yields of the refinancing bonds. The refinancing coupons change for every single simulated EBIT due to changes of the term structure of bankruptcy probabilities. However, for each parameter choice of $\mu$ and $\sigma_\eta$, the Kalman filter delivers the best estimates of the latent EBIT times series given that the par forward yields of the refinancing bond issues have been determined by these best estimates. Due to the unique relationship between EBIT and refinancing coupons, the refinancing coupons can as well be fixed for the simulation study.

Preliminary tests have shown that the log-likelihood function of equation (22) encounters problems with starting values for $\mu$ and $\sigma_\eta$ that are too far from the true parameters because the likelihood of the filter becomes close to zero. In contrast, the sum of absolute pricing differences of equation (25) is a less sensitive objective function for arbitrary starting values. It converges quickly to the global solution, i.e. the true parameters, as long as the starting parameters of the filter lie above the true parameters. If the starting parameters are chosen below the true parameters, the estimation procedure eventually converges to a local minimum whereas only very few runs could actually identify the true parameters. So, a two step estimation procedure seems appropriate: A first estimation
run is performed by minimizing the sum of absolute pricing errors with starting values far above the true parameters. A second run maximizes the log-likelihood function by starting the estimation at the estimates of the first run.

5.2 Parameter Estimation Results

Figure 1: Histogram of the estimated ABM-EBIT risk-neutral drift and volatility parameter from 500 simulation runs á 50 periods by minimizing the sum of absolute pricing errors. True parameter values are $\mu = 10$ and $\sigma_\eta = 40$.

Figure 1 collects the estimated ABM-risk-neutral drift and volatility parameters of the first estimation step in histograms. In contrast to Ericsson and Reneby (2004), we are able to identify both the volatility
and the risk-neutral drift.\textsuperscript{29} Both estimators appear to be unbiased and the true parameters lie within the 90\% confidence bounds of the simulated parameter distribution. The confidence bounds itself are sufficiently small given that we only used 50 data points for each simulation. The second round of estimation produced even closer confidence bounds (see Figure 2).

Panels A and B of Table 2 compare the statistics of the parameter, state variable and security price estimates of the two estimation steps.

\textsuperscript{29}To overcome their difficulties, Ericsson and Reneby (2004) need to set the EBIT-drift to the risk-less interest rate adjusted for an exogenously estimated constant cash payout ratio per period.

Figure 2: Histogram of the estimated ABM-EBIT risk-neutral drift and volatility parameter from 500 simulation runs á 50 periods by maximizing the likelihood function. True parameter values are $\mu = 10$ and $\sigma_{\eta} = 40$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{histogram.png}
\caption{Histogram of the estimated ABM-EBIT risk-neutral drift and volatility parameter from 500 simulation runs á 50 periods by maximizing the likelihood function. True parameter values are $\mu = 10$ and $\sigma_{\eta} = 40$.}
\end{figure}
Table 2: Simulated distribution summary statistics of the estimated ABM-Corporate Securities Framework. The table reports mean errors $ME$, mean absolute errors $MAE$, maximum $MAX$ and minimum errors $MIN$, as well as standard errors $STD$, skewness $SK EW$ and kurtosis $KU RT$ of the distribution. The Jarque-Bera test statistics and its p-value are also provided. The simulation comprised 500 runs á 50 periods.

### Panel A: Minimizing Absolute Pricing Errors

<table>
<thead>
<tr>
<th></th>
<th>Simulated Error Distribution</th>
<th>Jarque/Bera</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$ME$ $MAE$ $MAX$ $MIN$ $STD$ $SK EW$ $KU RT$</td>
<td>JBSTAT P-Val.</td>
</tr>
<tr>
<td>$\hat{\sigma}<em>\eta - \sigma</em>\eta$</td>
<td>-0.0017 0.0545 0.2132 -0.1904 0.0689 0.1732 3.0325</td>
<td>2.4921 0.2876</td>
</tr>
<tr>
<td>$\hat{\mu} - \mu$</td>
<td>-0.0022 0.0330 0.1363 -0.1206 0.0413 0.2612 3.0598</td>
<td>5.6995 0.0579</td>
</tr>
<tr>
<td>$\hat{\eta} - \eta$</td>
<td>0.0363 0.7045 3.5273 -3.2213 0.8836 0.2089 3.0966</td>
<td>191.4799 0.0000</td>
</tr>
<tr>
<td>$E - E$</td>
<td>0.0009 0.0531 0.2783 -0.2794 0.0673 -0.0047 3.1220</td>
<td>15.5345 0.0004</td>
</tr>
<tr>
<td>$D_{4.50,0.2} - D_{4.50,0.2}$</td>
<td>0.0003 0.0392 0.2122 -0.2047 0.0492 -0.0165 3.0440</td>
<td>2.0871 0.3522</td>
</tr>
<tr>
<td>$D_{5.00,0.4} - D_{5.00,0.4}$</td>
<td>0.0014 0.0363 0.1791 -0.1767 0.0459 -0.0077 3.0632</td>
<td>4.3822 0.1118</td>
</tr>
<tr>
<td>$D_{5.50,0.10} - D_{5.50,0.10}$</td>
<td>0.0016 0.0337 0.1969 -0.1674 0.0421 -0.0140 3.0012</td>
<td>0.8234 0.6625</td>
</tr>
<tr>
<td>$D_{6.00,0.\infty} - D_{6.00,0.\infty}$</td>
<td>0.0020 0.0313 0.1501 -0.1680 0.0394 0.0540 3.0307</td>
<td>11.5779 0.0031</td>
</tr>
</tbody>
</table>

The maximum-likelihood estimation improves parameter estimates considerably. We observe that the second step not only reduces the range of the estimates but also the mean errors and standard errors. Furthermore, the Jarque-Bera statistic for testing normality of the parameter estimation error distributions would reject the normality hypothesis for the risk-neutral drift in the first estimation step at the 5% level. The normality of the maximum-likelihood estimator cannot be rejected. The mean estimation error and standard error of the state variable and security prices are small although differences can become large in single periods (see the $MIN$- and $MAX$-columns in Table 2). Tests for normality of estimation errors of the state variable and long-term security must be rejected in the first estimation step at the 1% level. The maximum-likelihood estimation improves the situation. At least, the Jarque-Bera statistic of the equity pricing errors can no longer reject normality.

The results for the GBM-simulation study are different. Figure 3 illustrates that the true parameters fall within the confidence bounds. The
Figure 3: Histogram of the estimated GBM-EBIT risk-neutral drift and volatility parameter from 500 simulation runs á 50 periods by minimizing the sum of absolute pricing errors. True parameter values are $\bar{\mu} = 3.3\%$ and $\bar{\sigma}_\eta = 18\%$.

Parameter estimates are symmetrically distributed but slightly biased upwards. However, the maximum-likelihood estimation does not improve the parameter estimates (Figure 4). The bias moves downwards and Table 3 confirms that standard errors increase substantially. The same observation can be made for the state variable estimates. Estimated security prices are less affected. Although normality of the estimated parameters cannot be rejected in the first estimation step, the Jarque-Bera statistic exceeds the critical value at all reasonable confidence values in the second step.

The problems of the maximum-likelihood estimation in the GBM-case are most likely due to the small sample size of 50 time steps per simul-
Figure 4: Histogram of the estimated GBM-EBIT risk-neutral drift and volatility parameter from 500 simulation runs á 50 periods by maximizing the likelihood function. True parameter values are $\mu = 3.3\%$ and $\sigma_\eta = 18\%$.

In the GBM-case, the state variable $\eta_n$ is log-normally distributed. This non-linearity of the state variable translates $\hat{\Theta}$ of equation (23) into a quasi-maximum-likelihood estimator where normality is achieved only asymptotically as $N \to \infty$. Nevertheless, the proposed Kalman filter seems to work well even for the GBM-EBIT model.

In a second experiment, we test whether the filter is sensitive to a correctly specified observation error. The observation error of security prices is omitted. The two estimation steps are conducted as described above including the assumption that prices are observed with the indicated errors as of Table 1, Panel A. $^{30}$ The estimators of the risk-neutral EBIT-drift

$^{30}$See the Figures 5 to 8 and Tables 4 to 5 in the appendix.
Table 3: Simulated distribution summary statistics of the estimated GBM-Corporate Securities Framework. The table reports mean errors \( ME \), mean absolute errors \( MAE \), maximum \( MAX \) and minimum errors \( MIN \), as well as standard errors \( STD \), skewness \( SKEW \) and kurtosis \( KURT \) of the distribution. The Jarque-Bera test statistics and its p-value are also provided. The simulation comprised 500 runs á 50 periods.

### Panel A: Minimizing Absolute Pricing Errors

<table>
<thead>
<tr>
<th>Simulated Error Distribution</th>
<th>Jarque/Bera</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\sigma}<em>\eta - \sigma</em>\eta )</td>
<td>0.0001 0.0002 0.0008 -0.0005 0.0002 -0.0543 3.2203</td>
</tr>
<tr>
<td>( \mu - \hat{\mu} )</td>
<td>0.0001 0.0001 0.0004 -0.0003 0.0001 0.0127 2.9627</td>
</tr>
<tr>
<td>( \eta - \hat{\eta} )</td>
<td>0.3521 0.6970 3.1401 -2.6597 0.8080 0.0501 3.0901</td>
</tr>
<tr>
<td>( E - E )</td>
<td>-0.0033 0.0382 0.2688 -0.2273 0.0489 -0.0049 3.4543</td>
</tr>
<tr>
<td>( D_{4.50 %,2} - D_{4.50 %,2} )</td>
<td>0.0009 0.0384 0.2020 -0.1838 0.0482 -0.0177 3.0325</td>
</tr>
<tr>
<td>( D_{5.00 %,4} - D_{5.00 %,4} )</td>
<td>0.0023 0.0588 0.2992 -0.3393 0.0744 -0.0119 3.1024</td>
</tr>
<tr>
<td>( D_{5.50 %,10} - D_{5.50 %,10} )</td>
<td>0.0002 0.0553 0.2907 -0.2688 0.0489 -0.0049 3.4543</td>
</tr>
<tr>
<td>( D_{6.00 %,\infty} - D_{6.00 %,\infty} )</td>
<td>-0.0029 0.0531 0.2871 -0.2715 0.0672 0.0048 3.1208</td>
</tr>
</tbody>
</table>

### Panel B: Maximizing Log Likelihood

<table>
<thead>
<tr>
<th>Simulated Error Distribution</th>
<th>Jarque/Bera</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\sigma}<em>\eta - \sigma</em>\eta )</td>
<td>-0.0001 0.0005 0.0014 -0.0022 0.0006 -0.3128 3.3932</td>
</tr>
<tr>
<td>( \mu - \hat{\mu} )</td>
<td>-0.0001 0.0002 0.0006 -0.0009 0.0003 -0.2992 3.3159</td>
</tr>
<tr>
<td>( \eta - \hat{\eta} )</td>
<td>-0.4252 1.2506 4.3028 -5.9902 1.5576 -0.2781 3.2759</td>
</tr>
<tr>
<td>( E - E )</td>
<td>-0.0035 0.0383 0.2644 -0.2255 0.0486 0.0086 3.3984</td>
</tr>
<tr>
<td>( D_{4.50 %,2} - D_{4.50 %,2} )</td>
<td>0.0001 0.0386 0.2062 -0.1837 0.0485 -0.0212 3.0364</td>
</tr>
<tr>
<td>( D_{5.00 %,4} - D_{5.00 %,4} )</td>
<td>-0.0101 0.0635 0.3219 -0.3621 0.0792 -0.0289 3.1010</td>
</tr>
<tr>
<td>( D_{5.50 %,10} - D_{5.50 %,10} )</td>
<td>-0.0134 0.0606 0.3347 -0.3153 0.0747 -0.0040 3.0146</td>
</tr>
<tr>
<td>( D_{6.00 %,\infty} - D_{6.00 %,\infty} )</td>
<td>-0.0126 0.0590 0.2824 -0.3125 0.0732 -0.0326 3.0440</td>
</tr>
</tbody>
</table>

and EBIT-volatility of both the ABM- and GBM-filter are biased. The confidence intervals of the parameter estimates of the ABM-filter still contain the true values in both steps. Similarly to the first simulation experiment, the first step estimation of the GBM-filter produces estimators biased upwards whereas the maximum-likelihood estimators are biased downwards. Confidence intervals do not contain the true values. The GBM-filter is sensitive to a misspecification of observation errors in small samples. A preliminary test of the GBM-filter for time series of \( N = 200 \) price observations per security confirms that the distribution of the estimators moves closer to normality with its center approaching the true parameter values. The ABM-filter seems to be robust against misspecifications.

Although further research seems to be warranted on the small sample properties of the objective functions (23), (24), and (25), our results are promising. The objective functions converge in a way that the parameter estimators are close to the true parameters of the sampled time series

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although we introduce a quite substantial pricing error of several basis points in bond prices and several percent in equity prices.

6 Summary

In this chapter a state-space representation of the Corporate Securities Framework of Genser (2005a) is proposed. In contrast to the existing literature, the estimation procedure is able to incorporate multiple time series of corporate security prices. Since the EBIT-process can be chosen freely, we can test which type of EBIT-process fits security price movements best.

In a simulation study, we show that estimators based on the Kalman filter can identify the true parameters of the EBIT-process. For the ABM- and the GBM-filter, 500 time series of 50 daily observations of stock and bond prices are simulated where each observation is measured with a zero-mean error. The estimation is performed in two step. The first parameter estimates were achieved by minimizing the sum of absolute pricing errors. This objective function allows for almost arbitrary starting values as long as the estimation does not start in a bankruptcy state and the starting values of EBIT-drift and volatility lie above the true parameter values. A standard search algorithm is then able to approach the true parameter values relatively quickly. Given the first estimates, the log likelihood function of the filter is maximized which is expected to refine the parameter estimates. The Kalman filter is not only able to identify the EBIT-volatility but also the EBIT-risk-neutral drift from the imprecisely observed security prices which no other method was able to before. In previous empirical studies, the drift has been arbitrarily set to the risk-free rate adjusted for an estimate of a constant payout ratio disregarding the theoretical inconsistency that the payout ratio changes together with EBIT in a dynamic way.

Both filters work remarkably well despite our small sample size of only 50 data points per filter optimization. However, the GBM-estimators do not improve in the second estimation step which is attributable to the non-linearity of the state function. The small sample properties of the GBM-model are therefore less favorable. Larger sample time-series are needed for better convergence.

The filter even works under a second setting where security pricing errors are misspecified. Although the estimators are biased, both filters find optima close to the true parameter values. However, for the GBM-filter more observations are needed to get reasonable sample distributions.

Having succeeded in a simulation study, we can take the next step and use time series of bond and equity prices of individual firms to test the
model directly. Section 4 discusses some of the practical issues. Depending on the individual firm and the information available about the firm, additional assumption might be needed. However, the proposed model gives a sound foundation for reasonable and consistent adjustments.

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### A Additional Figures and Tables
Figure 5: Histogram of the estimated ABM-EBIT risk-neutral drift and volatility parameter from 500 simulation runs á 50 periods by minimizing the sum of absolute pricing errors. True parameter values are $\mu = 10$ and $\sigma_\eta = 40$. Observation errors are assumed in the estimation although the observation matrix is free of measurement error.
Figure 6: Histogram of the estimated ABM-EBIT risk-neutral drift and volatility parameter from 500 simulation runs à 50 periods by maximizing the likelihood function. True parameter values are $\mu = 10$ and $\sigma_\eta = 40$. Observation errors are assumed in the estimation although the observation matrix is free of measurement error.
Table 4: Simulated distribution summary statistics of the estimated ABM-Corporate Securities Framework. The table reports mean errors $ME$, mean absolute errors $MAE$, maximum $MAX$ and minimum errors $MIN$, as well as standard errors $STD$, skewness $SKEW$ and kurtosis $KURT$ of the distribution. The Jarque-Bera test statistics and its p-value are also provided. The simulation comprised 500 runs at 50 periods. Observation errors are assumed in the estimation although the observation matrix is free of measurement error.

### Panel A: Minimizing Absolute Mean Errors

<table>
<thead>
<tr>
<th></th>
<th>Simulated Error Distribution</th>
<th>Jarque/Bera</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$ME$ $MAE$ $MAX$ $MIN$ $STD$ $SKEW$ $KURT$</td>
<td>JBSTAT P-Val.</td>
</tr>
<tr>
<td>$\sigma_\eta - \hat{\sigma}_\eta$</td>
<td>0.0018 0.0024 0.0125 -0.0094 0.0026 0.4098 4.2998</td>
<td>48.1868 0.0000</td>
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<tr>
<td>$\mu - \hat{\mu}$</td>
<td>0.0020 0.0027 0.0088 -0.0129 0.0026 -0.6440 5.4828</td>
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<td>$\eta - \hat{\eta}$</td>
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<td>$E - \hat{E}$</td>
<td>0.0015 0.0043 0.0529 -0.0291 0.0026 1.7115 8.0112</td>
<td>48355.3198 0.0000</td>
</tr>
<tr>
<td>$D_{4.50,2} - \hat{D}_{4.50,2}$</td>
<td>0.0003 0.0007 0.0127 -0.0017 0.0010 2.4963 14.9075</td>
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<tr>
<td>$D_{5.00,4} - \hat{D}_{5.00,4}$</td>
<td>0.0015 0.0044 0.0533 -0.0126 0.0062 1.7010 7.1668</td>
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<tr>
<td>$D_{5.50,10} - \hat{D}_{5.50,10}$</td>
<td>0.0017 0.0065 0.0767 -0.0231 0.0091 1.6580 6.8205</td>
<td>26652.2479 0.0000</td>
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<tr>
<td>$D_{6.00,\infty} - \hat{D}_{6.00,\infty}$</td>
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### Panel B: Maximizing Log Likelihood

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<tr>
<td>$\eta - \hat{\eta}$</td>
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</table>
Figure 7: Histogram of the estimated GBM-EBIT risk-neutral drift and volatility parameter from 500 simulation runs à 50 periods by minimizing the sum of absolute pricing errors. True parameter values are $\bar{\mu} = 3.3\%$ and $\bar{\sigma}_\eta = 18\%$. Observation errors are assumed in the estimation although the observation matrix is free of measurement error.
Figure 8: Histogram of the estimated GBM-EBIT risk-neutral drift and volatility parameter from 500 simulation runs at 50 periods by maximizing the likelihood function. True parameter values are $\bar{\mu} = 3.3\%$ and $\bar{\sigma}_\eta = 18\%$. Observation errors are assumed in the estimation although the observation matrix is free of measurement error.
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### Panel A: Minimizing Absolute Mean Errors

<table>
<thead>
<tr>
<th>Simulated Error Distribution</th>
<th>Jarque/Bera</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\sigma}<em>\eta - \hat{\sigma}</em>\eta$</td>
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</tr>
<tr>
<td>$\mu - \hat{\mu}$</td>
<td>$0.0000$</td>
</tr>
<tr>
<td>$\eta - \hat{\eta}$</td>
<td>$0.2460$</td>
</tr>
<tr>
<td>$E - E$</td>
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<td>$D_{5.00 %, 4} - D_{5.00 %, 4}$</td>
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<td>$D_{5.50 %, 10} - D_{5.50 %, 10}$</td>
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<tr>
<td>$D_{6.00 %, \infty} - D_{6.00 %, \infty}$</td>
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### Panel B: Maximizing Log Likelihood

<table>
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<th>Simulated Error Distribution</th>
<th>Jarque/Bera</th>
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<tbody>
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<td>$\mu - \hat{\mu}$</td>
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<td>$\eta - \hat{\eta}$</td>
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