Optimal Investment Timing
When External Financing Is Costly

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Abstract

This paper bridges the gap between investment timing options and investment-cash flow sensitivities of financially constrained firms. In a real options model with costly external financing we derive optimal investment timing as a trade-off of present and expected future financing costs.

While a pure-quantity constrained firm always overinvests in investment states, we find that financing costs can induce both voluntary delay and acceleration of investment, and we show that both investment volume and investment-cash flow sensitivities are non-monotonic in financing constraints.

For high- and low-liquidity firms, we find that sensitivities are increasing in constraints, and that both dimensions of constraints, namely liquidity constraints and capital market frictions inducing financing costs, have similar effects on investment. In contrast, the effects are ambiguous for firms with intermediate liquidity.

JEL Classification: G31, G32

Keywords: investment timing, underinvestment, overinvestment, investment-cash flow sensitivities, real option valuation, costly external financing
1 Introduction

Since the seminal article of Modigliani and Miller (1958) it is a well-known fact in Corporate Finance theory that investment and financing decisions can be made independently on frictionless markets. In consequence, there is a wide range of literature analyzing either the investment or the financing decision of the firm.

More recent literature deals with the fact that investment and financing decisions cannot be separated in the presence of financing constraints. There has been empirical as well as theoretical research attempting to answer how a firm’s investment policy is affected by financing constraints.\(^1\) However, these studies have come to ambiguous conclusions. While early empirical research starting with Fazzari, Hubbard, and Petersen (1988) claimed that financially more constrained firms should have higher sensitivities of investment volume to cash flow, this was later questioned most prominently by Kaplan and Zingales (1997). Although the latter presented both theoretical and empirical evidence why investment-cash flow sensitivities do not need to increase in constraints, they still took for granted that investment volume itself is decreasing in constraints.

Two very recent modelling approaches show that investment levels need not be decreasing in liquidity constraints. In a 'now-or-never' investment decision framework with continuously adjustable investment levels Cleary, Povel, and Raith (2004) explain non-monotonic investment levels in liquidity constraints by non-monotonic marginal costs of debt financing. Boyle and Guthrie (2003) highlight the dynamic perspective of investment decisions while focusing on lumpy investment. Although investment volume is not a decision variable within their approach, their result that whenever firms do not face binding constraints, they are willing to invest in less favorable projects when liquidity constraints increase can be interpreted as evidence against the 'classical' view of investment volume being decreasing in constraints.

\(^1\)There is other literature dealing with the interaction of investment and financing decisions, but focusing on aspects other than financing constraints: Mauer and Triantis (1994) determine optimal investment, operating, and financing policies in a real-option framework given that there are switching costs for policy changes. They find that debt financing has a negligible influence on investment timing. More recently, Mauer and Sarkar (2005) and others discuss the agency cost of over- and underinvestment when there are stockholder-bondholder conflicts about investment policy. Mello and Parsons (2000) focus on the value that is created by the hedging policy of financially constrained firms, which is pioneered by Froot, Scharfstein, and Stein (1993) and is examined for our modelling framework by Boyle and Guthrie (2004).
As Boyle and Guthrie (2003) we focus on the timing aspect of investment. However, we do not only allow for liquidity constraints but consider market frictions also inducing financing costs. Interestingly, stronger constraints can lead to both more and less investment as the result of an endogenous trade-off of present and possible future financing costs, depending on the firm’s level of liquid funds. Thus, we can explain non-monotonicity of investment in financing constraints as a firm’s optimal decision in a dynamic investment framework.

More precisely, we show that for high-liquidity firms the fear of future costs is most important, therefore the firm makes less use of the value of waiting, and investment is increasing in constraints. In contrast, for low-liquidity firms avoiding present costs is most important, and investment is decreasing in constraints. Among either high- or low-liquidity firms, stronger constraints amplify these effects on investment, therefore the sensitivity of investment to cash flow is increasing in constraints. For firms with intermediate levels of liquidity, there are ambiguous effects since both present and future costs are important.

Following Cleary, Povel, and Raith (2004) and Lyandres (2005), we separately examine these effects for both dimensions of financing constraints, namely liquidity constraints and capital market frictions inducing financing costs. We show that – at least for high- and low-liquidity firms – both dimensions of financing constraints have similar effects on both investment volume and the associated sensitivities.

This work is organized as follows: Section 2 presents the framework of the model. In Section 3 we turn to the investment decision: We compare the investment threshold function and thus optimal investment timing, as well as today’s value of the firm, to both an unconstrained firm and a quantity constrained firm with costless financing. Section 4 provides a comparative statics analysis for the investment threshold and connects our results to existing findings on investment volume and investment-cash flow sensitivities. Section 5 concludes.

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2An empirical approach that directly tests investment timing is by Whited (2005). She focuses on lumpy investment and the timing of large investment projects rather than incremental investment, which suggests that the theoretical approach of Boyle and Guthrie (2003) is also of empirical importance.
2 Framework

2.1 Model of the Firm

Consider a financially constrained value-maximizing all-equity firm at time $t$ consisting of

- physical assets worth $G$,
- a cash balance of $X_t$, and
- the rights to an investment project worth $V_t$.

The physical assets create an income stream of

$$ \nu G dt + \phi G dZ_t $$

where $\nu$ and $\phi$ are constant parameters and $Z_t$ is a Wiener process. While $G$ remains constant over time, the income stream augments or reduces the firm’s cash balance $X_t$. As long as the investment project is not launched, the cash balance is invested in riskless securities yielding an interest rate of $r$, so it evolves according to

$$ dX_t = rX_t dt + \nu G dt + \phi G dZ_t. \tag{1} $$

The focus of the work is on the firm’s perpetual\(^3\) rights to a project worth $V_t$. At any point in time, the firm can decide to exercise its rights by paying an investment amount $I$, launching the project, and receiving the project value. The net payoff upon exercise of this investment option is $V_t - I$. Alternatively, the firm can delay investment and retain the rights, or sell both the existing assets and the project rights to outside investors, which causes liquidation costs discussed in more detail below. Prior to investment, the value of the investment option is $F_t$, and the underlying project value $V_t$ evolves according to the Geometric Brownian Motion

$$ dV_t = (\mu - \delta) V_t dt + \sigma V_t dW_t \tag{2} $$

where $\mu$, $\delta$, and $\sigma$ are constant parameters, and the correlation of the Wiener processes $Z_t$ and $W_t$ is given by $\rho$ (from now on we will leave out time $t$ subscripts). There is a

\(^3\)The assumption of perpetual project rights, i.e. an infinitely lived investment option, abstracts from effects of competition on the firm’s option value.
value of waiting to invest, since the firm can avoid sunk costs in case the project does not develop favorably. However, delay is not always optimal as the project will provide a cash-flow stream of $\delta V$ after investment, which is the opportunity cost of waiting.

In states in which the firm postpones investment, we will assume throughout this paper that both the existing assets $G$ and the project rights are accepted as a collateral for risk-free debt. Thus, we allow the cash balance to become negative while according to (1) the firm has to pay the risk-free rate on $X$. However, denoting the value of the project rights to the unconstrained firm by $F_u(V)$, the value to financially unconstrained outside investors is only $F_u(\gamma V)$: This means that although they accept the project rights as a collateral, the outside investors account for the fact that they would receive only a project value of $\gamma V$ upon investment of $I$. This discount reflects that they might not have the knowledge required to extract full project value, but only a fraction $\gamma \in [0, 1)$. This is the central assumption of our model, which makes financing constraints relevant for valuation. For $\gamma = 1$ the firm could sell the investment option to outside investors at the unconstrained value $F_u(V)$, and therefore the state of the firm’s cash balance would have no effect on the valuation of the investment option.

Due to these assumptions the firm has access to a short-term risk-free credit up to the amount of $G + F_u(\gamma V)$, and the firm can continue operation as long as

$$X \geq -[G + F_u(\gamma V)].$$

As soon as (3) is not satisfied any longer, liquidation is enforced: The risk-free debtholders receive both the existing assets and the project rights, which they can sell to outside investors at exactly the value of their loan, $|X|$, and the owners of the firm receive nothing. In the case of enforced liquidation, $\gamma$ can be interpreted as a measure of bankruptcy costs.

The firm may also liquidate voluntarily at any time before. Again, the existing assets and the project rights are sold to outside investors. The owners of the firm receive an amount of $X + G + F_u(\gamma V)$, where in case of a negative cash balance the risk-free debtholders receive the value of their loan, $|X|$. Permitting voluntary liquidation ensures that the liquidation value of the investment option, $F_u(\gamma V)$, is always a lower bound for the actual option value.

While the project rights $F$ are intangible and investment requires special knowledge incorporated in the existing assets, both $G$ and $V$ are physical assets after investment has
taken place, and their value is fully transferable to outside investors. So after investment, the firm’s owner is free to sell the assets at the full value of $G + V$.

### 2.2 Costly External Financing

We assume that the firm is subject to a market friction, and therefore it has only limited and possibly costly access to external financing. Therefore the state of the firm’s cash balance has a substantial impact on the firm’s investment decision.

While a firm that is restricted to internal funding can exercise its investment option only if $X \geq I$, we assume that the existing assets $G$ can serve as a collateral for risk-free debt, which is paid back immediately after investment. Therefore the firm can invest bearing no financing costs as long as $X + G \geq I$, or $\Delta X \leq 0$, where we define

$$\Delta X = I - (X + G)$$

(4)

as the part of the investment amount that exceeds the firm’s capacity of internal funds and risk-free debt. Since we consider a firm whose growth options form a substantial part of the total value, the investment amount $I$ needed to launch the project is of the same order as the value of the existing assets $G$. Thus the firm will usually not be able to fund its project using only costless financing. Then, in the case $\Delta X > 0$, the firm has to raise the additional funding $\Delta X$ by issuing claims written on the project value $V$ that will be extracted after investment. Following Boyle and Guthrie (2003) we assume that due to the market friction, the maximum available amount of funding is

$$\Delta X \leq \alpha V \quad \text{with} \quad \alpha \in [0, 1).$$

(5)

This can be motivated by ”uncertainty about the firm’s ability or willingness to extract full project value”.

In addition to this financing quantity constraint, we introduce financing costs to the model. Specifically, we assume that besides having to pay back $\Delta X$ after investment, the firm suffers issuance costs amounting to

$$IC(\Delta X, V, \alpha, k) = \max \left(\frac{\Delta X}{\alpha V}, 0\right)^k (1 - \alpha)V \quad \text{with} \quad k \geq 1.$$

(6)

Instead of imposing them directly, such costs could also be derived endogenously as the outcome of a financing contract subject to market frictions. Similar to Lyandres (2005)

and Whited (2005) we refrain from this – in order to keep the model tractable, we focus on the effect of exogenous costs on the firm’s investment policy. Nevertheless, we will give an economic interpretation of the cost function below.

The issuance costs (6) are increasing in the amount of funding $\Delta X$: While there are no costs for $\Delta X = 0$, the maximum cost of $(1 - \alpha)V$ is reached for a funding of $\Delta X = \alpha V$. Moreover, it can be shown that the $IC$ are decreasing in $\alpha$ for all $\Delta X$, i.e. the overall level of financing costs is increasing in the degree of market frictions, while the maximum available amount of funding $\alpha V$ is decreasing. In the extreme case of $\alpha = 0$, the firm cannot issue any claims written on the project value. On the other hand, for $\alpha \to 1$ there are no market frictions. Consequently, there are no financing costs, and the firm can invest whenever there is a non-negative firm value remaining after investment.

In accordance to recent literature, e.g. Cleary, Povel, and Raith (2004) and Lyandres (2005), we will distinguish the following two dimensions of financing constraints in the remainder of this paper: One dimension is the degree of market frictions measured by $\alpha$, and the second dimension is the degree of liquidity constraints. The most obvious approach is to measure the latter by the amount $\Delta X$ that the firm has to raise using costly external financing given that immediate investment is optimal. However, we point out that the actual severeness of liquidity constraints also depends on the firm’s maximum available costly financing capacity $\alpha V$. Therefore we introduce the firm’s relative funding capacity

$$\omega = \frac{\Delta X}{\alpha V} \quad (7)$$

as an even more expressive measure of liquidity constraints.\(^5\) As (6) shows, the $IC$ depend on both dimensions of financing constraints, and so does the firm’s investment policy.

The investment and financing conditions of a constrained firm are summarized in Figure 1 for the whole state space, where each state is fully determined by the cash balance $X$ and the project value $V$. In Area E internal funds are sufficient to permit investment. The firm can invest using internal funds and risk-free debt within Area D, but still there are no financing costs. Notice that such a firm is considered financially constrained, since the fear of future costs may induce an investment timing policy different from that of an unconstrained firm. In Area C, we have $\omega > 0\%$ and immediate investment causes issuance

\(^5\)However, $\omega$ as a measure of liquidity constraints is also dependent on the degree of market frictions $\alpha$. In order to have a clear distinction of the two dimensions of financing constraints, we will therefore measure liquidity constraints by $\Delta X$ rather than $\omega$ whenever we are varying $\alpha$.\(^6\)
costs according to (6), but at least it is still feasible. Within Area C, the dash-dot lines
denote states (X,V) in which the firm’s relative funding capacity is at an equal level \( \omega \).
In Area B the firm can survive, but \( \omega > 100\% \) and immediate investment is not possible.
Finally, as soon as Area A is reached, immediate liquidation takes place.

[Figure 1]

Depending on the parameter \( k \), issuance costs can either be interpreted as equity or debt
issuance costs. For \( k = 1 \), the \( IC \) are independent of the project value \( V \) and linear in the
amount of funding \( \Delta X \). Such a cost structure is quite commonly used in the literature in
order to model equity issuance costs.\(^6\) Therefore, for \( k = 1 \) we interpret our issuance cost
function as equity issuance costs

\[
EIC(\Delta X, \alpha) = IC(\Delta X, V, \alpha, 1) = \frac{1 - \alpha}{\alpha} \max(\Delta X, 0). \tag{8}
\]

For any \( k > 1 \), issuance costs are decreasing in \( V \) and they approach zero for \( V \to \infty \). In
this case, the \( IC \) can be interpreted as risky debt issuance costs with the project value \( V \)
being the collateral of the debt contract. Thus, for \( k > 1 \), issuance costs are called debt
issuance costs thereafter:

\[
DIC(\Delta X, V, \alpha, k) = IC(\Delta X, V, \alpha, k) = \max \left( \frac{\Delta X}{\alpha V}, 0 \right)^k (1 - \alpha) V \quad \text{with} \quad k > 1. \tag{9}
\]

Figure 2 depicts the shapes of \( EIC \) and \( DIC \) for representative parameter values. The
\( DIC \) are convex in the amount of funding \( \Delta X \), where the convexity is increasing in
the parameter \( k \).\(^7\) Since the \( DIC \) equal the linear \( EIC \) function for both \( \Delta X = 0 \) and
\( \Delta X = \alpha V \), they are strictly lower than the \( EIC \) for any \( \Delta X \in (0, \alpha V) \). This is consistent
with the Pecking Order Hypothesis put forward by Myers and Majluf (1984), who explain
why market frictions can make debt issuance a more favorable financing alternative than
equity issuance. Hirth and Uhrig-Homburg (2004) examine risky debt issuance along the

\(^6\)See e.g. Lyandres (2005) for a similar approach, also in the context of investment timing, and Hirth
and Uhrig-Homburg (2004) for a formal and economic explanation why these properties are reasonable
for equity issuance costs.

\(^7\)The assumption of a convex external financing cost function is not found unanimously in the litera-
ture. While convexity in the amount of funding is also assumed in Kaplan and Zingales (1997), they
comment this in a footnote as "a reasonable, but not obvious assumption". In their follow-up article Ka-
plan and Zingales (2000) even more distinctly question the convexity property. As Whited (2005) points
out in a recent paper, convexity does indeed no longer hold as soon as there are fixed costs of external
financing.
lines of Merton (1974). They show that risky debt issuance costs are indeed convex in the amount of funding, and that for low values of the firm’s relative funding capacity \( \omega \) the DIC should be negligible while they approach the EIC for \( \omega \to 100\% \). The intuitive explanation of their result is that in the risky debt financing case the market friction affecting the collateral value is relevant only in states in which the external investors expect default, and these states are more likely for higher values of \( \omega \). In contrast, the market friction is relevant in all states of the world for new equityholders.

[Figure 2]

3 Investment Policy

3.1 Problem

Now we will formally derive the firm’s investment policy. The policy is characterized by the free boundary \( \hat{V}^i(X) \) above which immediate investment is optimal, where \( i \) indicates a firm facing issuance costs according to (6). Due to the firm’s financing constraints, the boundary does not remain constant, but it is dependent on the cash balance \( X \). The firm chooses \( \hat{V}^i(X) \) in order to maximize the investment option value,\(^8\) subject to the financing quantity constraint (5) that can be written using (7) as

\[
\omega = \frac{\Delta X}{\alpha V} \leq 100\% \quad (10)
\]

in investment states. If immediate investment is optimal, we have

\[
F^i(X, V) = \bar{F}^i(X, V) \quad \forall V \geq \hat{V}^i(X), \quad (11)
\]

where the pure exercise value of \( V - I \) is reduced to

\[
\bar{F}^i(X, V) = V - I - IC(\Delta X, V, \alpha, k) \quad (12)
\]
due to the issuance costs. For all states in which it is optimal to postpone investment, i.e. for all \( V < \hat{V}^i(X) \), it can be shown\(^9\) using standard replication arguments that the option value \( F^i(X, V) \) satisfies

\[
\frac{1}{2}\sigma^2 V^2 F^i_{V^2} + \rho \sigma V G F^i_{XV} + \frac{1}{2} \phi^2 G^2 F^i_{XX} + (r - \delta)V F^i_V + r(X + G) F^i_X = r F^i. \quad (13)
\]

\(^8\)An investment policy \( \hat{V}^i(X) \) that maximizes the investment option value also maximizes total firm value, and is therefore optimal from the firm’s point of view before investment.

\(^9\)A detailed derivation can be found in the Appendix of Boyle and Guthrie (2003).
Determination of the free boundary $\hat{V}^i(X)$ in order to maximize the option value is equivalent to an optimal stopping problem, since the decision to invest can also be seen as a decision to stop waiting. Although our model does not explicitly depend on time, continuity of $V$ implies that a higher investment threshold will be reached later, given a specific current project value below the threshold. So we can interpret higher and lower investment thresholds as policies inducing later and earlier investment, respectively.

The other boundary conditions that determine the solution to (13) are as follows: For a worthless project, we also have a worthless option, therefore

$$F^i(X, 0) = 0 \quad \forall X.$$  

(14)

As soon as the cash balance becomes so negative that it cannot be covered by the firm’s risk-free debt capacity of $G + F^u(\gamma V)$, the firm suffers an enforced liquidation, and we have

$$F^i(X, V) = F^u(\gamma V) \quad \forall X < -[G + F^u(\gamma V)].$$  

(15)

On the other hand, a firm with extensive cash holdings has an investment option whose value approaches that of an unconstrained firm:

$$\lim_{X \to \infty} F^i(X, V) = F^u(V).$$  

(16)

Before we analyze the investment policy $\hat{V}^i(X)$ of a firm facing issuance costs, we will introduce two benchmark cases.

### 3.2 Benchmark Cases

#### 3.2.1 The Unconstrained Firm

We saw in (16) that the situation of an unconstrained firm can be interpreted as the limit case of a constrained firm with a cash balance $X$ being high enough that the probability of suffering financing restrictions or even a forced liquidation can be neglected. Then, the investment option value does not depend on the cash balance $X$, and (13) simplifies to

$$\frac{1}{2}\sigma^2 V^2 F^u_{VV} + (r - \delta)V F^u_{V} = r F^u \quad \forall V < \hat{V}^u.$$  

(17)

for the unconstrained option value $F^u(V)$. The value matching condition reduces to

$$F^u(V) = \tilde{F}^u(V) = V - I \quad \forall V \geq \hat{V}^u,$$  

(18)
and \( F^u(0) = 0 \) still holds. The firm then faces a simple trade-off between the value of waiting and saving the investment amount \( I \) and the costs of waiting due to the foregone project cash flow stream \( \delta V \).

McDonald and Siegel (1986), and later Dixit and Pindyck (1994) in a simplified version, derived a solution for this option value. Since the rights to the project are perpetual and the firm’s cash balance \( X \) does not affect valuation, the only remaining state variable is the project value \( V \). The firm chooses the critical project value \( \hat{V}^u \) above which immediate investment is optimal, i.e. the investment threshold, in order to maximize today’s value \( F^u(V) \). The solution to the problem is an investment threshold of

\[
\hat{V}^u = \frac{\beta I}{\beta - 1},
\]

which results in an investment option value of

\[
F^u(V) = \left( \frac{I}{\beta - 1} \right)^{1-\beta} \left( \frac{V}{\beta} \right)^{\beta} \quad \forall V < \hat{V}^u,
\]

with

\[
\beta = \frac{1}{2} - \frac{r - \delta}{\sigma^2} + \sqrt{\frac{2r}{\sigma^2} + \left( \frac{1}{2} - \frac{r - \delta}{\sigma^2} \right)^2}.
\]

### 3.2.2 The Pure-Quantity Constrained Firm

As a second benchmark case we consider a firm facing no financing costs, i.e. in Area C it can use a fraction \( \alpha \) of the project value at no cost as a collateral for raising external financing. Since it still can invest only if \( \omega \leq 100\% \), it faces the pure financing quantity constraint (10). This case was first analyzed by Boyle and Guthrie (2003) and lies in between the first benchmark case of an unconstrained firm and the situation of a firm facing financing costs considered in this paper. We denote its investment option value by \( F^c(X, V) \) and its threshold by \( \hat{V}^c(X) \). In states in which waiting is optimal, (13) still holds just as for \( F^i(X, V) \), as well as the boundary conditions (14) to (16). But since we have no financing costs,

\[
F^c(X, V) = \tilde{F}^c(V) = V - I \quad \forall V \geq \hat{V}^c(X),
\]

so the exercise value is the same as for an unconstrained firm. In states in which waiting is optimal, however, the pure-quantity constrained firm’s option value is below that of
an unconstrained firm: The option’s time value is lower for the pure-quantity constrained firm, since it follows exercise policies that are suboptimal compared to an unconstrained firm.

3.3 Investment Threshold and Investment Option Values for Constrained Firms

Our goal now is to analyze investment policies in the costly financing cases and compare them to the two benchmark cases. However, the interdependence of the two state variables $X$ and $V$ given by the boundary conditions precludes an analytical solution for all cases but the benchmark case of an unconstrained firm. Therefore we use a numerical method in order to solve (13). The procedure we use yields an iterative solution based on finite difference methods, and except for changes to account for financing costs according to (12) instead of (19), it follows the procedure described in Boyle and Guthrie (2003). The parameter values that we use are given in Table 1. In order to ensure comparability, we use the same values as in Boyle and Guthrie (2003) whenever possible.

Table 1: Parameter Values used in Numerical Examples.
Values are identical to those in Boyle and Guthrie (2003) (except for the risky debt convexity, which is not used there).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Project investment cost ($I$)</td>
<td>100</td>
</tr>
<tr>
<td>Project value volatility</td>
<td>$\sigma = 0.2$</td>
</tr>
<tr>
<td>Project cash-flow rate</td>
<td>$\delta = 0.03$</td>
</tr>
<tr>
<td>Project value – firm cash flow correlation</td>
<td>$\rho = 0.5$</td>
</tr>
<tr>
<td>Cash flow volatility</td>
<td>$\phi = 0.6$</td>
</tr>
<tr>
<td>Market value of existing assets ($G$)</td>
<td>100</td>
</tr>
<tr>
<td>Market friction</td>
<td>$\alpha = \gamma = 0.8$</td>
</tr>
<tr>
<td>Riskless interest rate</td>
<td>$r = 0.03$</td>
</tr>
<tr>
<td>Risky debt convexity</td>
<td>$k = 5$</td>
</tr>
</tbody>
</table>
The resulting investment thresholds are shown in Figure 3. The solid $\omega$ lines known from Figure 1 represent the boundaries of Area C, in which immediate investment is possible but implies financing costs. For $\omega > 100\%$, immediate investment is not possible, and for $\omega \leq 0\%$ there are no present financing costs. The threshold of the unconstrained firm, $\hat{V}^u$, is a horizontal line, since it does not depend on the cash balance $X$.

In what follows we analyze the by far more interesting thresholds of the three financially constrained cases, which are characterized by (1) pure quantity constraints and costless external financing, (2) convex issuance costs which we interpret as debt financing, and (3) linear issuance costs which we interpret as equity financing (superscripts $c$, $d$, and $e$, respectively). We point out that these thresholds are significantly different from what might be an intuitive guess, namely that the investment threshold should be monotonically decreasing in the firm’s initial cash balance, i.e. that less constrained firms always invest earlier.

First, we discuss the threshold of the pure-quantity constrained firm with costless financing, $\hat{V}^c(X)$. Compared to an unconstrained firm, we recognize two areas where the constrained firm is forced to invest suboptimally: For lower cash balance than at the intersection of $\hat{V}^u$ and $\hat{V}^c(X)$, notice that for project values above $\hat{V}^u$ and below $\hat{V}^c(X)$ the constrained firm is forced to delay investment, although an unconstrained firm would invest immediately. For higher cash balance than at that intersection, we observe just the opposite behavior: For project values above $\hat{V}^c(X)$ and below $\hat{V}^u$ the pure-quantity constrained firm invests earlier, i.e. for smaller project values, than an unconstrained firm. As Boyle and Guthrie (2003) point out, this voluntary acceleration occurs in order to avoid future funding shortfalls.

For low cash balance, the pure-quantity constrained firm invests as soon as possible, i.e. as soon as $\omega \leq 100\%$. Although this is an endogenous decision, notice that we have a boundary solution: The firm invests as soon as the exogenous constraint allows to do so. Above a certain level of cash balance we observe that the value of waiting becomes more important than the risk of future funding shortfalls, and therefore $\hat{V}^c(X)$ starts rising in $X$. As we expected, for very large $X$ the financing constraint is unlikely to become important anymore, and therefore the constrained firm’s threshold approaches that of an unconstrained firm.

Second, we compare the risky debt threshold denoted by $\hat{V}^d(X)$ to that of the pure-quantity constrained firm, $\hat{V}^c(X)$. We similarly start at the intersection of $\hat{V}^d(X)$ and
\( \hat{V}^c(X) \). For lower cash balance than at that intersection, voluntary investment delay takes place: The risky debt financing firm does not invest although the pure-quantity constrained firm does, and although investment is feasible, i.e. the financing constraint (10) is satisfied. Investment is postponed in order to avoid prohibitively high financing costs. Notice that this is an endogenous decision, which is in sharp contrast to the forced delay that we observed above. Graphically, this can be explained by the fact that the threshold \( \hat{V}^d(X) \) leaves the exogenous minimum threshold \( \omega = 100\% \) while it is still falling in \( X \), and that especially at the new critical level of cash balance where the value of waiting becomes more important than the risk of future financing costs and \( \hat{V}^d(X) \) starts rising from its minimum point, it is significantly above \( \omega = 100\% \). The policy for cash balance levels above the intersection of \( \hat{V}^d(X) \) and \( \hat{V}^c(X) \) can be described as accelerated investment: Even though the risky debt financing firm faces costs, immediate investment takes place in order to avoid even higher financing costs in the future. In contrast, the pure-quantity constrained firm does not invest immediately, although it has more favorable investment conditions in these states. Just as in the comparison of \( \hat{V}^u \) and \( \hat{V}^c(X) \), it may be surprising at first glance that the firm facing stronger constraints invests earlier. Finally, for very large \( X \) the financing constraint is again unlikely to become important anymore, and therefore the risky debt financing firm’s threshold approaches that of the pure-quantity constrained firm, and in the end that of an unconstrained firm.

For the third case of an equity financing firm, the shape of the optimal investment threshold looks quite different: The only obvious resemblance to the cases discussed above is that the threshold still approaches the two usual boundaries, namely the minimum possible level at \( \omega = 100\% \) for very low values of cash balance and the unconstrained threshold \( \hat{V}^u \) for sufficiently high cash balance. For low levels of cash balance we observe that the threshold is first close to a project value of about \( \hat{V}^{ul} = 278 \), then it is smoothly decreasing. At \( X = 0 \) it jumps down far below the unconstrained threshold \( \hat{V}^u \) and afterwards smoothly approaches this threshold for increasing cash balance. Note that the threshold \( \hat{V}^{ul} = \hat{V}^u / \gamma \) refers to the investment decision of an unconstrained outside investor holding the constrained firm’s investment option after liquidation.\(^{10}\)

Obviously, for low cash balance the equity financing firm follows an investment policy

\(^{10}\) For derivation we use the liquidation value \( F^u(\gamma V, I) = \gamma F^u(V, I/\gamma) \) of the constrained firm’s investment option. Here we explicitly mention the exercise price as an argument, and we use the fact that \( F^u(V, I) \) is homogenous of degree one.
very similar to that of an unconstrained outside investor who owns the constrained firm’s investment option after liquidation: Postpone investment for $V < \hat{V}^{ul}$, and on the other hand invest even for small cash balance as soon as $V \geq \hat{V}^{ul}$. This behavior can be explained by the fact that for low cash balance, equity issuance costs are so prohibitively high that the firm owns a value that is close to the liquidation value, and therefore also adopts the policy of an investor owning the liquidation value.

The jump of $\hat{V}^e(X)$ at $X = 0$ is due to a kink in the exercise value at $\Delta X = 0$:

$$\frac{\partial(V - I - EIC(\Delta X, \alpha))}{\partial X} = \frac{1 - \alpha}{\alpha} \cdot 1_{\{\Delta X > 0\}}.$$ 

An economic explanation for the policy switch is that for negative cash balance, each additional unit of cash balance raises the option value significantly by reducing financing costs for future investment, and therefore the value of waiting dominates the threat of future financing costs, should the cash balance further decrease. The firm therefore requires a relatively high project value to justify immediate investment for $\Delta X > 0$. For increasing cash balance, however, as soon as $\Delta X = 0$ is reached, each additional unit of cash balance does not raise the exercise value, but it only makes future financing costs less likely. In this situation, the value of waiting consists solely of the chance of better project values, and no longer of a possible reduction in financing costs. Should the cash balance decrease again by one marginal unit, however, the cost of waiting is unalteredly significant. Therefore there is a strong incentive for the firm to invest immediately as soon as $\Delta X = 0$ is reached, even for relatively low project values.

In all other cases discussed so far but the equity issuance case, there is no kink in the exercise value. This is obvious for the unconstrained and the pure-quantity constrained case, since the exercise value does not at all depend on $X$. For the risky debt financing firm, it can be seen in Figure 2 that the risky debt issuance costs smoothly approach the abscissa for $\Delta X \to 0$. Therefore we do not observe jumps in the investment threshold for all these cases.

Overall, we find that equity issuance is used only for very favorable project values. Besides an actual relevance as a source of funding, its more remarkable effect is that it significantly

\[\text{Notice that the usual value matching and smooth pasting conditions of the option value at the free boundary are still satisfied separately for positive and negative cash balance. That is, for negative cash balance the option value merges into a plane that has a slope of } \frac{1 - \alpha}{\alpha} \text{ in } X \text{ direction, while for positive cash balance the corresponding plane is parallel to the } X \text{ axis.}\]
drives down the investment threshold of a firm without access to debt markets even for the area of positive cash balance, where there are no financing costs at all, but the firm anticipates the financing costs in case of funding shortfalls in the future.

4 Investment and Changing Financing Constraints

4.1 Investment Volume and Investment-Cash Flow Sensitivity

Having discussed the optimal investment policy for a given set of parameter values, as well as starting values for the two state variables $X$ and $V$, this section examines how investment responds to changing financing constraints. Throughout the analysis, we confine ourselves to a convex issuance cost function with a fixed $k > 1$, interpreted as risky debt financing in Section 2.

So far we have characterized the optimal investment policy in terms of investment timing. However, in order to make our following implications comparable, we will coin them in terms of investment volume and investment-cash flow sensitivity. These notions are commonly used, particularly in the empirical literature on investment and financing constraints. They are related to the variables of our model if we use the following two definitions, which are similar to those used in previous work connecting theoretical models and empirical studies, e.g. Kaplan and Zingales (1997) and Lyandres (2005).

Firstly, we interpret a lower investment threshold $\hat{V}(X)$ as a higher investment volume. Although there is only one investment project with a fixed volume $I$ in our model, this interpretation can be justified in our context by the fact that given an ex-ante uncertain project value $V$, a firm that has chosen a lower $\hat{V}(X)$ will accept more projects and therefore has a higher expected investment volume $E(I \cdot 1\{V \geq \hat{V}(X)\})$.

Secondly, we define the investment-cash flow sensitivity as the change of the investment volume, which corresponds to the negative change of the investment threshold, given an

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12This explanation follows Lyandres (2005), another article that shares this interpretation is Mauer and Sarkar (2005). An alternative way of modelling variable investment volume consists of actually endowing the firm with a portfolio of investment options or using an option to expand with variable capacity (see e.g. Dixit and Pindyck (1994) as a reference). Yet another approach as shown e.g. in Kaplan and Zingales (1997) is the use of a simple concave production function in order to reflect variable capacity. However, this is not appropriate for our purpose, since it ends up in a one-period problem and neglects the dynamic investment timing option that is important to us.
increase in the firm’s liquid funds, i.e. $-\frac{d\hat{V}(X)}{dX}$. We will examine how the absolute value $|\frac{d\hat{V}(X)}{dX}|$ of this sensitivity changes for different levels of financing constraints.

4.2 Comparative Statics

We analyze the effects of the two dimensions of financing constraints that we introduced in Section 2, namely liquidity constraints and market frictions. For investment volume and afterwards for investment-cash flow sensitivity we first examine the effect of changing liquidity constraints, measured by the relative funding capacity $\omega$, while keeping the market friction $\alpha$ constant. In a second step we examine the effect of changing market frictions for a constant level of the amount $\Delta X$ that the firm has to raise using costly external financing.

4.2.1 Investment Volume

We first discuss the effect of a change in the firm’s liquidity constraints, measured by the relative funding capacity $\omega$, on investment volume. Graphically, increasing $\omega$ corresponds to rotating the $\omega = 0\% \ (\Delta X = 0)$ line in Figure 3 counter-clockwise around the origin towards the $\omega = 100\%$ line.

Notice the monotonicity of the pure-quantity constrained firm’s threshold: Whenever the funding capacity is sufficient to permit investment ($\omega < 100\%$), investment is increasing (the threshold is decreasing) in $\omega$. As soon as the exogenous quantity constraint $\omega = 100\%$ is crossed, the threshold jumps to infinity.

Examining the risky debt financing firm’s threshold, we recall that it features an interior minimum. As we explained above, it emerges from trading off waiting value and future expected financing costs against present financing costs. Interpreting a lower investment threshold $\hat{V}^{nd}$ as a higher investment volume leads to the conclusion that for low-liquidity firms (i.e. high levels of $\omega$) investment is decreasing in liquidity constraints, which is the usually expected behavior. However, the reverse is true for less constrained firms.

[Figure 4]

Next we discuss the effect of changes in the exogenous market friction parameter $\alpha$ on the investment policy. Notice that changing $\alpha$ does not only affect the applicable financing costs for the current $(X, V)$ state, but also affects the relative funding capacity $\omega$. In
Figure 4 we therefore examine the investment threshold as a function of $\alpha$ for selected amounts of costly financing $\Delta X$. For the sake of comparison, we also show the respective quantity constraint $\omega = \frac{\Delta X}{\alpha V} = 100\%$ as a thin line for each positive $\Delta X$.

For negative $\Delta X$ (e.g. $\Delta X = -50$), the quantity constraint is not binding for any $\alpha$ – even an investment threshold of zero would be feasible. Moreover, there are no financing costs for instantaneous investment, so there is no need to avoid present costs. Still, the fear of future costs should the cash balance decrease can make early investment favorable. These possible future costs are higher for firms facing stronger market frictions, therefore the threshold is monotonically increasing in $\alpha$, i.e. decreasing in the degree of market frictions.

For a modest required amount of funding (e.g. $\Delta X = 100$) we observe a trade-off between present costs and possible future financing costs. Among firms facing a negligible degree of market frictions ($\alpha$ close to one), those that face higher market frictions invest more (have a lower threshold) since they fear the higher future costs. Among firms facing severe market frictions (low $\alpha$), we observe the contrary, since the firms facing stronger frictions are prevented from immediate investment by prohibitively high present financing costs.

Finally for firms with large required amounts of funding (e.g. $\Delta X = 250$) the quantity constraint at $\omega = 100\%$ becomes more and more binding, and the threshold is monotonically increasing in the degree of market frictions (decreasing in $\alpha$). There is little scope left for the firms’ decisions, they simply cannot afford to invest although they are in ranges of project value where the opportunity costs of waiting are very high. The investment is made as soon as it is feasible, even in the view of high financing costs.

[Figure 5]

To summarize these results, Figure 5 illustrates both the effect of changing cash balance $X$ (and thus changing $\Delta X$) and changing degree of market frictions $\alpha$ on the investment threshold. It shows in a contour plot that the analysis carried out by holding either $\alpha$ (Figure 3) or $\Delta X$ (Figure 4) constant and varying the other can indeed be generalized: The investment threshold as a function of $X$ with fixed $\alpha$, which we obtain by cutting a horizontal line in Figure 5, is U-shaped for any choice of $\alpha$. On the other hand, the investment threshold as a function of $\alpha$ with fixed $X$ can indeed take exactly the three shapes depicted in Figure 4. Cutting a vertical line in Figure 5, we see that the threshold is strictly increasing in $\alpha$ for $X > 0$ ($\Delta X < 0$), but U-shaped as soon as the demand for external financing becomes positive, since the present financing costs raise the threshold
for firms facing severe market frictions. Finally, we see that low-liquidity firms have no choice but wait until \( \omega \leq 100\% \), and the threshold is strictly decreasing in \( \alpha \).

Figure 5 further shows that for low-liquidity firms relaxing either form of constraint (increasing \( X \) or \( \alpha \)) leads to more investment (a lower threshold). On the other hand, for firms that face no present financing costs (\( \Delta X \leq 0 \)) relaxing either form of constraint leads to less investment. Only for intermediate levels of liquid funds there are states in which relaxing either form of constraint can lead to more and to less investment.

### 4.2.2 Investment-Cash Flow Sensitivities

Having discussed how investment volume is related to financing constraints, we now turn to analyze the effects of constraints on investment-cash flow sensitivities, i.e. we ask whether a more constrained firm will adjust its investment volume more pronouncedly than a less constrained firm, given a change in liquid funds.

First we determine the effect of a change in the firm’s liquidity constraints, measured by the relative funding capacity \( \omega \), on the investment-cash flow sensitivity. Again there is a short story for the pure-quantity constrained firm’s threshold: While we found that investment volume is increasing in \( \omega \) for all \( \omega < 100\% \), Figure 3 shows that the absolute investment-cash flow sensitivity \( \left| \frac{d\hat{V}(X)}{dX} \right| \) is monotonically increasing in \( \omega \) as well, i.e. stronger liquidity constraints imply that a decline in the firm’s cash balance results in an even higher increase in investment.\(^{13}\)

For a risky debt financing firm facing low liquidity, i.e. \( \omega > 50\% \), we see in Figure 3 that an increase in the firm’s liquidity constraints not only reduces investment, but also amplifies the sensitivity. Similarly, when we consider immaterial liquidity constraints (negative \( \Delta X \) for positive \( X \)), the investment threshold approaches that of the pure-quantity constrained firm, and the sensitivity becomes negligible. Approaching \( X = 0 \) (\( \omega = 0\% \)) from positive cash balances and thus increasing liquidity constraints, we have an increasing sensitivity.

There is one interesting area remaining: When the firm’s liquidity constraints further increase from \( \omega = 0\% \) to 50%, there is an inflection point in the threshold: While the

\(^{13}\)However, Boyle and Guthrie (2003) originally defined the sensitivity for the pure-quantity constrained firm using the change in the value of the timing option, unlike in Kaplan and Zingales (1997), Lyandres (2005), or in this work. Therefore they come to the conclusion that "the sensitivity of investment to cash flow can be greatest for high-liquidity firms" (Boyle and Guthrie (2003), Abstract).
sensitivity keeps increasing when crossing $\omega = 0\%$ because losing an additional marginal unit of cash makes the fear of future costs relative to the value of waiting more and more important, the sensitivity finally switches from increasing to decreasing, as the present financing costs become predominant.

The second part of the analysis consists of examining the effect of a change in the firm’s market friction parameter $\alpha$ on the investment-cash flow sensitivity. It can be shown that the absolute slope of the investment threshold in Figure 3 is decreasing in $\alpha$ for both high and low cash balance. In our language, this means that the absolute investment-cash flow sensitivity is monotonically decreasing in $\alpha$ for both high and low cash balance. This is a reasonable behavior, since the effects of both future and present financing costs are less pronounced for firms facing more perfect capital markets.

We conclude that the absolute sensitivity of investment to cash flow is increasing in both forms of constraints for both high- and low-liquidity firms. However, we have to keep in mind that the sensitivities describe changes in investment volume that work in opposite directions for high- and low-liquidity firms, respectively.

### 4.3 Implications and Related Work

The empirically testable implications of our model for risky debt financing firms can be summarized as follows:

I. For any given level of market frictions, investment is inversely U-shaped in liquid funds.

II. For high-liquidity firms, investment and investment-cash flow sensitivity are both increasing in either form of financing constraints (liquidity constraints or market frictions).

III. For low-liquidity firms, investment is decreasing and investment-cash flow sensitivity is increasing in either form of financing constraints.

IV. For intermediate levels of liquidity, investment is inversely U-shaped in market frictions, and relaxing either form of constraint has an ambiguous effect on both investment and investment-cash flow sensitivity.
Implications I and IV state that investment can be non-monotonic in either form of financing constraints. We point out that most previous studies took for granted that an unconstrained firm invests at the highest level and that increasing constraints always reduce investment. Consequently, when examining investment-cash flow sensitivities, those studies aimed to compare the extent of investment reduction for different levels of financing constraints.

To give a prominent example, Kaplan and Zingales (1997) model the investment decision as a static profit maximization subject to a concave production function and a convex financing cost function. Investment is considered a 'now-or-never' opportunity, therefore an unconstrained firm has the highest level of investment, and any constraint drives the investment volume below the first-best level.

In contrast, we regard a firm as unconstrained when it has the opportunity to choose an investment timing policy that ensures the full value of waiting in the real option sense of McDonald and Siegel (1986). Since the optimal policy consists of postponing investment in certain states and a constrained firm may have to invest prematurely, Boyle and Guthrie (2003) show that investment can be increasing in financing constraints.

Obviously, it does not make sense to compare the extent of a change without first ascertaining its direction. When turning to sensitivities, we will therefore confine ourselves to states in which the other models come to equivalent conclusions concerning investment volume.

At first sight, our Implication I is very similar to the findings of Cleary, Povel, and Raith (2004), who propose that investment is U-shaped in liquid funds.\textsuperscript{14} However, although they also have a non-monotonic relation of investment and financing constraints, their U-shape is the opposite to the inverse U-shape proposed in this work. Their model setup is still a static 'now-or-never' framework as in Kaplan and Zingales (1997). The U-shape, i.e. the eventually increasing investment volume for low liquid funds, is due to the design of the debt contract: Since the external investors participate more and more in the success of the project, it is favorable to provide more capital for low internal liquid funds, and the firm’s marginal costs of debt financing are non-monotonic in liquid funds. In contrast, in our model a firm with given existing assets and investment opportunities faces higher financing costs for each additional unit of liquid funds required for investment. Since we also allow for postponing investment in a dynamic framework, immediate investment is

\textsuperscript{14}See Cleary, Povel, and Raith (2004), Proposition 2.
less favorable for low liquid funds.

What is more surprising at first glance is that Lyandres (2005) proposes that investment is strictly increasing in liquid funds, although he shares most of our assumptions: Like in Boyle and Guthrie (2003) and our model, he accounts for the option to postpone investment instead of thinking of a 'now-or-never' opportunity, and he considers both quantity constraints and market frictions, as in Cleary, Povel, and Raith (2004) and our work. The discrepancy can be explained by the fact that Lyandres (2005) uses a cost function that is linear in the amount of funding, which both he and we interpret as equity issuance, but our propositions are derived for the case of a convex cost function, which we interpret as debt issuance costs. Moreover, he only analyzes states in which the firm faces present financing costs, equivalent to states with $\Delta X > 0$ in our model. For these states, Figure 3 shows that for the equity financing firm, investment volume is strictly increasing in liquid funds in our model as well, since the effect of decreasing present costs is predominant. As to the second dimension of financing constraints, Lyandres (2005) finds in accordance to our Implication III that for sufficiently low liquidity, investment is decreasing in the degree of market frictions.

Now we come to discuss our implications for investment-cash flow sensitivities. Early empirical research by Fazzari, Hubbard, and Petersen (1988) claimed that financially more constrained firms have higher sensitivities of investment to cash flow, therefore those sensitivities are a good measure of financing constraints. Later this was questioned most prominently by Kaplan and Zingales (1997), who presented both theoretical and empirical evidence why investment-cash flow sensitivities do not need to increase in constraints, and pointed out in a follow-up note that this holds for both dimensions of constraints. We confirm the basic message of Kaplan and Zingales, in that our model accounts for the fact that sensitivities generally need not be increasing in constraints, and we show that there are indeed ambiguous effects for intermediate levels of liquidity. Moreover, we remind the reader that we are talking about absolute sensitivities, and that for high-liquidity firms the reduction of investment due to higher liquid funds becomes more pronounced with

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15 See Lyandres (2005), Proposition 1.
16 See Lyandres (2005), Proposition 2.
17 Kaplan and Zingales (2000) showed using specific production and cost functions that in their modelling framework investment-cash flow sensitivities can be non-monotonic in either form of financing constraints. They left open to examine for which parameters and areas of the state space the two forms of constraints have different or similar effects, respectively.
increasing constraints. However, our Implications II and III suggest that at least among either high- or low-liquidity firms, investment-cash flow sensitivities can be regarded as a good measure of financing constraints.

Comparing our sensitivity implications to those of Lyandres (2005), we confine the comparison to the case of low-liquidity firms, since in both models the investment volume of these firms is increasing in liquid funds. For those firms, we agree that the investment-cash flow sensitivity is decreasing in liquid funds and increasing in the degree of market frictions,\(^{18}\) i.e. increasing in both dimensions of financing constraints. We do not relate our sensitivity implications to Cleary, Povel, and Raith (2004) since there are no areas of the state space in which we come at least to similar conclusions on investment volume.

Finally we discuss the distinction of different forms of financing constraints. Recent work by Cleary, Povel, and Raith (2004) argues that the empirical contradictions of the first studies, namely Fazzari, Hubbard, and Petersen (1988) and Kaplan and Zingales (1997), can be explained by their failure to distinguish between different forms of financing constraints. Their view is shared by Lyandres (2005).

Our model setup takes the distinction of the two dimensions into account, and we examine the effect of both forms of financing constraints on investment volume and investment-cash flow sensitivities. Indeed, we find that for intermediate levels of liquidity, relaxing either form of constraint can result in both an increase and a decrease in investment volume, as stated in Implication IV. However, our Implications II and III lead to the conclusion that although both forms of financing constraints should be taken into account, the effects are similar in many cases: Within groups of firms with high and low liquid funds, respectively, we find that relaxing either form of financing constraints changes investment volume in the same direction. Similarly, Lyandres (2005) finds that at least for low-liquidity firms, changes in either form of financing constraints have the same effect on investment volume and investment-cash flow sensitivity.

5 Conclusion

In this paper we presented a real-option-based investment timing model that reflects two dimensions of financing constraints, namely liquidity constraints and capital market constraints.\(^{18}\)See Lyandres (2005), Propositions 3 and 4.
frictions inducing financing costs.

We first introduced a general issuance cost function. Then we interpreted a parametrization that leads to a linear cost function as equity issuance costs, while we interpreted the case of a convex cost function as risky debt issuance. By introducing financing costs into the investment timing decision, we could explain both voluntary delay and acceleration of investment as an endogenous trade-off of present and possible future financing costs.

Relating our implications for investment volume and investment-cash flow sensitivities of financially constrained firms to issues of recent research, we claim that:

- Investment can be non-monotonic in either form of financing constraints (according to Implications I and IV).

- Investment-cash flow sensitivities can be regarded as a good measure of financing constraints among high- and low-liquidity firms (according to Implications II and III). Still, sensitivities can be highly non-monotonic in constraints for intermediate levels of liquidity (according to Implication IV).

- Either dimension of financing constraints (liquidity constraints or market frictions) has similar effects on investment and sensitivities of high- and low-liquidity firms (according to Implications II and III). Again, this need not be true for intermediate levels of liquidity (according to Implication IV).

Kaplan and Zingales (2000) regard as a more important question for future research than whether investment-cash flow sensitivities are a good measure of financing constraints: "What causes this sensitivity? We do not pretend to have given an answer to this."\textsuperscript{19}

At this point, our model contributes to the understanding of the investment decision as a dynamic trade-off problem, and we provide an economic explanation why and how there should be non-monotonic effects of both dimensions of financing constraints on corporate investment policies.

The importance of sufficient internal funds gives rise to the question how it can be ensured that these are available when necessary. For example, Froot, Scharfstein, and Stein (1993) take into account the correlation between the firm’s cash flows and the value of investment opportunities and show how hedging can add value to the firm. For a possible extension

\textsuperscript{19}Kaplan and Zingales (2000), p.711.
of our modelling framework in that direction, the work of Boyle and Guthrie (2004) could be a starting point. In general, endogenizing the dynamics of the cash balance gives rise to a different strand of literature.

References


Figure 1: State Space – Investment and Financing Conditions.
In the two-dimensional state space (cash balance $X$ and project value $V$) we identify five areas characterized by different investment conditions. Parameter values are given in Table 1. Area A is bounded by the firm’s solvency constraint. Areas B and C are separated by the $\omega = 100\%$ line, which represents states in which the full funding capacity would be exhausted if immediate investment took place, therefore in Area B immediate investment is not possible. The border between Areas C and D is the $\omega = 0\%$ line, which means that in Area C there are costs of external financing, while there are no such costs in Area D. As soon as the firm’s cash balance $X$ exceeds the investment amount $I$, we are in Area E where internal funds are sufficient to permit investment. The dash-dot lines represent $(X, V)$ combinations for which $\omega = 25\%, 50\%, \text{ and } 75\%$, respectively, of the available costly financing capacity would be used for immediate investment.
The issuance costs $IC(\Delta X, V, \alpha, k)$ are plotted as a function of the amount of funding $\Delta X$. We use a project value of $V = 100$ and a market friction of $\alpha = 80\%$ (thick lines) and $\alpha = 75\%$ (thin lines), respectively. Moreover, we show the cases $k = 1$ (representing equity issuance costs $EIC$), as well as $k = 2$ and $k = 5$ (representing debt issuance costs $DIC$). While the $EIC$ are linear in the amount of funding, the $DIC$ are negligible for small amounts of funding, but their slope is increasing for every additional unit of funding. The convexity of the $DIC$ is increasing in $k$. For the maximum amount of funding $\Delta X = \alpha V = 80$ ($75$), the costs reach the same maximum value of $(1 - \alpha)V = 20$ ($25$) for any value of $k$. 

Figure 2: Comparison of Debt and Equity Issuance Costs.
Figure 3: The Constrained Firm’s Investment Threshold.

The value of the constrained firm’s investment threshold is plotted as a function of cash balance $X$. We distinguish the pure-quantity constrained firm with costless financing ($\hat{V}_c(X)$, dash-dot) and the costly financing cases using risky debt ($\hat{V}_d(X)$, dark) or new equity ($\hat{V}_e(X)$, light), respectively. Additionally, the lines of constant $\omega = 0\%$ ($\Delta X = 0$), 25%, 50%, 75%, and 100%, known from Figure 1, and the investment threshold of the unconstrained case, $\hat{V}_u$, are shown, as well as the threshold after liquidation, $\hat{V}_{ul}$. Parameter values are given in Table 1. For high-liquidity firms, the value of waiting dominates the risk of future financing constraints and raises the investment threshold. For low-liquidity firms, the exogenous lower bound $\omega = 100\%$ becomes binding in the pure-quantity constrained case, or present financing costs become prohibitively high and raise the investment threshold.
Figure 4: Risky Debt Investment Threshold for Changing Degree of Market Frictions.

The value of the risky debt financing firm’s investment threshold $\hat{V}_d$ (thick lines) and the quantity constraint $\omega = \Delta X / \hat{V} = 100\%$ (thin lines) are shown as functions of the market friction parameter $\alpha$ for different amounts of costly financing $\Delta X$. Parameter values are given in Table 1. The threshold is strictly increasing in $\alpha$ for negative $\Delta X$ (e.g. $\Delta X = -50$) since there are no present financing costs, and the quantity constraint is not binding for any $\alpha$ in that case. It is U-shaped for modestly positive amounts of costly financing (e.g. $\Delta X = 100$), and strictly decreasing in $\alpha$ if $\Delta X$ is high enough that the quantity constraint is binding for all $\alpha$ (e.g. $\Delta X = 250$).
Figure 5: Contour Plot of Risky Debt Investment Threshold.
The value of the risky debt financing firm’s investment threshold $\hat{V}^d$ is shown as a contour plot for different values of cash balance $X$ and market friction parameter $\alpha$. Parameter values are given in Table 1. It can be verified that the threshold as a function of $X$ with fixed $\alpha$ is always U-shaped as shown in Figure 3 for $\alpha = 0.8$. However, if we compare firms facing different $\alpha$ and equal cash balance $X$, we see that the threshold is strictly increasing in $\alpha$ for $X > 0$, U-shaped for modestly negative cash balance, and strictly decreasing in $\alpha$ for very low $X$, as shown for special values of $X$ (of $\Delta X$, to be more precise) in Figure 4.