Bond Portfolio Optimization:
A Risk-Return Approach

Olaf Korn* and Christian Koziol**

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*Prof. Dr. Olaf Korn (corresponding author), Chair of Corporate Finance, WHU, Otto Beisheim Graduate School of Management, Burgplatz 2, D-56179 Vallendar, Germany, Phone +49 261 6509 410, Fax +49 261 6509 419, Email olaf.korn@whu.edu.

**Dr. Christian Koziol, Chair of Finance, University of Mannheim, D-68131 Mannheim, Germany, Phone +49 621 181 1521, Fax +49 621 181 1519, Email c.koziol@uni-mannheim.de.

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Abstract

In this paper, we apply Markowitz's approach of portfolio selection to government bond portfolios. As a main feature of our analysis, we use term structure models to estimate expected returns, return variances, and covariances of different bonds. Our empirical study for the German market shows that a small number of risky bonds is sufficient to reach very promising predicted risk-return profiles. If the number of risky bonds in the portfolio is not too large and the term structure model does not contain more than two factors, these predictions are confirmed by the realized risk-return profiles.

1 Introduction

The portfolio approach pioneered by Markowitz is one of the cornerstones of modern portfolio management. A broad knowledge has been accumulated about the performance, the strengths, and the weaknesses of this approach when applied to equity portfolios. However, much less is known about portfolio optimization in bond markets.

There are at least two reasons for this observation: First, at the time when Markowitz's approach became more widely recognized as a useful tool for portfolio management, interest rates were not particularly volatile and a portfolio approach seemed somehow unnecessary. However, this observation has changed over the last decades. Even if one concentrates on government bonds of highly rated countries and leaves default risk aside, there are very substantial risks in bond investments due to possible changes in interest rates. Given that many different bonds with different maturities are available, it is a natural issue to think about the potential for risk diversification.

Second, severe difficulties to implement Markowitz's approach might have discouraged further work on $\mu - \sigma$ optimization of bond portfolios.[1] In particular, there are two major problems, one general problem and one bond-specific problem. The general issue concerns the large number of parameters needed in the Markowitz approach if the number of assets increases. Here, recent research has shown that restrictions with respect to the parameters and portfolio weights are useful for improving the performance of optimized portfolios.[2] The bond specific issue concerns the variation of moments over time, which precludes simple historical estimation based on the assumption of stationarity. For example, in a high-interest-rate period the returns of bonds are supposed to be higher than in a low-interest-rate period. Therefore, if interest rates decline during a period, the average return during this period seems to overstate the return during the succeeding period with (probably)

[1] In the 1960s and 1970s, several papers dealt with bond portfolio problems in the context of $\mu - \sigma$ analysis, e.g. Cheng (1962), Roll (1971), Yawitz, Hempel and Marshall (1976), Yawitz and Marshall (1977), and Kaufman (1978). Later on, the literature has taken a different route and has mainly focussed on risk management and immunization strategies. See e.g. Fabozzi (2004) for a discussion of these latter issues.

[2] One approach has applied shrinkage estimators, as proposed e.g. by Jobson and Korkie (1980), Jorion (1986) and Frost and Savarino (1986). Another suggestion put forward by Pastor (2000) and Pastor and Stambaugh (2000) is the usage of additional information from equilibrium asset pricing models in a Bayesian framework. The effect of short sale constraints has been studied e.g. by Frost and Savarino (1988) and Jagannathan and Ma (2003).

lower interest rates. Moreover, bonds change their characteristics over time. For example, the local riskiness of a default-free bond depends crucially on the time to maturity.

To mitigate the problems mentioned above, term structure models could be very useful. These models have an economic foundation either based on the no-arbitrage or the equilibrium concept and have been successfully applied to price interest rate dependent securities over the last decades. For the purpose of bond portfolio optimization, term structure models have two major advantages. First, depending on the complexity of the model, i.e. the number of factors and the complexity of their dynamics, term structure models might impose severe restrictions on the moments of bond returns. However, they can also be very flexible with respect to the implied moments. Second, term structure models consider that moments of bond returns are time varying and capture the effects of a decreasing time to maturity.

This paper proposes the use of term structure models for constructing optimized bond portfolios according to the Markowitz approach. Such a suggestion is not a new one. As early as in 1980, Brennan and Schwartz wrote:[3] "It is hoped that conditional prediction models (i.e. term structure models) such as this will play the same role in bond portfolio management as Sharpe's [1963] diagonal model and subsequent elaborations thereof have played in the management of stock portfolios." Wilhelm (1992) also propagates using term structure models for portfolio selection and derives the required means, variances, and covariances of discrete holding period returns for a CIR type model. However, to our knowledge, our paper is the first one which implements such portfolio strategies and tests them in an empirical study.

Our empirical study for the German government bond market over the period of 1974 to 2004 addresses several important questions: Which risk-return trade-off can we achieve if we restrict our attention to investments in government bonds? What are the diversification effects if we use bonds with different times to maturity? How many bonds do we need? How much complexity should we allow for the term structure model? When do we need more flexibility and when do we need more restrictions?

To answer these questions, Section 2 sets the ground by introducing the class of multi-factor Vasicek type term structure models we use in our empirical study. Based on these models, we derive the implied expected values, variances, and covariances of discrete holding period bond returns that are needed to solve the portfolio optimization problem. Section 3 provides the empirical study. Following a description

[3]See Brennan and Schwartz (1980), p. 406.

of the bond data set in Section 3.1, the estimation procedure for the parameters of the term structure models and the design of the empirical study is outlined in Section 3.2. Section 3.3 provides first results concerning the risk-return profiles of optimized portfolios by looking at the predictions made by different specifications of the term structure model. Section 3.4 contrasts these predictions with the realized risk-return profiles obtained in an out-of-sample study.

The main result of the paper is that optimized bond portfolios exhibit very attractive risk-return profiles. As long as the number of risky bonds in a portfolio is moderate and the number of stochastic factors of the term structure model is limited, this result holds both for the predictions made by the model and for the out-of-sample performance.

2 Portfolio Selection Based on Term Structure Models

We analyze the portfolio problem of an investor who uses the traditional Markowitz approach, i.e. he or she seeks to select a $\mu - \sigma$ efficient portfolio. The portfolio is set up at time $t = 0$ and held until the planning horizon $t = T$ without rebalancing. Such a static strategy has relatively low transactions and monitoring costs. The investor can invest in government bonds with different maturities. For concreteness and to highlight maturity diversification, we assume that the investment opportunity set comprises $N+1$ default-free zero-coupon bonds which mature at times $T_0, T_1, ..., T_N$, with $T = T_0 < T_1 < ... < T_N$. The bond that matures at time T_0 can be understood as a risk-free investment from time $t = 0$ to the planning horizon T.

To determine the required values of the expected return and the variance-covariance matrix of returns of these bonds, we use a multi-factor term structure model of the Vasicek type. The main advantages of this model are the closed-form solutions for zero-bond prices and the straightforward maximum likelihood estimation of the unknown model parameters by means of the Kalman filter algorithm.

We assume that each factor $X_k(t)$, $k = 1, ..., K$, follows an Ornstein-Uhlenbeck process and has a long-term mean of zero. Moreover, each factor exhibits a constant risk premium λ_k. As a consequence, the process followed by each factor under the risk-neutral measure reads

$$dX_k(t) = \kappa_k \cdot (\lambda_k - X_k(t)) \cdot dt + \sigma_k \cdot dz_k(t), \text{ for all } k = 1, ..., K.$$

As usual, κ_k is the speed of reversion, σ_k stands for the local volatility of the increment of factor k, and $z_k(t)$ denotes a standard Wiener process under the risk-neutral measure. We further assume that all factors $X_k(t)$, $k = 1, ..., K$, are mutually independent. The current value of the short rate $r(t)$ is assumed to equal the sum of a fixed term $\overline{r}$ and the values of the factors $X_k(t)$:

$$r(t) = \overline{r} + \sum_{k=1}^{K} X_k(t). \tag{1}$$

The price at time t of a zero bond $P(t,\tau)$ that pays its face value of one at time τ results from the expectation $E_t^{\mathbb{Q}}(\cdot)$ under the risk-neutral measure in a well-known manner:

$$P(t,\tau) = E_t^{\mathbb{Q}}\left(e^{-\int_t^\tau r(s)ds}\right).$$

The evaluation of this expectation leads to the following price of a zero bond:

$$P(t,\tau) = e^{-A(t,\tau)-\overline{r}\cdot(\tau-t)-\sum_{k=1}^{K}\{X_k(t)\cdot B_k(t,\tau)\}}, \tag{2}$$

with

$$\begin{aligned} A(t,\tau) &= \sum_{k=1}^{K}\left\{\left(\frac{\sigma_k^2}{2\cdot\kappa_k^2} - \lambda_k\right)\cdot\left(B_k(t,\tau) - (\tau - t)\right) + \frac{\sigma_k^2}{4\cdot\kappa_k}B_k(t,\tau)^2\right\}, \\ B_k(t,\tau) &= \frac{1 - e^{-\kappa_k\cdot(\tau-t)}}{\kappa_k}. \end{aligned}$$

To solve the investors portfolio problem, we need the expected return μ_i, $i = 1, \ldots, N$, of every risky bond and the covariance matrix $\Omega = \left\{s_{i,j}^2\right\}$, $i, j = 1, \ldots, N$, under the physical rather than the risk-neutral measure. The (discrete) expected return μ_i and the return covariance $s_{i,j}^2$ of bond i and bond j for the period from $t = 0$ to T are given by

$$\begin{aligned} \mu_i &= \frac{E_0^{\mathbb{P}}\left(P(T,T_i)\right)}{P(0,T_i)} - 1, \\ s_{i,j}^2 &= Cov_0^{\mathbb{P}}\left(\frac{P(T,T_i)}{P(0,T_i)}, \frac{P(T,T_j)}{P(0,T_j)}\right). \end{aligned}$$

To determine the above moments, we need to know the moments of the factors. For the dynamics of the factors $X_k(t)$ under the real measure, we consider the following representation:

$$dX_k(t) = \kappa_k \cdot (-X_k(t)) \cdot dt + \sigma_k \cdot dz_k'(t), \text{ for all } k = 1, ..., K,$$

where $z_k'(t)$ is a standard Wiener process under the real measure.

The expectation $E_0^{\mathbb{P}}\left(X_k\left(T\right)\right)$ and the variance $Var_0^{\mathbb{P}}\left(X_k\left(T\right)\right)$ of the value of the $k-$th factor at the planning horizon $t = T$ under the physical measure result from the well-known properties of the O-U process:

$$\begin{aligned} E_0^{\mathbb{P}}\left(X_k\left(T\right)\right) &= X_k\left(0\right) \cdot e^{-\kappa_k \cdot T}, \\ Var_0^{\mathbb{P}}\left(X_k\left(T\right)\right) &= \frac{\sigma_k^2}{2 \cdot \kappa_k}\left(1 - e^{-2\kappa_k \cdot T}\right). \end{aligned}$$

We recall that the factors are mutually independent and therefore the covariance of two distinct factors equals zero.

Given the realizations of the factors and a set of constant model parameters, the term structure model completely determines all bond prices. In the real world, however, there will in general be pricing errors at the planing horizon. These pricing errors, which reflect the model risk, are another source of uncertainty that should be taken into account in the portfolio problem. Therefore, we assume that the zero bond prices at the planning horizon T for maturities $\tau > T$ are affected by a further source of uncertainty, denoted by ε_τ, in the following way:

$$P\left(T, \tau\right) = e^{-A(T,\tau) - \bar{r}\cdot(\tau - T) - \sum_{k=1}^{K}\left\{X_k(T)\cdot B_k(T,\tau)\right\} + \varepsilon_\tau(T)}. \tag{3}$$

The error terms $\varepsilon_\tau(T)$ are assumed to be normally distributed with zero mean and variance $s^2\left(\varepsilon_\tau\right)$, and error terms that refer to different maturities τ are mutually independent by assumption. When discussing the estimation of the term structure model in Section 3.2, we show how the pricing error variances $s^2\left(\varepsilon_\tau\right)$ can be jointly estimated with the other model parameters by maximum likelihood.

Under the stated assumptions, the expected return μ_i, the variance $s_{i,i}^2$, and the covariance $s_{i,j}^2$ can be calculated directly. The resulting moments are given by

$$\begin{aligned} \mu_i &= \frac{e^{M^{(1)}(T_i) + \frac{1}{2}S^{(1)}(T_i)^2}}{P\left(0, T_i\right)} - 1, \\ s_{i,i}^2 &= \frac{e^{2\cdot M^{(1)}(T_i) + S^{(1)}(T_i)^2} \cdot \left(e^{S^{(1)}(T_i)^2} - 1\right)}{P\left(0, T_i\right)^2}, \\ s_{i,j}^2 &= \frac{e^{M^{(2)}(T_i,T_j) + \frac{1}{2}S^{(2)}(T_i,T_j)^2} - e^{M^{(1)}(T_i) + M^{(1)}(T_j) + \frac{1}{2}\left(S^{(1)}(T_i)^2 + S^{(1)}(T_j)^2\right)}}{P\left(0, T_i\right) \cdot P\left(0, T_j\right)}, \text{ for } i \neq j, \end{aligned} \tag{4}$$

with

$$
\begin{aligned}
M^{(1)}(T_i) &= A(T,T_i) - \overline{r}\cdot(T_i - T) - \sum_{k=1}^{K}\left\{E_0^{\mathbb{P}}(X_k(T))\cdot B_k(T,T_i)\right\},\\
S^{(1)}(T_i) &= \sqrt{\sum_{k=1}^{K}\left\{Var_0^{\mathbb{P}}(X_k(T))\cdot B_k(T,T_i)^2\right\} + s^2(\varepsilon_{T_i})},\\
M^{(2)}(T_i,T_j) &= -A(T,T_i) - A(T,T_j) - \overline{r}\cdot(T_i + T_j - 2T)\\
&\quad - \sum_{k=1}^{K}\left\{E_0^{\mathbb{P}}(X_k(T))\cdot(B_k(T,T_i) + B_k(T,T_j))\right\},\\
S^{(2)}(T_i,T_j) &= \\
&\sqrt{\sum_{k=1}^{K}\left\{Var_0^{\mathbb{P}}(X_k(T))\cdot(B_k(T,T_i) + B_k(T,T_j))^2\right\} + s^2(\varepsilon_{T_i}) + s^2\left(\varepsilon_{T_j}\right)}.
\end{aligned}
$$

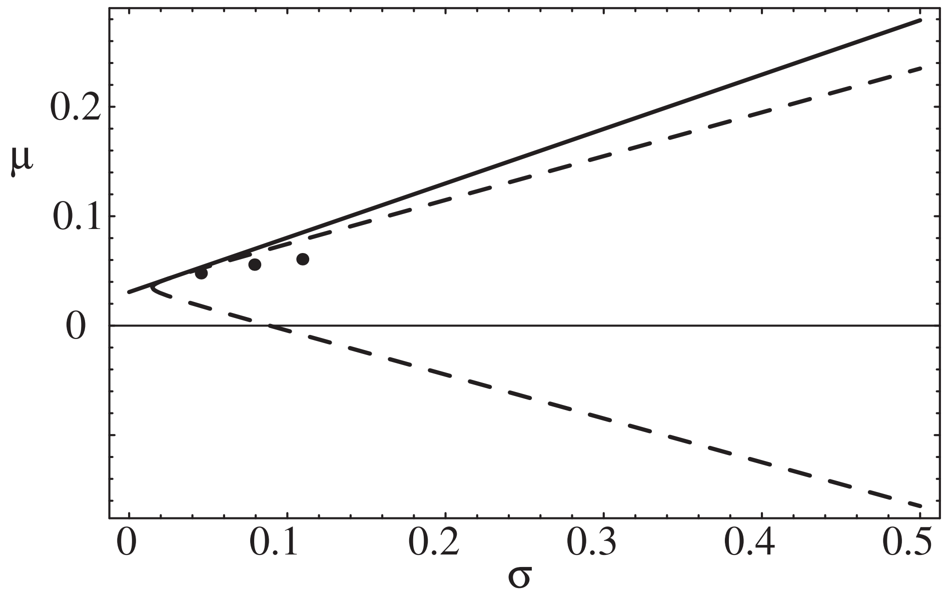
If the vector of expected returns and the variance-covariance matrix of returns is available, $\mu - \sigma$ efficient portfolios can easily be determined, as outlined e.g. by Merton (1972) or Huang and Litzenberger (1988), Chapter 3.

To provide some intuition for the kind of efficient frontiers that result from the term structure model, Figure 1 shows a typical example of a $\mu - \sigma$ diagram in the case of two factors and an investment horizon of one year. The investment opportunity set contains three risky bonds (four, seven, and ten years to maturity) and a risk-free bond (one year to maturity). The parameters of the interest rate process were estimated by using the whole data set described in the next section. The values of the factors $X_1(0)$ and $X_2(0)$ are set to zero, which equals the factors' long-term means.

Like in this example, we generally find in our empirical study that zero bonds with a longer time to maturity have a higher standard deviation of discrete holding period returns. This finding is quite intuitive, as bonds with a longer time to maturity have a higher exposure to the common factors $X_k(T)$. If we interpret the coefficients $B_k(T,T_i)$ as factor loadings, we see that for a longer time to maturity T_i these coefficients are higher and therefore the bond price is more strongly affected by changes in $X_k(T)$. Moreover, we find in our empirical study that different risky zero bonds usually have a rather high correlation above 0.9. Therefore, the risky bonds almost lie on a line. Whether this line has a positive slope or a negative slope depends on the risk premia λ_k. Usually, the higher λ_k is, the greater is the slope of the line. As a further consequence of the high correlations between risky bonds, the standard deviation of the global minimum variance portfolio of risky bonds is close to zero.

Figure 1: $\mu - \sigma$ diagram

The figure depicts $\mu - \sigma$ combinations for three risky zero bonds (thick dots), that mature in four, seven, and ten years, respectively. An investment horizon of one year is assumed. The zero bonds with longer time to maturity correspond to the higher standard deviation σ. The dashed curve shows the frontier portfolios obtained form the three risky bonds. The solid line shows the efficient portfolios if in addition to the risky bonds a risk-free bond with time to maturity of one year is available. The following parameter values are used: $\bar{r} = 0.0256$, $\lambda_1 = 0.0210$, $\kappa_1 = 0.4203$, $\sigma_1 = 0.0177$, $\lambda_2 = 0.0533$, $\kappa_2 = 0.0311$, $\sigma_2 = 0.0126$, $s(\varepsilon_{T_1}) = 0.00229$, $s(\varepsilon_{T_2}) = 0.00148$, $s(\varepsilon_{T_3}) = 0.000366$, $X_1(0) = 0$, *and* $X_2(0) = 0$.



3 Empirical Study

3.1 Government Bond Data

Our empirical study analyzes investment strategies in German government bonds. According to a bond market study by Merril Lynch (2002), the German market has established itself as the third biggest bond market in the world, having an outstanding total nominal value equal to € 2350 billion in 2001. Within the important Euroland bond market, the German market is by far the biggest one, with a volume almost twice as high as that of the second biggest market.

The German government bond market is well known as the benchmark bond market in Euroland. Since 1985, government bonds have been attributing between 30 to 40 percent to the whole German bond market. The continuous issue process ensures the availability of a sufficient number of bonds in all relevant maturity classes and a high liquidity. Government bonds are typically issued up to three times a year with a nominal volume between € 15 billion to € 25 billion and a time to maturity of ten years. Recently, some 30-year bonds have also been placed. In addition, futures contracts on government bonds such as the prominent Bund Futures and the Bobl Futures are actively traded, which facilitates taking short positions.

In our study, we use data from the period between 1974/1/1 to 2004/9/30. Before 1974, yield curves were strongly affected by the regulated currency regime of Bretton Woods. Therefore, we exclude earlier observations. The results of monthly term structure estimations are provided by the Deutsche Bundesbank. They are based on the approach suggested by Svensson (1994), which presents the yield curve as a function of six parameters, β_0, β_1, β_2, β_3, τ_1, and τ_2. The current spot rate $y(l)$ for an investment over l years takes the following form:

$$y(l) = \frac{1}{100} \cdot \left(\beta_0 + \beta_1 \frac{1 - e^{-\frac{l}{\tau_1}}}{\frac{l}{\tau_1}} + \beta_2 \left(\frac{1 - e^{-\frac{l}{\tau_1}}}{\frac{l}{\tau_1}} - e^{-\frac{l}{\tau_1}} \right) + \beta_3 \left(\frac{1 - e^{-\frac{l}{\tau_2}}}{\frac{l}{\tau_2}} - e^{-\frac{l}{\tau_2}} \right) \right).$$

From the yield curves provided by the Deutsche Bundesbank, we use the spot rates

$$y(2), y(3), ..., y(10)$$

for maturities from two to ten years to determine the corresponding zero bond prices

$$\frac{1}{(1 + y(2))^2}, \frac{1}{(1 + y(3))^3}, ..., \frac{1}{(1 + y(10))^{10}}.$$

The one-year rate $y(1)$ acts as the risk-free rate. Since typically ten-year bonds are issued, our choice covers the available set of government bonds. Although primarily coupon bonds are traded in the market, we base our analysis of investment strategies on the prices of 'synthetical' zero bonds. The use of zero bonds is the most intuitive way to analyze the effect of diversification with respect to different times to maturity.

3.2 Parameter Estimation

The risk-return profiles of bond portfolios derived from term structure models depend on the unknown model parameters that have to be estimated. To estimate these parameters we use a maximum likelihood approach that combines time series and cross sectional information. It is based on the state space representation of a term structure model and the Kalman filter algorithm.[4] Assume that there is an estimation period from time $\underline{t}$ to time $\overline{t}$, preceeding the investment period. The estimation period is split into m equally spaced intervals of length h, and at each point $t = \underline{t}, \underline{t} + h, \ldots, \underline{t} + mh$ we observe the (N+1)-vector $[\,P(t, t + T_0),\, P(t, t + T_1), \ldots, P(t, t + T_N)\,]$ of zero bond prices with N+1 different times to maturity $T_0, T_1, \ldots, T_N$.

The state space representation of a model consists of measurement equations and transition equations. The measurement equations show how observed prices are related to the unobserved factors. Based on the pricing formula (3) of the multi-factor Vasicek model, we obtain the following measurement equations, one for each maturity:[5]

$$ln(P\,(t, t + T_i)) = -A\,(T_i) - \bar{r} \cdot T_i - \sum_{k=1}^{K} \{X_k\,(t) \cdot B_k\,(T_i)\} + \varepsilon_{T_i}(t), \quad i = 0, \ldots, N. \tag{5}$$

The transition equations describe the evolution of the stochastic factors over discrete time intervals of length h. For factors following a zero mean Ornstein-Uhlenbeck process, the following transition equations result, one for each factor:

$$X_k(t) = e^{-\kappa_k h}\, X_k(t - h) + \omega_k(t), \quad k = 1, \ldots, K. \tag{6}$$

As a consequence of our assumptions about the term structure model, the error terms $\omega_k(t)$ are serially and cross-sectionally uncorrelated normal random variables with zero means and variances equal to $(1 - e^{-2\kappa_k h})\sigma_k^2/2\kappa_k$.

The likelihood function of the above state space model can be computed recursively by means of the Kalman filter algorithm, which allows for straightforward numerical maximization with respect to the unknown model parameters. If the model parameters have been obtained, the values of the unobserved factors can be estimated in a second step.[6]

Actual estimation is carried out using a rolling data window of ten years and a monthly data frequency. The rather long estimation period of ten years is chosen to capture the cyclical behavior of interest rates. Given our data set, the first estimation period covers data from January 1974 to December 1983. For each month

[4] See e.g. Chen and Scott (1993), Pearson and Sun (1994), Duffie and Singleton (1997), and Babbs and Nowman (2001) for similar applications of the Kalman filter algorithm to the estimation of term structure models.

[5] For the state space representation of the multi-factor Vasicek model see also Babbs and Nowman (2001).

[6] See e.g. Harvey (1989) and Gourieroux and Monfort (1997), Chapter 15, for details.

ten different contracts with maturities from one year to ten years are used. Thus, combining time series and cross-sectional information delivers 1.200 data points for estimation. We estimate three model variants, with one, two, and three stochastic factors, respectively. Once a model has been estimated, we calculate the vector of expected returns and the variance-covariance matrix of returns according to equations (4) for an investment horizon of one year. These moments are used to form different optimized portfolios, as explained in the next section. Then the estimation window is moved one month further, i.e. the second estimation period covers data from February 1974 to January 1984. The whole process is repeated until September 2003, the starting date of our final investment period. This procedure ultimately delivers for each model variant 238 estimates of the model parameters, the mean vectors, and the variance-covariance matrices of yearly zero bond returns.

Table 1 provides some information on the size of the model parameters. The table shows averages taken over the parameter estimates of all 238 estimation periods. Two observations are worth mentioning. First, the error variances σ_ε considerably decline if we move from the one-factor model to the two-factor model and further to the three-factor model, i.e. the in-sample explanation of zero bond prices clearly improves with the number of factors. Whether or not this observation translates into a better portfolio choice of multi-factor models is a central question of the empirical study. Second, the estimates of the κ-parameters show that for all three model variants at least one factor possesses considerable mean reversion. Mean reversion is highly relevant for our mean-variance analysis, because it implies at least some degree of state dependence and predictability of expected returns. An important issue for portfolio choice is whether the mean reversion of certain factors is stable over time. Figure 2 provides some information on this question. The figure shows the time series of the parameter estimates for the 238 estimation periods. As can be seen, there is some variation of the different κ-parameters over time. However, qualitatively, we always obtain a medium size κ-parameter for the one-factor model, one medium size and one small κ-parameter for the two-factor model, and one medium size, one large, and one small parameter for the three-factor model.

3.3 Predicted Risk-Return Profiles

Table 2 shows the ex ante attractiveness of efficient bond portfolios. In this table we report the average (over the 238 starting dates of the investment period) expected return of portfolios that maximize expected return for a fixed standard deviation of 20%. In addition we report the Sharpe ratios. The portfolio construction is based

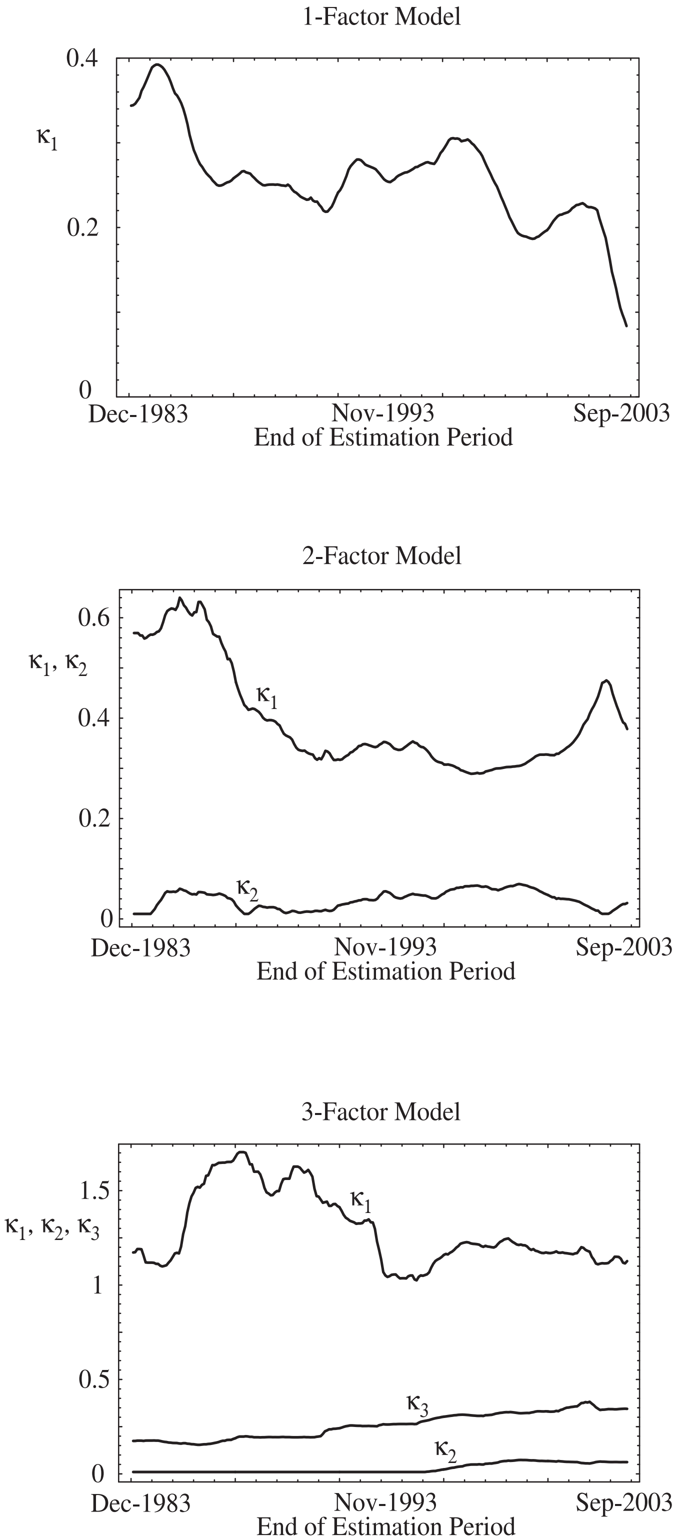
Table 1: Estimated parameters of the term structure models

This table shows the average parameter values obtained from the rolling estimation procedure (average over the 238 values from the different estimation periods).

1-Factor Model		2-Factor Model		3-Factor Model	
$\overline{r}$	0.0488	$\overline{r}$	0.0272	$\overline{r}$	0.0438
λ_1	0.0311	λ_1	0.0159	λ_1	0.00271
κ_1	0.258	κ_1	0.397	κ_1	1.29
σ_1	0.0124	σ_1	0.0176	σ_1	0.0288
		λ_2	0.0611	λ_2	0.348
		κ_2	0.0385	κ_2	0.0286
		σ_2	0.0131	σ_2	0.0235
				λ_3	-0.00104
				κ_3	0.256
				σ_3	0.0236
σ_{ε_2}	0.00246	σ_{ε_2}	0.00173	σ_{ε_2}	0.00102
σ_{ε_3}	0.00358	σ_{ε_3}	0.00200	σ_{ε_3}	0.00073
σ_{ε_4}	0.00420	σ_{ε_4}	0.00183	σ_{ε_4}	0.00031
σ_{ε_5}	0.00460	σ_{ε_5}	0.00153		
σ_{ε_6}	0.00488	σ_{ε_6}	0.00121	σ_{ε_6}	0.00017
σ_{ε_7}	0.00511	σ_{ε_7}	0.00090	σ_{ε_7}	0.00024
σ_{ε_8}	0.00529	σ_{ε_8}	0.00060	σ_{ε_8}	0.00022
σ_{ε_9}	0.00546	σ_{ε_9}	0.00030	σ_{ε_9}	0.00014
$\sigma_{\varepsilon_{10}}$	0.00562				

Figure 2: Evolution of mean-reversion parameters (κ)

The three diagrams show the evolution of the estimated κ-coefficients over time for the three model variants (one-factor, two-factor, three-factor). The horizontal axis refers to the end dates of the 238 ten-year estimation periods.



on moments obtained from interest rate models with either one, two or three factors. To analyze the impact of the number of different bonds combined in a portfolio, we also varied this number. Since government bonds in the German market typically have a time to maturity of up to ten years, we considered bond portfolios within this maturity range. As the risky asset in a portfolio consisting of the risk-free instrument and one risky bond, we chose the seven-year zero bond. The portfolio with two risky bonds includes the four-year and ten-year zero bonds as risky instruments, i.e. the average time to maturity of these two risky bonds is equal to that of the risky bond in the portfolio with one risky asset. The portfolio with three risky bonds contains all instruments from the portfolios with one and two risky bonds, which are the one-year, four-year, seven-year, and ten-year zero bonds. Finally, we added the results for a portfolio with one risk-free and nine risky bonds, which covers all maturities in the data set.

The expected portfolio returns in Table 2 range from 8.56% to 16.91% and the corresponding Sharpe ratios from 0.17 to 0.59. Especially for the one-factor model, the risk return profile is very attractive. Also the two-factor model and the three-factor model produce high Sharpe ratios if the number of risky bonds is not very low. This result might be quite surprising. Since bonds are supposed to be less profitable than e.g. stocks, one might expect less attractive expected returns and Sharpe ratios. However, we find that with only a few risky zero bonds we can achieve more attractive predicted risk-return profiles than with many well diversified bond and equity portfolios, as analyzed e.g. by DeMiguel, Garlappi and Uppal (2004). These authors report Sharpe ratios between 0.16 and 0.4.

Clearly, the more bonds we include in our portfolio the more attractive the portfolio becomes and the higher the Sharpe ratios must be. A closer examination of Table 2 shows that attractive risk-return profiles can be achieved for any variant of the interest rate model if at least as many risky bonds are used as there are stochastic factors. For the one-factor model, we see that the Sharpe ratios are almost unaffected by the number of risky bonds we consider. For the two-factor model there is a strong increase of the Sharpe ratio from 0.3 to 0.43 when we move from one to two risky bonds. However, the potential for further improvements of the Sharpe ratio by adding more risky bonds is rather small. Similarly, for the three-factor model, we find that the Sharpe ratio increases from 0.17 to 0.39 if three risky bonds are used instead of one. However, adding six more risky bonds increases the Sharpe ratio by no more than 0.07.

Table 2: Expected returns and Sharpe ratios

This table shows the mean (over the 238 investment periods) expected annual return in percentage points and the Sharpe ratio of efficient portfolios with an annual standard deviation of 20%. All portfolios contain the risk-free instrument. In addition they either contain a zero bond that matures at time $T_1 = 7$ *(one risky bond), zero bonds that mature at times* $T_1 = 4$ *and* $T_2 = 10$ *(two risky bonds), zero bonds that mature at times* $T_1 = 4$, $T_2 = 7$, *and* $T_3 = 10$ *(three risky bonds), or zero bonds with maturities* $T_1 = 2$, $T_2 = 3$, ..., $T_9 = 10$ *(nine risky bonds).*

# Risky Bonds	**Return**	**Sharpe Ratio**
Panel A (1-Factor Model)		
9	16.91	0.59
3	16.19	0.55
2	16.15	0.55
1	16.10	0.55
Panel B (2-Factor Model)		
9	13.88	0.44
3	13.76	0.43
2	13.73	0.43
1	11.21	0.30
Panel C (3-Factor Model)		
9	14.37	0.46
3	12.91	0.39
2	9.18	0.20
1	8.56	0.17

This observation is quite intuitive. Bond portfolios are exposed to one or more risk factors, depending on the variant of the term structure model. If the number of available zero bonds is below the number of factors, one cannot achieve arbitrary combinations of exposures for each factor. If the relation between risk exposures and portfolio weights were linear, the number of non-redundant risky bonds would have needed to be equal to the number of factors to span every combination of risk exposures. In other words, as long as the number of risky bonds is below the number of risk factors, every additional bond is likely to lead to a substantial improvement with respect to the possible choices of risk exposures. Therefore, the attractiveness of the portfolios, i.e. the expected return and Sharpe ratio, will substantially increase. However, note that additional bonds do generally increase the Sharpe ratio, at least to some extent, even if the number of available bonds already exceeds the number of risk factors. One reason for this finding is that the relation between bond prices and risk factors is not linear but log linear, as can be seen from equation (3). A second reason is the existence of the pricing errors ε_{T_i}.

Another interesting empirical result states that as long as the number of different risky bonds is fixed, it does not matter which bonds (i.e. which maturities) are chosen. This conclusion is based on additional analyses with varying sets of bonds.[7] The stated robustness result is in line with our notion that risky zero bonds are primarily used to set the risk exposure with respect to different factors. Since all zero bonds depend on the same factors, the choice among them is almost irrelevant.

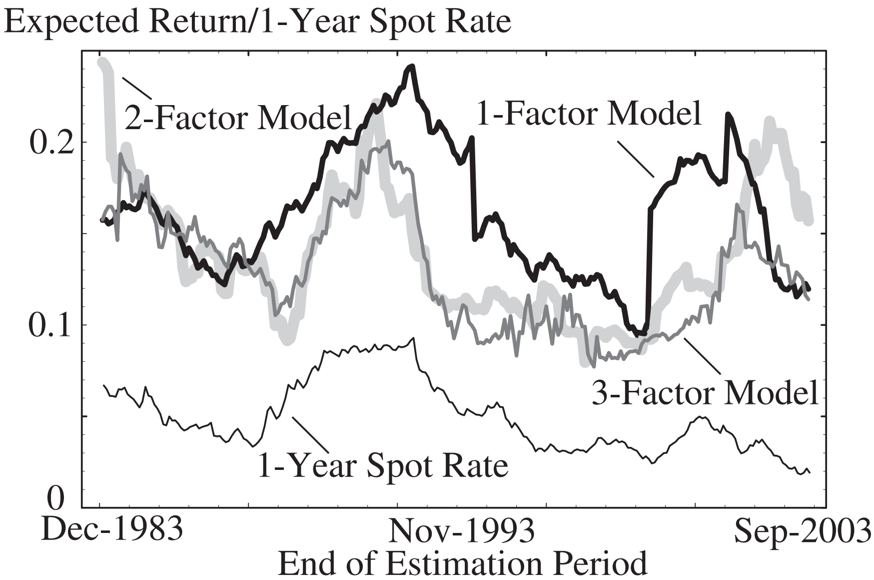
Finally, it is interesting to see how the predicted expected return behaves over time, i.e. how it changes with the estimation period and the starting date of the investment period. Figure 3 provides the corresponding results for portfolios with three risky bonds. In addition to the expected returns derived from the three model variants (one to three factors), the figure shows the one-year interest rate. As can be seen, the expected portfolio return strongly varies over time and takes values between 7.7% and 24.4%. The average expected return over all investment periods and all three model variants is 14.3%. Moreover, we observe that the evolution of expected returns over time is similar for the three model variants. The strongest relation is between the two-factor model and the three-factor model, with a correlation between expected returns of 0.8.

In addition, it turns out that the expected return of the considered bond portfolios is much higher than the one-year interest rate. The average one-year spot rate lies at about 5.1%, which is far below the average expected return of 14.3%. However,

[7] The corresponding detailed results are available upon request from the authors.

Figure 3: Evolution of expected portfolio returns

The diagram shows the evolution of yearly expected portfolio returns over time, as they are predicted by the three model variants (one-factor, two-factor, three-factor). The horizontal axis refers to the start dates of the 238 one-year investment periods. The investment opportunity set contains a risk-free bond (one-year bond) and three risky bonds (four-, seven-, and ten-year bond). The portfolios are efficient ones with a predicted return volatility equal to 20%.



the one-year interest rate clearly moves together with the expected return derived from each of the three model variants. For example, the correlation of the expected portfolio return derived from the one-factor model and the one-year interest rate equals 0.53. In this sense, we can say that the optimized portfolios have a higher expected return if interest rates are high.

3.4 Realized Risk-Return Profiles

The essential question concerning the success of different term structure models for portfolio optimization is whether the predicted risk-return profiles are attainable out of sample. Therefore, we compare predicted and realized returns. For every month from January 1984 to September 2003 different optimized portfolios were set up corresponding to different model variants and different numbers of risky bonds. We then calculated for all portfolios the returns they earned over the following year, which is the investment period. Then we subtracted the predicted expected returns from the realized returns in each period. We recall that — as seen in Figure 3 — the predicted expected return does not only depend on the model variant but is also state dependent, i.e. it varies over time. The resulting excess returns, however, have the same expected value of zero and the same volatility of 20% in every period, provided that the portfolio optimization procedure works perfectly. Deviations from a mean of zero can be detected by looking at the average excess return, with averages taken over the 238 excess returns of a portfolio.

Detecting deviations from the volatility of 20% is a bit more difficult. To achieve this, we first took a potential bias (deviation of the average excess return from zero) of the portfolio strategies into account and subtracted the average excess return from the excess returns. Second, we calculated the absolute values of these excess returns corrected for bias and averaged over the 238 observations. This procedure provides us with an estimate of the volatility of the strategies. Standard deviations for the mean excess returns and the mean absolute deviations were calculated with the Newey and West (1987) estimator. We used 11 lags to take into account that the return intervals have an overlap of 11 months. The Newey-West standard errors allow us to test whether the realized values are significantly different from the predicted values.

Table 3 provides our results. To facilitate a comparison with the predictions presented in Table 2, we added the average predicted return to the average excess return. Thus, the bold numbers in the second column of Table 3 provide the total average return in percentage points. The predictions of Table 2 are repeated below. The third column provides the absolute deviations and the fourth column shows the Sharpe ratios, calculated from the values of the second and third columns, using the average one-year interest rate of 5.11%. Again, realized values are in bold face, predicted values are given below.

Panel A shows the results of the portfolio strategies derived from the one-factor model. A striking point to note is that the realized returns of the portfolio with nine risky bonds strongly deviate from the predictions with respect to both the expected return and the volatility. The performance of this strategy is clearly unacceptable. With as many as nine risky assets we are close to a situation where some assets are redundant. The near singularity[8] of the variance-covariance matrix leads to rather extreme portfolios with a high volume of short sales. Table 4 supports this statement

[8]Note that the variance-covariance matrix can never be exactly singular because bond prices are non-linear functions of the factor and the pricing errors ε_{T_i} are assumed to be uncorrelated between different maturities.

Table 3: Realized returns, absolute deviations, and Sharpe ratios

This table shows the mean (over the 238 investment periods) annual realized returns, the mean absolute deviation of realized returns from expected returns, and the Sharpe ratios of the different portfolios (bold numbers). For comparison reasons the predicted values from Table 2 are repeated. In addition, we report the results of the strategies using the DAX and REX. An asterisk indicates whether the realized value significantly differs from the predicted value on a 5% significance level.

# Risky Bonds	Return	Abs. Dev.	Sharpe Ratio
Panel A (1-Factor Model)			
9	**43.48**	**122.52***	**0.31**
	16.91	20	0.59
3	**22.20**	**25.19**	**0.68**
	16.19	20	0.55
2	**21.47**	**23.49**	**0.69**
	16.15	20	0.55
1	**19.43**	**19.32**	**0.74**
	16.10	20	0.55
Panel B (2-Factor Model)			
9	**10.61**	**18.42**	**0.29**
	13.88	20	0.44
3	**13.87**	**20.96**	**0.41**
	13.76	20	0.43
2	**13.45**	**19.72**	**0.42**
	13.73	20	0.43
1	**13.55**	**11.44***	**0.73**
	11.21	20	0.30
Panel C (3-Factor Model)			
9	**2.88***	**23.62**	**-0.10**
	14.37	20	0.46
3	**9.00**	**21.00**	**0.18**
	12.91	20	0.39
2	**7.41**	**9.74***	**0.23**
	9.18	20	0.20
1	**9.49**	**6.24***	**0.69**
	8.56	20	0.17
DAX	**12.19**	**25.75**	**0.24**
	12.09	20	0.43
REX	**12.75**	**19.06**	**0.40**
	19.75	20	0.72

by showing the short sale volume of different strategies. Strategies with nine risky bonds require on average short positions with an absolute value that is between 45 and 211 times higher than the total portfolio value. The rather extreme portfolios composed of nine risky bonds lead to some ex ante improvement in performance. However, it is not very surprising that their performance is not robust out of sample. The same problem arises for the two-factor model and the three-factor model. As we see from the last column of Table 3, portfolios with nine risky assets have the highest ex ante Sharpe ratio by construction, but always have the lowest out-of-sample Sharpe ratio within their panel. Therefore, a first important result from the out-of-sample study is that one should not use too many different bonds with different maturities.

With respect to the other portfolios shown in Panel A, we see that the out-of-sample return is generally much higher than the predicted value. This result also shows up in the very high Sharpe ratios of the strategies. If a one-factor model is used for portfolio selection, the differences between strategies with one, two, or three risky bonds are minor. Basically, we find that as long as a moderate number of different bonds is used, we obtain very successful strategies with respect to the Sharpe ratio even out of sample.[9] However, the realized returns are somehow not in line with the predictions.

Panel B reports the results for the two-factor model. If the number of risky bonds is kept reasonably low, these results are very promising. For portfolios with two and three risky bonds we achieve both high Sharpe ratios and very small deviations between predicted and realized values. It is striking that the high accuracy of the predictions not only refers to the Sharpe ratio. Also the values for the expected return and volatility, that determine the Sharpe ratio, are accurately predicted by the two-factor model. If only one risky bond is used, the results are similar to the ones for the one-factor model. The reason for this finding is that the corresponding strategies are quite similar too. If one risky bond is used, most of the time one takes a short position in the risk-free bond and a long position in the risky bond.

The results for the three-factor model are provided in Panel C. The portfolio with nine risky assets shows a very poor performance like that of the one-factor and two-factor model, but the portfolios with two and three risky bonds also have relatively low Sharpe ratios. Moreover, realizations are often not in line with predictions. Note that for the three-factor model we can reject the hypothesis of no difference

[9] Such high out-of-sample Sharpe ratios are very difficult to achieve. For example, none of the strategies analyzed by DeMiguel, Garlappi and Uppal (2004) does come close to a value of 0.7.

Table 4: Short sale volume

This table shows the mean (over the 238 investment periods) short sale volumes of the different portfolio strategies. The short sale volume of a portfolio is defined as the value of all short positions as a multiple of the total portfolio value.

# Risky Bonds	1-Factor Model	2-Factor Model	3-Factor Model
9	45.44	45.05	211.23
3	6.68	11.90	63.05
2	6.10	9.83	3.87
1	4.84	1.44	0.31

between prediction and realization in three cases, compared to only one case for the one-factor model and the two-factor model. Note also that the three-factor model leads to portfolio strategies with the highest and second highest short sale volume according to Table 4. If only one risky bond is used, we obtain similar strategies and similar results as for the other model variants. Therefore, the results of Panel C indicate that a three-factor model might already be over-parameterized. It seems that rather strong restrictions both in terms of model complexity and the number of risky bonds are necessary to obtain stable out-of-sample results.

Finally, it is instructive to compare the optimized bond portfolios with some simple benchmark strategies. The lines below Panel C show the corresponding results. A first strategy uses the German stock index DAX as the single risky asset. The expected return and variance of this index are estimated historically for the rolling data window under the assumption of stationarity. Once the expected return and variance is available for the 238 periods, portfolio optimization is done in the same way as for the bond portfolios, i.e. the risky asset is combined with the risk-free asset in order to maximize the expected return for a given volatility of 20%. The second benchmark strategy uses the German bond index REX[10] as the single risky asset. Although the assumption of stationarity is clearly wrong in bond markets, such a benchmark strategy might still be interesting for diversified bond portfolios because of its simplicity. Table 3 shows that the DAX strategy promises a quite high Sharpe ratio of 0.43 but realizes only a Sharpe ratio of 0.24, a value that is much lower than the values obtained for rather simple bond portfolios based on the one-factor and two-factor models. The REX strategy predicts the best risk-return profile of all strategies (Sharpe ratio of 0.72), but cannot keep the promise out of

[10]The index consists of 30 different bonds and is available as a price index and a performance index. We used the performance index in our study.

sample. In this respect strategies based on a one-factor or two-factor model using up to three risky bonds are clearly preferable. Therefore, bond portfolio optimization adds some value compared to the benchmark strategies.

4 Conclusions

This paper has proposed to use term structure models for bond portfolio optimization. We believe that such an approach has clear merits. By choosing a particular term structure model, i.e. selecting the number of factors and the complexity of their dynamics, one can impose reasonable restrictions on the bond price dynamics. Because term structure models do consistently price bonds with different maturities, they are particularly suited for portfolio considerations.

On the theoretical side, we have shown how the inputs needed for portfolio selection — expected returns, variances and covariances of discrete holding period bond returns — can be calculated for a multi-factor Vasicek model. On the empirical side we have tested the performance of different portfolio strategies in a study based on German government bond data. By means of this study we address a variety of interesting questions. Which risk-return trade-off can we achieve with German government bonds? What are the diversification effects if we use bonds with different times to maturity? How many different bonds should be used? How should we choose the complexity of the term structure model, i.e. how many stochastic factors should it contain? How reliable are the predicted risk-return profiles out of sample?

Our results show that the predicted risk-return profiles are quite promising. For example, if three risky bonds and a risk-free asset are included in a portfolio, rather high Sharpe ratios between 0.39 (three-factor model) and 0.55 (one-factor model) result. From an ex ante perspective it is advisable to have at least as many risky bonds in the portfolio as there are factors in the term structure model. However, if we look at the out-of-sample performance, one should not use too many different bonds with different maturities. If many different bonds are considered, the corresponding strategies become rather extreme and require a large volume of short sales. These extreme portfolios perform very poorly out of sample. Another important result states that one should also not rely on a term structure model with too many factors, since the three-factor model has the worst out of sample performance and its predictions can be rejected in three out of four cases. For the one-factor model and the two-factor model, however, the out-of-sample results are more accurate. The two-factor model comes very close to the prediction and achieves an out-of-sample

Sharpe ratio of 0.41 with three risky bonds. The one-factor model does understate the expected return, however, very high out-of-sample Sharpe ratios of about 0.7 are reached. Finally, comparisons with simple benchmark strategies based on the DAX stock index and the REX bond index show that optimized portfolios perform better as long as not too many bonds and not too many factors are used.

This paper presents only a first step towards an understanding of bond portfolio selection by means of term structure models. Our goal was to demonstrate the potential of such an approach and therefore we kept the empirical setting as simple as possible. Of course, this procedure ignores a couple of practical issues that have to be addressed in the future. These issues primarily concern the implementation of portfolio strategies with securities actually traded in the market and not with "synthetical" zero bonds. At this point problems like the liquidity of certain bonds and the different tax treatment of coupon payments and capital gains come into play. The requirement of short sales is another challenging issue that might be dealt with by means of derivatives products.

We can think of several extensions of the approach presented and empirically tested in this paper. Many term structure models have been developed in the literature, which potentially differ in their power to explain prices in sample and in their stability out of sample. In principle, every term structure model could be used as the basis for portfolio selection. Obviously, there is still much to learn about the predictions that different models make about expected returns, variances and covariances and about the out-of-sample performance of the corresponding strategies. Another field for extensions is to look at different national bond markets. In this respect one could also think of portfolio strategies that invest in different markets simultaneously. Such strategies could be derived from international term structure models. Further extensions could add equity to the investment opportunity set. As first results show, our optimized bond portfolios have a rather low correlation with the stock market. Therefore, further valuable opportunities for diversification would arise.

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