Warrant Exercise and Bond Conversion in Large Trader Economies

by

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Abstract

It is well known that the sequential (premature) exercise of American-type warrants may be advantageous for large warrantholders, even in the absence of regular dividends, because using exercise proceeds to repurchase stock or to expand the firm's scale increases the riskiness of an equity share. We present an upper bound on this advantage and show that this advantage is negligible for a realistic parameter setting. This result, however, does not justify in general the simplifying restriction that warrants or convertible securities are valued as if exercised as a block. It turns out that the option to exercise only a fraction of the outstanding convertibles at the maturity date (partial exercise option) has a positive value in large trader economies. Moreover, we show that there is a gain from hoarding warrants in the presence of at least two large warrantholders.

Key words: Warrants, Convertible Bonds, Large Trader, Sequential Exercise, Partial Exercise Option

JEL: G12, G13, G32
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1 Introduction

Warrants unlike call options are issued by companies and when exercised new shares are created with the exercise proceeds increasing the firm’s assets. Because of this, there is some dilution of equity and dividend when warrants are exercised. The value accruing to one warrantholder therefore is not independent of what other warrantholders do. Under certain conditions, the premature exercise of a warrant can increase the value of the warrants that remain outstanding, because using exercise proceeds to expand the firm’s scale increases the riskiness of an equity share. Emanuel (1983), Constantinides (1984) and Constantinides and Rosenthal (1984) demonstrate the potential advantage of a sequential exercise strategy assuming a firm without senior debt. All these papers compare a sequential exercise strategy with an exercise strategy, called block exercise, where all warrantholders completely exercise their warrants simultaneously or not at all. Emanuel (1983) studies the monopolistic case, and Constantinides and Rosenthal (1984) pricetaking warrantholders. Constantinides (1984) shows that the warrant price in a competitive equilibrium is smaller than or equal to the warrant price under the block exercise constraint, if all projects of the firm have a zero net present value and the firm pays dividends and coupons. In the absence of dividend payments, Ingersoll (1987) demonstrates that a sequential exercise policy is never optimal for a pricetaker, while it can be beneficial to a monopoly warrantholder. Spatt and Sterbenz (1988) generalize this result to oligopoly warrantholders and show that there are reinvestment policies of the firm for which sequential exercise is not advantageous. Their analysis helps to justify the frequent simplifying restriction that warrants or convertible securities are valued as if exercised as a block. Articles on warrant valuation which rely on the reasonableness of block exercise include Ingersoll (1977), Brennan and Schwartz (1977, 1980), Schulz and Trautmann (1994), and Crouhy and Galai (1994).

Unfortunately, the analysis of Spatt and Sterbenz (1988) is restricted to a firm without senior debt in its capital structure. However, the existence of senior debt causes a positive value for the option to exercise only a fraction of the outstanding warrants at maturity in large trader economies. For competitive markets, Bühler and Koziol (2002) have demonstrated that allowing senior debt in the capital structure causes a partial conversion of convertible bonds to be optimal. This result is primarily driven by a wealth transfer from the stockholders to the (senior) debtholders. Both, the values of common stock and the values of the senior debt can differ for both — block and partial conversion. However, the value of the convertible bond is never below the corresponding value in the block conversion case (and above only in case of premature exercise due to dividend payments). Koziol (2002) finds similar results when examining exercise strategies for warrants in a competitive market.

This paper extends the analysis of Bühler and Koziol (2002) and Koziol (2002) to
large trader economies. We present and compare exercise strategies and the corresponding warrant values for a competitive economy (with pricetakers), a monopoly, an economy with one large trader and a competitive fringe, and an economy consisting only of two large traders. We present an upper bound on the advantage of a sequential exercise strategy. This bound decreases with an increasing interest rate and it is positive only for unrealistic parameter settings. Therefore, it turns out that from a theoretical perspective the potential advantage of sequential exercise strategies is not the main obstacle against the use of the block exercise assumption. The latter assumption, however, is questionable on the ground that it may be advantageous not to exercise all warrants if they finish in the money. It turns out that partial exercise strategies compared to block exercise strategies are beneficial for all warrantholders if and only if one or more warrantholders are non-pricetakers. The warrant values increase with the concentration of the warrant ownership distribution in the economy. Moreover, we show that there is a gain from hoarding warrants if there are at least two non-pricetaking warrantholders.

For the sake of simplicity we assume a firm which pays no regular dividends or coupon payments. In this case the partial exercise option of warrants has the same value like the partial exercise option of convertible bonds in case of European-type convertibles. Because the premature conversion of convertible bonds does not change the capital structure in contrast to the conversion at maturity, the value of the partial exercise option of American-type convertible bonds equals the value of European-type convertible bonds. Therefore, we analyse the value of the partial exercise option in case of warrants and compare it later with the case of convertible bonds.

The paper is organized as follows: In Section 2 we specify the model and define the different exercise policies. Section 3 looks at the partial exercise policies of European-type warrants and compares the warrant prices with and without the block exercise constraint. Section 4 examines the optimality of sequential exercise strategies under three different firm policies with respect to the use of the exercise proceeds of American-type warrants. Section 5 summarises the results in case of convertible bonds. Section 6 concludes the paper. All proofs are given in the Appendix.
2 Model

We consider a firm with value $V_t$ at time $t$ following a Geometric Brownian Motion. The firm is financed by issuing equity, warrants and debt and pays no regular dividends. Furthermore we assume throughout the paper that there are no taxes or transaction costs, and no arbitrage opportunities in the project market. At $t = 0$ the warrantholders know the firm value $V_0$ and the parameters of the log-normal distribution of $V_t$ at maturity $T$. The risk neutral probability measure is denoted by $Q$ with $e^{-rT} \int V_T^- dQ = V_0$.

2.1 Capital structure

At time 0 the equity is split into $N$ outstanding shares and $n$ warrants with maturity $T$ and strike price $K$. Every warrant entitles its owner to get one share of common stock when exercising the warrant at times 0 and $T$ (American-type warrant) or only at maturity (European-type warrant). The debt is a zero coupon bond with a common face value of $F$ and maturity $T_D$ with $0 < T < T_D$. At $t \in [0, T_D]$ we denote the price of one stock by $S_t$, one warrant by $W_t$ and the debt by $D_t$. The valuation of the shares of common stock, the warrants and the debt depend on the firm’s value. According to Modigliani and Miller (1958) the firm value is equal to the sum of all shares, all warrants and the debt: 

$$V_t = NS_t(V_t) + nW_t(V_t) + D_t(V_t) \quad \text{for all } t \in (0, T).$$

We denote the exercise policy of the warrantholders at time $T$ by $m \in [0, n]$. The exercise proceeds $mK$ are used to rescale the firm’s investment, so the firm value increases to $V_T = V_{T^-} + mK$, where $V_{T^-}$ denotes the last value immediately before the maturity of the warrants at time $T$. After the maturity of the warrants the firm value is

$$V_t = (N + m)S_t(V_t) + D_t(V_t) = \bar{S}_t(V_t) + D_t(V_t) \quad \text{for all } t \in [T, T_D),$$

where $\bar{S}_t$ denotes the value of the total common stock. If at time $T_D$ the firm value is less than the face value of the debt (i.e. $V_{T_D} < F$), a default occurs and the stocks get worthless, i.e. $S_{T_D}(V_{T_D}) = 0$ and $D_{T_D}(V_{T_D}) = V_{T_D}$. Otherwise the common stock equals the firm value minus the face value of the debt, so we get the equation

$$V_{T_D} = \bar{S}_{T_D}(V_{T_D}) + \min\{F; V_{T_D}\},$$

\footnote{Some results can be generalised to a firm whose value follows an arbitrary stochastic process.}

\footnote{This representation assumes that in $t = 0$ no warrant is exercised. Otherwise if $m_0$ warrants are exercised in $t = 0$ the number of stocks increases to $N + m_0$, the number of warrants decreases to $n - m_0$ and the firm value increases in dependence of the reinvestment policy of the firm.}
i.e. the value of the total common stock equals a call option on the firm value with maturity $T_D$ and strike price $F$. Since $V_t$ follows a geometric Brownian motion, $\tilde{S}(V_t)$ behaves similarly as the Black/Scholes-value of a European call option. For all $V \in \mathbb{R}^+$ we have $\Delta_T(V) = \frac{\partial}{\partial V} \tilde{S}_T(V) \in (0,1)$ and $\Gamma_T(V) = \frac{\partial^2}{\partial V^2} \tilde{S}_T(V) \geq 0$, respectively.

### 2.2 Warrantholders and their payoff functions

The set of the warrantholders is denoted by $I$ and $P$ is a measure on $I$. Every warrantholder $i \in I$ holds $n_i$ warrants with $\int_I n_i dP = n$. Furthermore, we assume that warrantholders do not own shares of common stock of the firm and that every warrantholder knows the number of warrants of each other warrantholder (complete information on the distribution of warrant ownership).

**European-type warrants**

In the case of European-type warrants the set of strategies of warrantholder $i \in I$ are all possible exercise policies $m_i \in [0,n_i]$ at time $T$. The number of warrants exercised by all warrantholders is $m = \int_I m_i dP \in [0,n]$. We denote the exercise policy of all warrantholders without $i$ by $m_{-i}$, so the exercise policy of all warrantholders equals $m = m_i P(\{i\}) + m_{-i}$. We call warrantholder $i \in I$ a non-atomic player if $P(\{i\}) = 0$. The exercise policy of a non-atomic player does not affect the exercise policy of all other warrantholders, i.e. warrantholder $i$ is a pricetaker. The payoff of warrantholder $i$ is defined as the exercise value of warrants exercised by warrantholder $i$, i.e.\(^3\)

$$\pi_i(m_i, m_{-i}, V_{T-}) = \frac{m_i}{N + m} \tilde{S}_T(V_{T-} + mK) - m_i K.$$  

As the payoff function of each pricetaking warrantholder $i$ is a function which is linear in the number of warrants exercised by himself, his payoff function is maximised at $m_i = 0$ or $m_i = n_i$. Only if 

$$\tilde{S}_T(V_{T-} + mK) - (N + m)K = 0$$  

holds, every exercise policy of $i$ maximises his payoff. Looking at equation (1), we note: If $m$ is a solution of equation (1) it is unique. If $m(V_{T-})$ describes all solutions of equation (1), so $m(V_{T-})$ is strictly increasing with respect to the firm value.

\(^3\)Clearly, we know that the exercise value of one warrant is represented by the positive part of the function $\tilde{S}_T(V_{T-} + mK)/(N + m) - K$. Defining the payoff as an exercise value assumes therefore rational warrantholders exercising warrants only if they finish in the money.
In contrast to the non-atomic player we call warrantholder \( A \in I \) with \( P(\{A\}) = 1 \) an atomic player. As the exercise policy of the atomic player influences the prices, warrantholder \( A \) is a non-pricetaker and his payoff function is defined by

\[
\pi_A(m_A, m_{-A}, V_T-) = \frac{m_A}{N + m_A + m_{-A}} \bar{S}_T(V_T^- + m_A K + m_{-A} K) - m_A K.
\]

While the payoff function of a pricetaker is linear in the number of warrants he exercises, the payoff function of a non-pricetaker is strictly concave.

**Lemma 1** When the firm uses the warrant exercise proceeds to rescale the firm’s project, and if \( \hat{m} \geq 0 \) denotes the critical number of warrants exercised at time \( T \) such that the exercise value of the warrants is zero, \( \bar{S}_T(V_T^- + \hat{m} K) - (N + \hat{m}) K = 0 \), the following two statements hold:

(a) The stock price as a function of the total number of warrants exercised is strictly decreasing:

\[
\frac{\partial}{\partial m} S_T(V_T^- + m K) < 0 \quad \text{for all} \ m \in [0, \hat{m}].
\]

(b) The payoff function of an atomic player as a function of the number of warrants exercised, \( m_A \in (0, \hat{m}_A - m_{-A}] \), where \( m_{-A} \in [0, n_{-A}] \), is strictly concave:

\[
\frac{\partial^2}{\partial m_A^2} \pi_A(m_A, m_{-A}, V_T-) < 0.
\]

The proof is given in the appendix.

**American-type warrants**

In the case of American-type warrants we assume that at time \( t = 0 \) the warrantholders have two options: either they exercise warrants or they sell warrants. \(^4\)

\(^4\)As it is well known, holders of American-type warrants have usually at every trading date three options: they can exercise, sell or hold the warrants. For the sake of tractability we do not consider the latter option and assume that all non-pricetaker exit the warrant market at time \( t = 0 \). This simplified framework avoids a time-consuming numerical analysis to calculate the current values of stocks and warrants in dependence of the market structure. Furthermore this is tantamount to consider only the warrantholders’ real wealth in the spirit of Jarrow (1992).
and selling the remaining \( n - m \) warrants to pricetakers, the payoff function of a pricetaking warrantholder \( i \in I \) is

\[
\pi_i(m_i, m_{-i}, V_0) = m_i \left( S_0(V_0, m) - K \right) + (n_i - m_i)W_0(V_0, m),
\]

where \( V_0 \) and \( m \) denote the firm value at time \( t = 0 \) and the total number of warrants exercised at time \( t = 0 \), respectively, and \( S_0(V_0, m) \) and \( W_0(V_0, m) \) are the stock price and the warrant price in \( t = 0 \), if at the warrants’ maturity date all warrantholders are pricetakers. The corresponding payoff function of a non-pricetaking warrantholder \( A \in I \) reads now as follows

\[
\pi_A(m_A, m_{-A}, V_0) = m_A \left( S_0(V_0, m_A + m_{-A}) - K \right) + (n_A - m_A)W_0(V_0, m_A + m_{-A}).
\]

(2)

So we denote the sequential exercise strategy with \( m \) (the number of warrants exercised in \( t = 0 \)), since the exercise strategy in \( t = T \) is well known by the behavior of pricetakers. Furthermore we have to make an assumption about the use of the exercise proceeds in \( t = 0 \). We distinguish between three assumptions: A rescaling of the firm’s investment, an investment in zero-bonds and the payment of an extraordinary dividend (The firm uses the exercise proceeds in \( t = T \) to rescale the firm’s investment as before).

### 2.3 Block exercise, partial exercise and sequential exercise

Stock prices rationally reflect anticipation of the number of warrants exercised and the assumed use of the exercise proceeds. We distinguish between three kinds of exercise policies:

**Definition 1** Warrantholders follow a so-called block exercise strategy if the number of warrants exercised at the maturity date is given by

\[
m = \begin{cases} 
0 & \text{for} \quad \frac{1}{N+n}S_T(V_{T^-} + nK) \in [0, K) \\
n & \text{for} \quad \frac{1}{N+n}S_T(V_{T^-} + nK) \in [K, \infty).
\end{cases}
\]

Otherwise the warrantholders follow a so-called partial exercise strategy at the maturity date, or they follow a so-called sequential exercise strategy if they exercise American-type warrants before maturity.

We model the warrantholders’ exercise behavior as a noncooperative game and consider a Nash equilibrium as an optimal exercise strategy for the warrantholders. The noncooperative game is defined by the set of warrantholders, the exercise policies as the strategies sets, and the payoff functions. While Constantinides (1984)
analyses a zero-sum game between the warrantholders and the stockholders (as passive players), our game is not zero-sum, because there is a wealth transfer from the stockholders and the warrantholders to the debtholders by the exercise of a warrant.

**Definition 2** In case of European-type warrants the exercise strategy \((m_i^*)_{i \in I}\) in time \(t = T\) is a Nash equilibrium if for every warrantholder \(i \in I\)

\[
\pi_i(m_i^*, m_{-i}^*, V_T^-) \geq \pi_i(m_i, m_{-i}^*, V_T^-) \quad \text{holds for all } m_i \in [0, n_i].
\]

In case of American-type warrants the exercise strategy \((m_i^*)_{i \in I}\) in time \(t = 0\) is a Nash equilibrium if for every warrantholder \(i \in I\)

\[
\pi_i(m_i^*, m_{-i}^*, V_0) \geq \pi_i(m_i, m_{-i}^*, V_0) \quad \text{holds for all } m_i \in [0, n_i].
\]

In a Nash equilibrium each warrantholder takes the other warrantholders’ exercise policy as given and maximises his payoff function. We show that a Nash equilibrium exists, although it may not be unique (e.g. if all warrantholders are pricetakers, the optimal exercise strategy is not unique).

### 3 Partial exercise of European-type warrants

#### 3.1 Exercise policies in a competitive economy

If every warrantholder is a pricetaker, we call this kind of market structure a *non-atomic game*. For the sake of consistency the measure of all pricetakers must be positive, e.g. \(P(I) = 1\), whereas each single warrantholder has a measure of zero. From the linearity of all warrantholders’ payoff functions we get immediately the optimal exercise policy for all warrantholders:

**Proposition 1** If all warrantholders are pricetakers, then the following exercise strategy is a Nash equilibrium:

\[
m_i^* = \begin{cases} 
0 & \text{for } V_{T^*} \in [0, V) \\
x_i^* & \text{for } V_{T^-} \in [V, V_0) \\
n_i & \text{for } V_{T^-} \in (V, \infty)
\end{cases}
\]

for all \(i \in I\), where \(V\) and \(V_0\) fulfill \(\frac{I}{N} \bar{S}_T(V) = K\) and \(\frac{1}{N+n} \bar{S}_T(V + nK) = K\), respectively, and \(x^* = \int_I x_i^* dP\) solves the equation

\[
\frac{1}{N + x^*} \bar{S}_T \left( V_{T^-} + x^* K \right) = K.
\]
Figure 1: **Stock price in a non-atomic game**

The figure shows the stock price as a function of the firm value at time $T$ in a non-atomic game (dashed line) and under the block exercise constraint (dotted line). We assume the parameters $r = 5\%$, $\sigma = 0.25$, $F = 80,000$, $T_D - T = 4$, $N = 100$, $n = 100$ and $K = 100$. The critical firm values are $V = 60,330.53$ and $\overline{V} = 66,258.47$.

The optimal exercise strategy in proposition 1 is not unique: Although equation (3) has a unique solution $m^*$, any exercise strategy $(m^*_i)_{i \in I}$ with $m^* = \int_I m^*_i dP$ is a Nash equilibrium.

According to this proposition, for $V_T^- \leq V$ and $V_T^- \geq \overline{V}$ the optimal partial exercise policy equals the block exercise strategy. Only if $V_T^- \in (V, \overline{V})$ the warrantholders exercise so many warrants that the stock price equals the strike price, whereas the stock price under the block exercise strategy is higher than the strike price (see figure 1). Nevertheless, the warrant price equals zero in both cases: Under the optimal partial exercise strategy the stock price equals the strike price, so that the warrantholders make no profit by exercising warrants, and under the block exercise strategy no warrant is exercised.
3.2 Exercise policies in large trader economies

Exercise policies when one non-pricetaker exists

First we look at a market structure with exactly one large warrantholder $A \in I$. We call this market structure a one-atomic game. Again, let $P$ be the measure on the set of warrantholders with $P\{\{A\}\} = 1$ and $P\{\{i\}\} = 0$ for all $i \in I, i \neq A$. Non-pricetaker $A$ owns $n_A \in (0, n]$ warrants. Please note that the monopoly is a special case of a one-atomic game with $n_A = n$ and $n_{-A} = 0$.

The total number of warrants exercised by all pricetakers is denoted by $m^*_A = \int_{\{A\}} m_i^* dP$.

**Proposition 2**  (a) In the presence of one non-pricetaker the following strategy is a Nash equilibrium:

\[
(m_A^*, m_{-A}^*) = \begin{cases} 
(0, 0) & \text{for } V_{T_-} \in [0, \bar{V}) \\
(0, x_{-A}^*) & \text{for } V_{T_-} \in [\bar{V}, V_A) \\
(x_A^*, n_{-A}) & \text{for } V_{T_-} \in [V_A, \bar{V}_A) \\
(n_A, n_{-A}) & \text{for } V_{T_-} \in (\bar{V}_A, \infty) 
\end{cases}
\]

where $\bar{V}$ solves $\frac{1}{N} S_T(\bar{V}) = K$, $\bar{V}_A$ solves $\frac{1}{N + n_{-A}} \bar{S}_T(\bar{V}_A + n_{-A}K) = K$ and $\bar{V}_A$ solves $\frac{N + n_{-A}}{N + n_{-A} + x_A^*} \bar{S}_T(\bar{V}_A + n_{-A}K + x_A^*K) + \frac{n_{-A}}{N + n} \Delta_T(\bar{V}_A + nK) = K$. The exercise policies $x_{-A}^*$, $x_A^*$ are the solutions of

\[
\frac{1}{N + x_{-A}^*} \bar{S}_T(V_{T_-} + x_{-A}^*K) = K
\]

\[
\frac{N + n_{-A}}{N + n_{-A} + x_A^*} \bar{S}_T(V_{T_-} + n_{-A}K + x_A^*K) + \frac{x_A^*}{N + n_{-A} + x_A^*} \Delta_T(V_{T_-} + n_{-A}K + x_A^*K) = K
\]

respectively.

(b) Let $m^*$ be the total number of warrants exercised in a competitive market (non-atomic game). Then for all $V_{T_-} \in (\bar{V}_A, \bar{V}_A)$ we have

\[
m_A^* + m_{-A}^* < m^*.
\]

The proof is given in the appendix.
**Figure 2: Stock price in a one-atomic game**

The figure shows the stock price as a function of the firm value at time $T$ in a one-atomic game. We assume the parameters $r = 5\%$, $\sigma = 0.25$, $F = 80,000$, $T_D - T = 4$, $N = 100$, $n = 100$, $n_A = 40$ and $K = 100$. The critical firm values are $V_0 = 60,330.53$, $V_A = 63,225.29$ and $V_A = 69,372.27$.

In contrast to a competitive market warrantholders exercise less warrants in the presence of a non-pricetaker while according to lemma 1 the stock price is higher in the presence of a non-pricetaker.

If the stock price is above the strike price, pricetakers exercise all their warrants so that their exercise policy is known to all non-pricetakers. If the stock price is below or equal the strike price, the non-pricetakers exercise no warrants. If the firm value $V_T – V$ is not below $V$, the pricetakers exercise so many warrants that the stock price equals the strike price.

**Exercise policies when two non-pricetakers exist**

We now assume a market structure with two non-pricetaking warrantholders without a competitive fringe. We call this a *two-atomic game*. The two non-
pricetakers $b, B$ own $n_b$ and $n_B$ warrants with $n_b + n_B = n$ where $n_b \leq n_B$. The optimal exercise policies are given by

**Proposition 3**  
(a) In the presence of two non-pricetakers, the following strategy is a Nash equilibrium:

\[
(m_b^*, m_B^*) = \begin{cases} 
(0, 0) & \text{for } V_T^- \in [0, V] \\
(x^*, x^*) & \text{for } V_T^- \in [V, V_b] \\
(n_b, x_B^*) & \text{for } V_T^- \in [V_b, V_B] \\
(n_b, n_B) & \text{for } V_T^- \in [V_B, \infty) 
\end{cases}
\]

where $V$ solves $\frac{1}{N} \bar{S}_T(V) = K$, $V_b$ solves $\frac{N+n_b}{(N+2n_b)} \bar{S}_T(V_b + 2n_b K) + \frac{m_b}{N+2n_b} \Delta_T(V_b + 2n_b K) = K$ and $V_B$ solves $\frac{N+n}{(N+n_A)} \bar{S}_T(V_B + nK) + \frac{n}{N+n} \Delta_T(V_B + nK) = K$ and $x^*$ and $x_B^*$ solve the equations

\[
\begin{align*}
\frac{N+x^*}{(N+2x^*)^2} \bar{S}_T \left( V_T^- + 2x^* K \right) + \frac{x^*}{N+2x^*} K \Delta_T \left( V_T^- + 2x^* K \right) &= K \\
\frac{N+n_b}{(N+n_b+x_B^*)^2} \bar{S}_T \left( V_T^- + n_b K + x_B^* K \right) + \frac{x_B^*}{N+n_b+x_B^*} K \Delta_T \left( V_T^- + n_b K + x_B^* K \right) &= K,
\end{align*}
\]

respectively.

(b) Let $(m_A^*, m_{-A}^*)$ be the optimal exercise strategy in the presence of one non-pricetaker (one-atomic game) and $n_{-A} = n_b$. For all $V_T^- \in (V, V_b)$ we have

\[
\begin{align*}
m_b^* &= m_B^* < m_{-A}^* \\
m_b^* &= m_B^* > m_A^* \\
m_b^* + m_B^* &= m_A^* + m_{-A}^*.
\end{align*}
\]

(c) Let $n_A = n$ and $m_A^*$ be the optimal exercise policy in a monopoly. For all $V_T^- \in (V, V_A)$ we have

\[m_A^* < m_b^* + m_B^*.
\]

The proof is given in the appendix. Proposition 3 can be generalised to a market structure with arbitrary many non-pricetakers.
Surprisingly, the warrantholder B exercises as much warrants as warrantholder b, although he owns more warrants if $V_T - \in [V, V_b)$. This is due to the fact that the payoff function of a non-pricetaker is strictly concave and does not depend on the total number of warrants he holds. So if an optimal exercise policy is a inner solution for one warrantholder, the same exercise policy is optimal for another warrantholder even if he holds a different number of warrants.

According to relation (7), two non-pricetakers exercise less warrants than one non-pricetaker plus a competitive fringe if the latter holds as much warrants as one of the two non-pricetakers. So the stock price and the warrant price are higher. On the other hand, if only one monopoly warrantholder exists, he exercises less warrants than in the presence of two warrantholders. So the stock price and the warrant price in the monopoly is higher than in the presence of two non-pricetakers.

Figure 3: **Stock price in a two-atomic game**

The figure shows the stock price as a function of the firm value at time $T$ in a two-atomic game. We assume the parameters $r = 5\%$, $\sigma = 0.25$, $F = 80,000$, $T_D - T = 4$, $N = 100$, $n = 100$, $n_b = 40$ and $K = 100$. The critical firm values are $V = 60,330.53$, $V_b = 67,581.81$ and $V_B = 69,372.27$. 
3.3 Comparison of exercise policies and exercise values

Figure 4 illustrates the differences of optimal exercise policies and their corresponding exercise values due to four different market structures. According to the figure in panel A, 100% of the outstanding warrants will be exercised in a competitive market (non-atomic game) at the critical firm value $V = 66,258.47$ (the same percentage as with the block exercise strategy) while only a percentage between 40 and 66 will be exercised in large trader economies for the same firm value. The figure in panel B confirms, first of all, the well-known fact that there is no difference between warrant values in a competitive economy and a block exercise-constrained economy although the optimal exercise strategy in a competitive market deviates from the block exercise strategy. Moreover, this figure demonstrates that an increasing concentration of the warrant ownership distribution may lead to substantially higher exercise values of the outstanding warrants.

3.4 Gains from hoarding warrants

We now approach the question, how a warrantholder can arise a monopoly position (e.g. posed by Ingersoll, 1987) or what is the warrantholder’s gain of hoarding warrants (posed by, e.g., Spatt and Sterbenz, 1988). A non-pricetaker (or a potential non-pricetaker) cannot buy a sufficient number of warrants from pricetakers. An offer of the non-pricetaker to buy a certain number of warrants is always rejected by the pricetakers for the following reason: the offered price is smaller than the present value of a warrant if the offer when accepted does not lead to a negative net-present-value for the non-pricetaker (recall that a non-pricetaker would only exercise a fraction of the warrants he could buy). This is due to the fact that the pricetaker’s decision has no impact on the stock price and therefore on the warrants’ exercise value. So every pricetaker wants to be a free rider, and in sum no pricetaker sells his warrants to the non-pricetaker. Also no non-pricetaker will sell his warrants to pricetakers, because the present value of a warrant will decrease if he does.

In the presence of two non-pricetakers, one non-pricetaker will always sell his warrants to the other, as they will both profit from the additional value due to the merger of their position (see statement (c) in proposition 3). The following example illustrates this effect for the case of three non-pricetakers; it can be generalized to arbitrary many non-pricetakers:
Figure 4: Exercise policies and exercise values

The figure shows the exercise rate of all players as a function of the firm value and the exercise value of a warrant as a function of the firm value at time $T$. We assume the parameters $r = 5\%$, $\sigma = 0.25$, $F = 80,000$, $T_D - T = 4$, $N = 100$, $n = 100$, $n_A = n_b = 40$ and $K = 100$. The critical firm values are $\underline{V} = 60,330.53$ and $\overline{V} = 66,258.47$.

Panel A: Optimal exercise policies

Panel B: Exercise values of European-type warrants
Example 1  We assume an interest rate of 5 % and a firm with the following parameters: The volatility of the asset return is 25 %, the debt has a face value of 65,000 and a maturity date in 4 years. The firm has issued 100 stocks and 100 warrants with a strike price of 100. The warrantholders A, B, C hold each 20 warrants while the remaining warrants are held by pricetakers (the pricetakers’ payoff are considered as one entity).

(a) Let us assume a firm value of $V_{T-} = 67,000$. Without any trade warrantholders’ payoffs are as follows:

<table>
<thead>
<tr>
<th></th>
<th>pricetaker</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>exercise policy</td>
<td>40.00</td>
<td>19.12</td>
<td>19.12</td>
<td>19.12</td>
</tr>
<tr>
<td>of 40</td>
<td>of 20</td>
<td>of 20</td>
<td>of 20</td>
<td></td>
</tr>
<tr>
<td>stock price</td>
<td>103.06</td>
<td>103.06</td>
<td>103.06</td>
<td>103.06</td>
</tr>
<tr>
<td>payoff</td>
<td>122.40</td>
<td>58.51</td>
<td>58.51</td>
<td>58.51</td>
</tr>
</tbody>
</table>

Warrantholder A offers B to buy his warrants for a price between 58.51 and 72.17, since B will sell his warrants only if the price is higher than his payoff (58.51), and A will only buy warrants if his new payoff (see the next table) minus the price is higher than his original payoff ($130.68 - 72.17 = 58.51$). As warrantholder B will not refuse this offer, all warrantholders profit from this trade:

<table>
<thead>
<tr>
<th></th>
<th>pricetaker</th>
<th>A</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>exercise policy</td>
<td>40.00</td>
<td>26.61</td>
<td>20.00</td>
</tr>
<tr>
<td>of 40</td>
<td>of 20</td>
<td>of 20</td>
<td></td>
</tr>
<tr>
<td>stock price</td>
<td>104.91</td>
<td>104.91</td>
<td>104.91</td>
</tr>
<tr>
<td>payoff</td>
<td>196.40</td>
<td>130.68</td>
<td>98.20</td>
</tr>
</tbody>
</table>

In the next step warrantholder A buys also the warrants of C.

(b) Let us now assume a firm value of $V_{T-} = 65,000$. Without any trade the payoffs of the warrantholders are:

<table>
<thead>
<tr>
<th></th>
<th>pricetaker</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>exercise policy</td>
<td>40.00</td>
<td>7.83</td>
<td>7.83</td>
<td>7.83</td>
</tr>
<tr>
<td>of 40</td>
<td>of 20</td>
<td>of 20</td>
<td>of 20</td>
<td></td>
</tr>
<tr>
<td>stock price</td>
<td>101.69</td>
<td>101.69</td>
<td>101.69</td>
<td>101.69</td>
</tr>
<tr>
<td>payoff</td>
<td>67.60</td>
<td>13.23</td>
<td>13.23</td>
<td>13.23</td>
</tr>
</tbody>
</table>

Warrantholder B will not sell his warrants to A, as after such a trade the payoff of A is less than the expected payoff of A and B before the trade:
But in the initial situation warrantholder C could give up 12 warrants to A without any remuneration since otherwise this 12 warrants would expire worthless:

<table>
<thead>
<tr>
<th></th>
<th>pricetaker</th>
<th>A</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>exercise policy</td>
<td>40.00</td>
<td>10.31</td>
<td>10.31</td>
</tr>
<tr>
<td>of 40</td>
<td>of 40</td>
<td>of 20</td>
<td></td>
</tr>
<tr>
<td>stock price</td>
<td>102.33</td>
<td>102.33</td>
<td>102.33</td>
</tr>
<tr>
<td>payoff</td>
<td>93.20</td>
<td>24.02</td>
<td>24.02</td>
</tr>
</tbody>
</table>

Now warrantholder A can buy the warrants of B for a price between 13.23 and 16.53, because C acts like a pricetaker before and after the trade: He exercises (nearly) all his warrants. So A can maximize his payoffs without a wealth transfer to another warrantholder (C and the pricetakers are shareholders).

<table>
<thead>
<tr>
<th></th>
<th>pricetaker</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>exercise policy</td>
<td>40.00</td>
<td>7.83</td>
<td>7.83</td>
<td>7.83</td>
</tr>
<tr>
<td>of 40</td>
<td>of 32</td>
<td>of 20</td>
<td>of 8</td>
<td></td>
</tr>
<tr>
<td>stock price</td>
<td>101.69</td>
<td>101.69</td>
<td>101.69</td>
<td>101.69</td>
</tr>
<tr>
<td>payoff</td>
<td>67.60</td>
<td>13.23</td>
<td>13.23</td>
<td>13.23</td>
</tr>
</tbody>
</table>

In the last step warrantholder A also buys the remaining warrants of C for a price between 20.96 and 25.07:

<table>
<thead>
<tr>
<th></th>
<th>pricetaker</th>
<th>A</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>exercise policy</td>
<td>40.00</td>
<td>11.36</td>
<td>8.00</td>
</tr>
<tr>
<td>of 40</td>
<td>of 52</td>
<td>of 8</td>
<td></td>
</tr>
<tr>
<td>stock price</td>
<td>102.62</td>
<td>102.62</td>
<td>102.62</td>
</tr>
<tr>
<td>payoff</td>
<td>104.80</td>
<td>29.76</td>
<td>20.96</td>
</tr>
</tbody>
</table>

In sum, this example shows that the warrants of non-pricetakers will be finally (i.e. at the warrants’ maturity) held by just one non-pricetaker. So, in an informationally efficient market the current warrant price will reflect the fact that there is only one large warrantholder just before maturity. Therefore, the warrant price for pricetakers is unique under all initial market structures, as long as the non-pricetakers do not trade with pricetakers. The condition that all non-pricetakers eventually sell

---

5Recall that we have assumed that all warrantholders know the number of warrants held by non-pricetakers.
their warrants to one large warrantholder just before maturity $T$ must only hold for the range of firm values where a partial exercise is beneficial (compared to the block exercise) for the warrantholders.

If we assume that warrants are indivisible (and if the number of the warrants is finite), every warrantholder is a non-pricetaker and sells his warrants to the potential monopolist. Then all warrantholders behave as if there is a monopoly market in $T$. Surprisingly, the warrant price depends on whether a warrantholder who holds only a few warrants has a positive measure or not. The reason is that only a pricetaker who has per definition a measure of zero, can be a free rider. In sum, it turns out that contrary to Spatt and Sterbenz (1988), the warrantholders have a gain from hoarding warrants in a large trader economy if the firm has issued additional debt. More precisely, all warrantholders have a gain if one warrantholder hoards warrants. If warrants are priced under the assumption of one large warrantholder, a pricetaking warrantholder can hedge his portfolio with shares of the common stock and risk-free bonds, as the warrant can be duplicated by these securities.

3.5 Price impact of the block exercise constraint

As the warrant price in a competitive market equals the warrant price under the block exercise constraint, we will compare this price to the warrant price in the presence of a monopoly warrantholder to see the maximum price impact. Figure 5 illustrates the absolute and the relative price differences in example.

Since at maturity the prices differ only if $V_T^- \in (\underline{V}, \overline{V})$, the price difference decreases as the probability $Q(\{V_T^- \in (\underline{V}, \overline{V})\})$ decreases. This is shown in figure 5. On the other hand a warrant price in the presence of a non-pricetaker is strictly positive at time $T$ if $V_T^- \geq \underline{V}$, whereas a warrant price under the block exercise constraint is strictly positive, if $V_T^- \geq \overline{V}$. If the warrant is out-of-the-money and the probability $Q(\{V_T^- \geq \underline{V}\})$ is much higher than $Q(\{V_T^- \geq \overline{V}\})$, the warrant price in the presence of a non-pricetaker is greater than under the block exercise constraint.

Finally, we will take a look at the volatility of the equity. A main assumption of our model is that the asset return volatility $\sigma_V = \sigma$ is constant. Obviously the shareholders take a greater part of the risk of the firm than the buyer of the debt, so the volatility of the equity $\sigma_S$ must be higher. On the other hand a part of the risk held by the shareholders is shifted to the warrantholders by issuing warrants.
Figure 5: **Monopoly versus competitive market**

The figures show the absolute and relative differences between warrant prices in a competitive market and a monopoly. We assume the parameters $r = 5\%$, $\sigma = 0.25$, $F = 80,000$, $T_D - T = 4$, $N = 100$, $n = 100$ and $K = 100$. The critical firm value is $V_T = 60,330.53$.

Panel A: Absolute price difference ($W^{\text{mono}} - W^{\text{comp}}$)

Panel B: Relative price difference ($W^{\text{comp}}/W^{\text{mono}}$)
Figure 6: **Stock volatility and market structures**

The figures show stock volatilities in a competitive market and in a monopoly. We assume the parameters $r = 5\%$, $\sigma = 0.25$, $F = 80,000$, $T_D - T = 4$, $N = 100$, $n = 100$ and $K = 100$.

**Panel A: Competitive Market**

**Panel B: Monopoly**
The nonstationarity of stock volatility is reflected by the well-known relationship

\[ \sigma_S = \sigma_V \frac{\partial S_t(V_t)}{\partial V_t} \frac{V_t}{S_t(V_t)}. \]

This is a standard result in option pricing theory where the stock’s elasticity gives the percentage change in the stock’s value for a percentage change in the firm’s value. At maturity \( T \) of the warrants we get

\[ \sigma_S = \sigma_V \frac{\partial S_T(V_T^- + m^*(V_T^-)K)}{\partial V_T^-} \frac{V_T^-}{S_T(V_T^- + m^*(V_T^-)K)}, \]

where \( m^*(V_T^-) \) describes the optimal total number of warrants exercised by all warrantholders. At time \( t < T \) stock volatility is given by

\[ \sigma_S = \sigma_V \left( \int_0^\infty \frac{\partial S_T(V_T^- + m^*(V_T^-)K)}{\partial V_T^-} dQ \right) \frac{V_t}{\int_0^\infty S_T(V_T^- + m^*(V_T^-)K)dQ}. \]

Figure 6 shows that the stock volatility goes down immediately before maturity if the stock price is near the strike price (in a competitive market even to zero), and recovers if the stock price is outside the proximity of the strike price. This is similar to the results of Schulz and Trautmann (1994). Figure 6 demonstrates that the change of the volatility at maturity is less dramatic in a market monopoly compared to a competitive market. ⁶

4 Sequential exercise of American-type warrants

Emanuel (1983) and Constantinides (1984) emphasize the potential advantage of sequential exercise strategies by holders of warrants and convertible bonds, even absent regular dividend payments. The examples developed in the literature illustrate the potential optimality of sequential exercise based upon differing assumptions about the firm’s policy regarding the use of warrant exercise proceeds and about the distribution of warrant ownership. All these examples disregard straight debt in the capital structure of the firm which is, however, considered in the following analysis.

⁶The stock price at maturity as a function of the firm value, \( S_T(V_T^-) \), does also behave differently. In contrast to the situation in a block exercise constrained market (see e.g. Crouhy and Galai (1994) or figure 1) the stock price at maturity \( S_T(V_T^-) \) does not jump when varying the firm value in any market structure.
4.1 Rescaling the firm’s investment

Ingersoll (1987) and Spatt and Sterbenz (1988) demonstrate that a sequential exercise can be optimal, even if the firm does not pay a regular dividend, if the proceeds from exercising the warrants prematurely are used to rescale the firm’s investment. Without additional debt a wealth transfer from the stockholders to the warrantholders is possible when exercising warrants sequentially. The following analysis shows that in a model with additional debt the situation is more complex: The value of the debt can both increase and decrease due to the exercise of a warrant.

\[
\bar{S}_T(V_T, m) = \bar{S}_T \left( \frac{V_0 + mK}{V_0} V_T \right)
\]

\[
\bar{S}_T(V_T, m) = \bar{S}_T \left( \frac{V_0 + mK}{V_0} V_T + (n - m)K \right)
\]

denote the total value of common stock when no warrant and all warrants are exercised, respectively. We write \( \Delta_T \) and \( \bar{\Delta}_T \) for the partial derivative of \( \bar{S}_T \) and \( \bar{S}_T \) with respect to the firm value, respectively. Assuming that warrants not exercised at \( t = 0 \) are sold to pricetakers, we get according to proposition 1 two critical firm values \( V_T(m) \) and \( \bar{V}_T(m) \) with

\[
\bar{S}_T(V_T(m), m) = (N + m)K \quad \text{and} \quad \bar{S}_T(\bar{V}_T(m), m) = (N + n)K.
\]

If the firm value \( V_T \) is less than \( V_T(m) \), no warrant is exercised and the stock price is less than the strike price, whereas if \( V_T \geq \bar{V}_T(m) \) all warrants are exercised in a competitive market. So the stock price can be written as

\[
S_T(V_T, m) = \begin{cases} 
\frac{1}{N + m} \bar{S}_T(V_T, m) & \text{for } V_T \in (0, V_T(m)) \\
K & \text{for } V_T \in [V_T(m), \bar{V}_T(m)) \\
\frac{1}{N + n} \bar{S}_T(V_T, m) & \text{for } V_T \in [\bar{V}_T(m), \infty).
\end{cases}
\]

The stock price and warrant price in \( t = 0 \) are given by

\[
S_0(V_0, m) = e^{-rT} \int_{\mathbb{R}_+} S_T(V_T, m) dQ \quad \text{(8)}
\]

\[
W_0(V_0, m) = e^{-rT} \int_{\tilde{V}_T(m)}^{\infty} \left( S_T(V_T, m) - K \right) dQ. \quad \text{(9)}
\]

As it is well known, a rational pricetaker will never exercise a warrant before maturity in the absence of dividend payments. Now we consider a non-pricetaking
warrantholder $A$ holding $n_A \in (0, n]$ warrants. The payoff function of warrantholder $A$ is now defined by the equations (2), (8) and (9). The following example illustrates the wealth transfer from the debtholder to the stockholders and warrantholders.

**Example 2** We assume a current firm value of $V_0 = 65,000$, an asset return volatility of 30% and an interest rate of zero. Furthermore, we assume that the firm has issued a zero coupon bond with a face value of 15,000 and a maturity of 5.5 years. The firm has also issued 50 stocks and 50 warrants with a strike price of $K = 250$ and maturity $T = 0.75$ years. In this example we assume that the warrants are indivisible.

- We simplify the non-atomic game in the following way: We assume 50 warrantholders each holding one warrant. The optimal exercise policy for every warrantholder is not to exercise his warrant.
- In the one-atomic game each of 25 warrantholders hold one warrant and the remaining warrants are held by one non-pricetaking warrantholder. While the optimal strategy of pricetakers is not to exercise their warrants, the optimal exercise policy of the non-pricetaker is to exercise 23 of his 25 warrants.
- In the two-atomic game two warrantholders hold each 25 warrants. The optimal exercise strategy is to exercise each 17 warrants, totally 34 warrants.
- A warrantholder with monopoly power will exercise all of his 50 warrants.

<table>
<thead>
<tr>
<th></th>
<th>Non-atomic game</th>
<th>One-atomic game</th>
<th>Two-atomic game</th>
<th>Monopoly</th>
</tr>
</thead>
<tbody>
<tr>
<td>stock price</td>
<td>625.63</td>
<td>625.68</td>
<td>625.70</td>
<td>625.76</td>
</tr>
<tr>
<td>warrant price</td>
<td>375.64</td>
<td>375.74</td>
<td>375.81</td>
<td>375.96</td>
</tr>
<tr>
<td>debt value</td>
<td>14,936.41</td>
<td>14,932.54</td>
<td>14,930.05</td>
<td>14,924.49</td>
</tr>
</tbody>
</table>

In the foregoing example like in the examples of the related literature (e.g., Ingersoll (1987) and Spatt and Sterbenz (1988, proof of theorem 3)) the assumed interest rate of $r = 0$ was mainly responsible for the optimality of a sequential exercise strategy. Furthermore, we see in the example only a low price change of the warrants when changing the warrant ownership distribution. So we have two questions: How likely is a sequential exercise and what is the impact on the warrants’ price? Both questions are answered by the next proposition.
Proposition 4 In the absence of regular dividends payments and when using the exercise proceeds to rescale the firm’s investment, then the following two statements hold:

(a) For all (sequential) exercise strategies \((m_i)_{i \in I}\) the marginal payoff of the non-pricetaking warrantholder \(A\) is bounded by

\[
\frac{\partial}{\partial m_A} \pi_A(m_A, m_{-A}, V_0) < K \left( \frac{n_A}{N + n} \frac{W_{0}^{am}(V_0)}{V_0} - (1 - e^{-rT}) \right),
\]

where \(W_{0}^{am}\) is an at-the-money warrant on the firm value with maturity \(T\). For pricetaking warrantholders the marginal payoff is always negative.

(b) The warrant price is an increasing and convex function with respect to the number of warrants exercised. For an optimal sequential exercise policy \((m^*_i)_{i \in I}\) and the non-pricetaker \(A \in I\) with \(m^*_A > 0\) and \(\frac{\partial}{\partial m_A} \pi_A(m^*_A, m_{-A}, V_0) = 0\) in \(t = 0\) a lower and an upper bound of the partial derivative is given by

\[
\frac{1}{n_A} K (1 - e^{-rT}) \leq \frac{\partial}{\partial m} W_0(V_0, m^*_A) \leq \frac{1}{n_A} K (1 - e^{-rT} + Q^{om}),
\]

where \(Q^{om} \equiv Q(\{V_T - \leq V_T(m^*)\})\) is the risk-neutral probability that in \(T\) no warrant is exercised.

The proof is given in the appendix.

The upper bound in statement (a) of proposition 4 does not depend on the exercise policy, the firm value \(V_0\) and the debt characteristics. If the interest rate \(r\) increases, the upper bound decreases. \(^7\) If the interest rate is sufficiently high, the marginal payoff for warrantholder \(A\) is negative and it is therefore not optimal to exercise warrants. This upper bound represents a reasonable tradeoff between the sharpness of the bound and the simplicity of its calculation. Nonetheless this bound is good enough to show that in many parameter settings a sequential exercise policy is not optimal (see example 3).

Statement (b) in proposition 4 presents an upper bound and a lower bound on the warrant price’s sensitivity with respect to the number of warrants exercised. The lower bound of this sensitivity decreases with decreasing interest rates, since \(\lim_{r \to 0} K (1 - e^{-rT}) = 0\). A sequential exercise policy can be optimal if the interest rate is low, but the price impact of a sequential exercise to the warrants is high if

\(^7\) The derivative of the upper bound with respect to the interest rate is given by

\[
\frac{\partial}{\partial r} \left[ K \left( \frac{n_A}{N + n} \frac{W_{0}^{am}(V_0)}{V_0} - (1 - e^{-rT}) \right) \right] = Te^{-rT} K \left( \frac{n_A}{N + n} \Phi(d_2) - 1 \right) < 0.
\]
and only if the interest rate is high. So either a sequential exercise is not optimal to the warrantholders or the price impact to the warrants is negligible.  

**Example 3** In this example we demonstrate (a) that a sequential exercise policy is not optimal for realistic parameters, and (b) that if a sequential exercise policy is optimal the price impact is not significant.

(a) We assume that a firm has issued 50 stocks and 50 warrants with a strike price of 250 and a maturity in 1 year. By the realistic parameters of an interest rate of 3 % and a volatility of the firm value of 25 % a non-pricetaking warrantholder $A$ who holds $n_A = 25$ warrants does not sequentially exercise warrants (by an arbitrary firm value $V_0$ and independent of the debt of the firm), as the upper bound of the marginal payoff is always negative:

$$\frac{\partial}{\partial m_A} \pi_A(\cdot, \cdot, V_0) < 7.093 - 7.389 = -0.296.$$  

By this parameters no pricetaker and no non-pricetaker with less than 25 warrants will exercise a warrant.

(b) Now we assume that the firm value is 63,000 in $t = 0$, the volatility of the firm value is 40 % and the interest rate is 1 %. Further we assume that the firm has issued a zero coupon bond with a face value of 15,000 and a maturity date in 7 years. The optimal exercise policy of a monopolistic warrantholder is to exercise $m^* = 4$ warrants. The price change of the warrants by increasing the number of warrants exercised from 3 to 4 equals

$$W_0(V_0, 4) - W_0(V_0, 3) = 373.64 - 373.57 = 0.07.$$  

The lower bound of the partial derivative of the warrant price under the optimal exercise policy is given by

$$\frac{1}{n_A} K \left(1 - e^{-rT}\right) = 0.0498$$

and with $U_T(m^*) = 24,182.43$ the upper bound is given by

$$\frac{1}{n_A} K \left(1 - e^{-rT} + Q^{em}\right) = 0.1127.$$  

So the absolute difference from the warrant price under the optimal sequential exercise strategy to the warrant price under the block exercise constraint is less than 0.4508. This is 0.12% of $W_0(V_0, m^*)$.

---

8Clearly, if the firm pays a regular dividend, a sequential exercise may be beneficial to the warrantholders. See Koziol (2002).
4.2 Investment in zero-bonds

In this section we assume that the firm uses the proceeds of any sequential exercise to buy zero-coupon bonds with maturity $T$. If $m \in [0,n]$ warrants are exercised in $t = 0$, the strike price $mK$ is invested in zero-coupon bonds. In $T$ the zero-coupon bonds $e^{rT}mK$ and the proceeds of further warrant exercises are reinvested in the firm investment. Obviously no warrantholder profits from the warrant exercise in $t = 0$, so we can treat this section very shortly.

Proposition 5 In the absence of regular dividend payments and if the firm invests the exercise proceeds in zero-bonds, then the payoff functions for all warrantholders (pricetakers or non-pricetakers) are decreasing with respect to the number of warrants exercised by themselves. For all warrantholder the optimal exercise policy is to exercise no warrant prematurely.

The proof is given in the appendix.

4.3 Extraordinary dividend payments

The third case assumes that the firm uses the proceeds of warrant exercises in $t = 0$ to pay an extraordinary dividend to all shareholders while the firm reinvests the proceeds of warrant exercises in $T$ in their regular investments. So if $m \in [0,n]$ warrants are exercised in $t = 0$, every shareholder gets a dividend of $mK/(N + m)$. The dividend payment has the impact that the proceeds of a premature exercise are distributed among the shareholder and warrantholder, whereas the proceeds of an exercise at maturity are distributed among the debtholders, stockholders and warrantholders.

The payoff function of player $A \in I$ with exercise rate $m_A \in [0,n_A]$ is given by

$$
\pi_A(m_A, m_{-A}, V_0) = m_A \frac{mK}{N + m} - m_A K \left(1 - e^{-rT}\right) + e^{-rT} \int_0^{\nabla_T(m)} \left( m_A \frac{S_T(V_{T-})}{N + m} - m_A K \right) dQ + e^{-rT} \int_{\nabla_T(m)}^{\infty} \left( \frac{n_A}{N + n} \bar{S}_T(V_{T-} + (n - m)K) - n_A K \right) dQ,
$$

\footnote{see e.g. Spatt and Sterbenz (1988)}
where $V_T(m)$ and $\bar{V}_T(m)$ solve the equations
$$\bar{S}_T(V_T(m)) = (N + m)K \quad \text{and} \quad \bar{S}_T(\bar{V}_T(m) + (n - m)K) = (N + n)K.$$ 

The partial derivative of the payoff function of a pricetaker with respect to the number of warrants exercised by himself is
$$\pi'(m) \equiv \frac{\partial}{\partial m_i} \pi_i(m_i, m_{-i}, V_0) \quad (11)$$

$$= e^{-rT} \int_0^{\bar{V}_T(m)} \left( \frac{1}{N + m} \bar{S}_T(V_{T-}) - K \right) dQ + e^{-rT}K - \frac{N}{N + m}K.$$ 

Since $\pi'(0) < 0$, it is optimal to hold all warrants until maturity in a competitive market. Nonetheless, there can be other equilibria. The following strategy
$$m^*_i = \begin{cases} 
0 & \text{if} \quad \pi'(0) < 0 \\
x^*_i & \text{if} \quad \pi'\left(\int x^*_i dP\right) = 0 \\
n_i & \text{if} \quad \pi'(n) \geq 0 
\end{cases}$$

is a Nash equilibrium for all $i \in I$, as the payoff function of a pricetaker is a linear function in the number of warrants exercised by the pricetaker.

If $\pi'(n) \geq 0$, the optimal strategy is $m^* = n$, even if $\pi_i(n_i, n_{-i}, V_0) < \pi_i(0, 0, V_0)$. As all pricetakers believe that all other warrantholders would exercise their warrants, they do best to exercise their own warrants to benefit from the extraordinary dividend. This is a “panic equilibrium”, where all warrantholders are worse off compared to other equilibria. The total level of early exercise $m^* = 0$ can be a “panic equilibrium” too, if $\pi_i(n_i, n_{-i}, V_0) > \pi_i(0, 0, V_0)$ (see example 4).

In contrast to a result of Spatt and Sterbenz (1988, Theorem 4), there are parameters such that the optimal exercise policy of a monopoly warrantholder is to exercise all his warrants. 

**Proposition 6** If the firm pays (in the absence of regular dividend payments) an extraordinary dividend to the equityholder with the proceeds of any warrant exercise, then for a non-pricetaker the optimal exercise strategy is either to hold all his warrants until maturity or to exercise all his warrants immediately.

The proof is given in the appendix.

The optimal exercise strategy of a monopoly warrantholder is obvious: If $\pi_A(n_A, V_0) \geq \pi_A(0, V_0)$ he exercises all his warrants, otherwise none. This is demonstrated in the next example:

---

\textsuperscript{10}This is caused by the additional debt. The sequential exercise leads to a wealth transfer from the debtholders to the shareholders and warrantholders.
Example 4  We assume a firm with a firm value of 40,000 in $t = 0$ and an asset volatility of 30 %. Further we assume that the firm has issued a zero coupon bond with a face value of 15,000 and a maturity date in 7 years. The firm has also issued 50 stocks and 50 warrants with a strike price of 250 and a maturity in 1 year. The exercise value of a warrant is the stock price cum dividend minus the strike price.

(a) The interest rate is $r = 3\%$. Then the optimal exercise policy of a monopoly warrantholder is to exercise no warrant ($m^* = 0$), because the warrant price for $m = 0$ is higher than the exercise value for $m = n$, whereas the optimal exercise policy in a competitive market is either to exercise or not to exercise ($m^*_i = n_i$ or $m^*_i = 0$).

<table>
<thead>
<tr>
<th></th>
<th>$m^* = 0$</th>
<th>$m^* = n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>exercise value of a warrant</td>
<td>151.29</td>
<td>158.26</td>
</tr>
<tr>
<td>warrant price</td>
<td>160.67</td>
<td>67.03</td>
</tr>
</tbody>
</table>

(b) The interest rate is $r = 1\%$. Then the optimal exercise policy of a monopoly warrantholder is to exercise all warrants ($m^* = n$), because the exercise value of a warrant for $m = n$ is higher than the warrant price for $m = 0$, whereas the optimal exercise policy in a competitive market is either to exercise or not to exercise ($m^*_i = n_i$ or $m^*_i = 0$).

<table>
<thead>
<tr>
<th></th>
<th>$m^* = 0$</th>
<th>$m^* = n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>exercise value of a warrant</td>
<td>136.59</td>
<td>142.86</td>
</tr>
<tr>
<td>warrant price</td>
<td>142.65</td>
<td>55.33</td>
</tr>
</tbody>
</table>

If the discount factor $e^{-rT}$ is low and as long as there are much more stocks than warrants, the warrantholders do best to exercise no warrant independently of the market structure, since the partial derivative (11) is bounded by

$$\pi'(m) \leq K \left( e^{-rT} - \frac{N}{N + m} \right).$$

5 Convertible Bonds

In this section we assume a firm financed by issuing equity, debt and convertible bonds which pays no regular dividends and coupons. Again at time 0 the equity is split into $N$ outstanding shares and $n$ convertible bonds. Every convertible allows for a conversion into one stock. If there is no conversion, each convertible bond pays $K$ at time $T$ and as long as the firm value is sufficiently high for the redemption. If the firm value is not sufficiently high to cover the redemption payment, the firm
is liquidated and the firm value is distributed without bankruptcy costs among the convertible bondholders in proportion to their holdings. This bankruptcy rule implies that the additional debt is subordinated in accordance to Bühler and Koziol (2002). The debt has a common face value $F$ and maturity $T_D$ with $0 < T < T_D$.

According to the bankruptcy rule the payoff function of a pricetaking holder of convertible bonds is defined by

$$
\pi_i(m_i, m_{-i}, V_{T^-}) = \frac{m_i}{N + m} \bar{S}_T \left( V_{T^-} - (n - m)K \right) + (n_i - m_i) \min \left\{ \frac{V_{T^-}}{n - m}, K \right\},
$$

where $\bar{S}_T(V_T)$ equals zero, if $V_T$ is negative. Since a default can occur at time $T$, the redemption value of a non-converted bond, i.e. $\min\{V_{T^-}/(n - m), K\}$, is risky. If $V_{T^-} \leq (n - m)K$ the payoff function collapses to $\pi_i(m_i, m_{-i}, V_{T^-}) = (n_i - m_i)V_{T^-}/(n - m)$ and the conversion of a bond can never be the optimal strategy for a convertible bondholder. Otherwise, if $V_{T^-} > (n - m)K$ the payoff function in case of convertibles is similar to the payoff function in case of warrants with the difference of two constants. The payoff function of a non-pricetaker is defined by

$$
\pi_A(m_A, m_{-A}, V_{T^-}) = \frac{m_A}{N + m_A + m_{-A}} \bar{S}_T \left( V_{T^-} - (n - m_A - m_{-A})K \right) + (n_A - m_A) \min \left\{ \frac{V_{T^-}}{n - m_A - m_{-A}}, K \right\}.
$$

Like a pricetaker a non-pricetaker does not convert a bond if $V_{T^-} \leq (n - m_A - m_{-A})K$. So for European-type convertible bonds the optimal conversion strategy in large trader economies or in a competitive market is similar to the optimal exercise strategy in case of warrants.

Without dividends and coupon payments the sequential conversion of a convertible bond does not change the capital structure of the firm at time $t < T$. Therefore it is not beneficial for the holder of convertible bonds to convert his bonds sequentially. The optimal conversion strategy of an American-type convertible bond is the same like the optimal conversion strategy of an European-type convertible bond.
6 Conclusion

This paper investigates the impact large traders have on the optimal exercise strategies for warrants and their corresponding market values. As distinguished from the existing literature, our analysis considers a firm that issued (additional) senior debt beside shares of common stock and warrants. We present exercise strategies and the corresponding warrant values for a three large traders economies and compare them to those of a competitive economy.

The upper bound on the advantage of sequential exercise of warrants, developed in this paper, decreases with increasing interest rates and it is only positive for unrealistic parameter settings. Hence, it turns out that from a theoretical perspective the potential advantage of sequential exercise strategies is not the main obstacle against the use of the block exercise condition. The latter condition is however questionable on the ground that it may be advantageous not to exercise all warrants if they finish in the money. It turns out that the option to exercise only a fraction of the outstanding convertible at the maturity date has a positive value if and only if one or more warrantholders are non-pricetakers. This option value increases with the concentration of the warrant ownership distribution in the economy. Moreover, we show that there is a gain from hoarding warrants if there are at least two non-pricetaking warrantholders.

This investigation is preliminary in nature, concentrating on characterizing the properties of warrant exercise strategies in large trader economies. As such, many directions for future research remain open to investigation. For instance, a model to determine a fair price for the warrants traded between large traders is needed. Furthermore, there are at least two facts not considered in the model: The pricetakers do not know the distribution of the warrant ownership and, secondly, warrantholders are often at the same time stockholders.
Appendix

Proof of Lemma 1:

Proof of part (a): The stock price is below the strike price for all exercise policies \( m > \hat{m} \). For \( m \in [0, \hat{m}] \) the first derivative of the stock price with respect to the number of warrants exercised reads:

\[
\frac{\partial}{\partial m} S_T(V_{T-} + mK) = \frac{K \Delta_T(V_{T-} + mK)}{N + m} - \frac{\tilde{S}_T(V_{T-} + mK)}{(N + m)^2} \leq \frac{K \Delta_T(V_{T-} + mK) - K}{N + m} < 0.
\]

Proof of part (b): When using the conventions \( \tilde{S}_T(m_A) \equiv \tilde{S}_T(V_{T-} + m_A K + m_A K) \), \( \Delta_T(m_A) \equiv \Delta_T(V_{T-} + m_A K + m_A K) \) and \( \Gamma_T(m_A) \equiv \Gamma_T(V_{T-} + m_A K + m_A K) \) we can define the auxiliary function

\[
v(m_A) = (N + m_A) \frac{\partial}{\partial m} \pi_A(m_A, m_A, V_{T-})
= \frac{N + m_A}{N + m_A + m_A} \tilde{S}_T(m_A) - (N + m_A + m_A)K + m_A K \Delta_T(m_A).
\]

With Taylor’s formula we have also

\[
\tilde{S}_T(0) = \tilde{S}_T(m_A) - m_A K \Delta_T(m_A) + \frac{1}{2} \int_{m_A}^{0} (m_A - t)K^2 \Gamma_T(t) dt. \quad (A1)
\]

Let us now assume, that \( m^1_A, m^2_A \in [0, \hat{m} - m_A] \) with \( m^1_A < m^2_A \). We show that \( v(m^1_A) - v(m^2_A) > 0 \) holds:

\[
v(m^1_A) - v(m^2_A) = \left( \frac{N + m_A}{N + m_A + m^1_A} \tilde{S}_T(m^1_A) - \frac{N + m_A}{N + m_A + m^2_A} \tilde{S}_T(m^2_A) \right)
+ m^1_A K \Delta_T(m^1_A) - \left( \frac{N + m_A}{N + m_A + m^2_A} \tilde{S}_T(m^2_A) \right)
+ \left( \tilde{S}_T(0) + m^1_A K \Delta_T(m^1_A) \right) - \left( \tilde{S}_T(0) + m^2_A K \Delta_T(m^2_A) \right).
\]
According to part (a) of the lemma the first term is greater than zero. With equation (A1) we get the result:

\[
v(m_A^1) - v(m_A^2) > (m_A^2 - m_A^1)K + \bar{S}_T(m_A^1) - \bar{S}_T(m_A^2) + \frac{1}{2} \int_{m_A^1}^{m_A^2} (m_A^2 - t)K^2 \Gamma_T(t) dt - \frac{1}{2} \int_{m_A^1}^{m_A^2} (m_A^2 - t)K^2 \Gamma_T(t) dt > 0.
\]

This proves that the first derivative of the payoff function with respect to the number of warrants exercised is strictly decreasing. That is, the payoff function of the non-pricetaker is strictly concave.

\[\square\]

**Proof of Proposition 2:**

*Proof of part (a):* Let \((m_i, (m_i)_{i \in I \setminus \{A\}})\) be an exercise strategy in the presence of one non-pricetaker. Differentiating the payoff function of the warrantholders results in

\[
\frac{\partial}{\partial m_i} \pi_i(m_i, m_{-i}, V_T^-) = \frac{1}{N + m} \bar{S}_T(V_T^- + mK) - K,
\]

\[
\frac{\partial}{\partial m_A} \pi_A(m_A, m_{-A}, V_T^-) = \frac{N + m_{-A}}{(N + m_A + m_{-A})^2} \bar{S}_T(V_T^- + m_AK + m_{-A}K)
+ \frac{m_A}{N + m_A + m_{-A}} K \Delta_T(V_T^- + m_AK + m_{-A}K) - K,
\]

and

\[
\frac{\partial}{\partial m_i} \pi_i(m_i, m_{-i}, V_T^-) - \frac{N + m_A + m_{-A}}{N + m_{-A}} \frac{\partial}{\partial m_A} \pi_A(m_A, m_{-A}, V_T^-)
= -\frac{m_A}{N + m_{-A}} K \left( \Delta_T(V_T^- + m_AK + m_{-A}K) - 1 \right) \geq 0 \tag{A2}
\]

with equality for \(m_A = 0\) and inequality for \(m_A > 0\).

Furthermore, please note: If \((m_j^* , m_{-j}^*)\) is a Nash equilibrium for any player \(j \in I\), we have

\[
\frac{\partial}{\partial m_j} \pi_j(m_j^* , m_{-j}^*, V_T^-) \begin{cases} 
< 0 & \Rightarrow m_j^* = 0 \\
= 0 & \Rightarrow m_j^* \in [0, n_j] \\
> 0 & \Rightarrow m_j^* = n_j.
\end{cases} \tag{A3}
\]

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Let \((m^*_A, m^*_{\bar{A}})\) be a Nash equilibrium. By the equations \((A2)\) and \((A3)\) we can conclude the following:

If \(m^*_{A} < n_{\bar{A}}\) \(\Rightarrow\) \(\frac{\partial}{\partial m_{\bar{A}}} \pi_A(m^*_A, m^*_{\bar{A}}, V_{T-}) \leq 0 \Rightarrow m^*_A = 0\), and \((A4)\)

If \(m^*_A > 0\) \(\Rightarrow\) \(\frac{\partial}{\partial m_{\bar{A}}} \pi_A(m^*_A, m^*_{\bar{A}}, V_{T-}) \geq 0 \Rightarrow m^*_A = n_{\bar{A}}\). \((A5)\)

According to equation \((A4)\) in the Nash equilibrium the non-pricetaker does not exercise warrants at all, if the pricetakers do not exercise all warrants they hold. From proposition 1 we know if they finish out of the money \((V_{T-} \leq \bar{V})\) the pricetakers let expire their warrants. If \(V_{T-} \in [\bar{V}, \bar{V}_A)\) the pricetakers exercise as many warrants as necessary to equalize stock and strike price. (For \(V_{T-} = \bar{V}_A\) the pricetakers exercise all their warrants, \(m^*_A = n_{\bar{A}}\), and the non-pricetaker none, \(m^*_A = 0\).)

According to equation \((A5)\) in the Nash equilibrium the pricetakers exercise all of their warrants, if the non-pricetaker exercises some of his warrants. The optimal exercise policy of warrantholder \(A\) is the solution of \(\frac{\partial}{\partial m_A} \pi_A(m^*_A, n_{\bar{A}}, V_{T-}) = 0\). If \(V_{T-} \in [\bar{V}_A, \bar{V}_A)\) the optimal exercise policy of \(A\) is a partial exercise, \(m^*_A < n_{\bar{A}}\), and if \(V_{T-} \geq \bar{V}_A\) the optimal exercise policy is to exercise all warrants.

**Proof of part (b):** Let \(V_{T-} \in [\bar{V}, \bar{V}_A)\). Then we have

\[
\frac{m^*_A + m^*_{\bar{A}} < n_{\bar{A}} + n_{\bar{A}} = n = m^*}.
\]

Let \(V_{T-} \in (\bar{V}_A, \bar{V})\) and \((m^*_A, n_{\bar{A}})\) the optimal exercise policy in the one-atomic game with \(m^* = m^*_A + n_{\bar{A}}\). The requirement \(\frac{\partial}{\partial m_A} \pi_A(m^*_A, n_{\bar{A}}, V_{T-}) = 0\) leads to

\[
\bar{S}_T(V_{T-} + m^*_A K + n_{\bar{A}}K) \\
= (N + m^*_A + n_{\bar{A}})K \left(1 + \frac{m^*_A}{N + n_{\bar{A}}}(1 - \Delta_T(V_{T-} + m^*_A K + n_{\bar{A}}K))\right).
\]

From this and the fact that \(m^* \) equalize stock and strike price, it follows

\[
K = \frac{1}{N + m^*} \bar{S}_T(V_{T-} + m^* K) \\
= \frac{1}{N + m^*_A + n_{\bar{A}}} \bar{S}_T(V_{T-} + m^*_A K + n_{\bar{A}}K) \\
= K \left(1 + \frac{m^*_A}{N + n_{\bar{A}}}(1 - \Delta_T(V_{T-} + m^*_A K + n_{\bar{A}}K))\right).
\]

Since \(m^*_A > 0\) this is not possible. So \(V_{T-}\) cannot have the assumed properties. Because \(m^*\) and \(m^*_A\) are continuous in \(V_{T-}\), the inequality \((4)\) must be correct. \(\square\)
Proof of Proposition 3:

Proof of part (a): Recall that according to lemma 1 the payoff functions \( \pi_b(\cdot, m_B, V_T- \cdot) \) and \( \pi_B(\cdot, m_b, V_T- \cdot) \) are concave and let \((m_b^*, m_B^*)\) be a Nash equilibrium. We now consider all four cases. If \( V_T- \in [0, V] \) the stock price is below the strike price, rational warrantholders let all warrants expire, i.e. \((m_b^*, m_B^*) = (0,0)\).

Let \( V_T- \in [V_A, V_b] \). We assume that the optimal exercise strategy of both warrantholder is an inner solution, i.e. both derivatives of the payoff functions equal zero. Then it holds:

\[
\frac{\partial}{\partial m_b} \pi_b(m_b^*, m_B^*, V_T-) - \frac{\partial}{\partial m_B} \pi_B(m_b^*, m_B^*, V_T-) = 0
\]

\[
(m_B^* - m_b^*) \left( \frac{S_T(V_T- + m_b^*K + m_B^*K)}{N + m_b^* + m_B^*} - K \Delta_T(V_T- + m_B^*K + m_b^*K) \right) = 0
\]

\[
m_B^* = m_b^*
\]

Accordingly, if it is optimal for one non-pricetaker to exercise only a fraction of his holdings, the same is true for the other non-pricetaker, and both exercise the same number of warrants. This is a Nash equilibrium for all firm values in the given range.

Finally consider cases 3, 4 where \( V_T- \geq V_b \). Then warrantholder \( b \) exercises all his warrants (for \( V_T- = V_b \) warrantholder \( b \) exercises already all his warrants), i.e. \( m_b^* = n_b \), so the situation of the warrantholder \( B \) is that of warrantholder \( A \) in the one-atomic game, if we set \( n_A = n_b \ (V_T- \geq V_A) \).

Proof of part (b): Let \( V_T- \in (V_A, V_b] \). Then inequality (6) is obviously correct, since according to proposition 2 \( m_A^* = 0 \) and because of part (a) of proposition 3 \( m_B^* \) and \( m_b^* \) are positive. The optimization of the payoff functions in the two-atomic game leads to a payoff of the non-pricetakers in the two-atomic game greater than zero. This is only possible if

\[
S_T(V_T- + m_b^*K + m_B^*K) > K = S_T(V_T- + m_A^*K).
\]

As the share price is strictly decreasing in \( m \), it follows \( m_b^* < m_b^* + m_B^* < m_A^* \). This proves the inequalities (5) and (7).

Let \( V_T- \in (V_A, V_b] \). Since \( m_A^* = n_A > m_b^* \) inequality (5) is correct.

We assume that \( n_A + m_A^* = 2m_B^* < 2n_b = 2n_A \) holds. Then

\[
\frac{N + m_A^* + n_A}{N + n_A} \frac{\partial}{\partial m_A} \pi_A(m_A^*, n_A, V_T-) - \frac{N + m_A^* + n_b}{N + m_b^*} \frac{\partial}{\partial m_B} \pi_B(m_B^*, m_b^*, V_T-) = 0
\]

\[
\frac{m_b^*}{N + m_A^*} (\Delta_T(V_T- + m_A^*K + n_A^*K) - 1)K
- \frac{m_B^*}{N + m_b^*} (\Delta_T(V_T- + m_B^*K + m_b^*K) - 1)K = 0
\]

\[
\frac{m_b^*}{N + n_A} - \frac{m_B^*}{N + m_b^*} = 0.
\]
Since \( n_A > m_b^* \) it must hold \( m_A^* > m_B^* \). As \( n_A > m_B^* \) the assumption cannot be correct. Since \( m_A^* \) and \( m_B^* \) are continuous in \( V_{T-} \) and inequality (7) is correct for \( V_{T-} = \bar{V}_A \), inequality (7) must be correct for all firm values in the given range.

We assume \( m_A^* = m_B^* < n_A \). As \( (m_b^*, m_B^*) \) and \( (m_A^*, n_A) \) are Nash equilibria, it holds

\[
\frac{\partial}{\partial m_B} \pi_B(m_b^*, m_B^*, V_{T-}) = \frac{\partial}{\partial m_A} \pi_A(m_A^*, n_A, V_{T-}) = 0.
\]

We use the conventions \( \tilde{S}_A = \tilde{S}_T(V_{T-} + m_A^* K + n_A K) \), \( \tilde{S}_B = \tilde{S}_T(V_{T-} + 2m_B^* K) \), \( \Delta_A = \Delta_T(V_{T-} + m_A^* K + n_A K) \) and \( \Delta_B = \Delta_T(V_{T-} + 2m_B^* K) \). Since

\[
0 = (N + m_b^* + m_B^*) \frac{\partial}{\partial m_B} \pi_B(m_b^*, m_B^*, V_{T-})
\]

\[
= (N + m_b^*) \left( \frac{1}{N + m_b^* + m_B^*} \tilde{S}_B - K \right) + m_B^* (K \Delta_B - K)
\]

\[
< (N + n_A) \left( \frac{1}{N + m_b^* + m_B^*} \tilde{S}_B - K \right) + m_A^* (K \Delta_A - K)
\]

\[
= (N + m_A^* + n_A) \frac{\partial}{\partial m_A} \pi_A(m_A^*, n_A, V_{T-})
\]

\[
+ (N + n_A) \left( \frac{1}{N + m_b^* + m_B^*} \tilde{S}_B - \frac{1}{N + m_A^* + n_A} \tilde{S}_A \right)
\]

< 0

the assumption must be wrong. As \( m_A^* \) and \( m_B^* \) are continuous in \( V_{T-} \), inequality (6) must be correct for all firm values of the given range.

**Proof of part (c):** If in the presence of two non-pricetakers \( V_{T-} \in \overline{(\bar{V}_B, \bar{V}_A)} \) then both non-pricetakers exercise all their warrants while a monopolistic warrantholder will only exercise a fraction of his holdings, i.e. \( m_b^* + m_B^* = \tilde{n} > m_A^* \).

Let us now assume \( V_{T-} \in (\bar{V}_B, \bar{V}_A) \) and \( m_A^* = m_b^* + m_B^* \). From

\[
S_T(V_{T-} + m_A^* K) = S_T(V_{T-} + m_b^* K + m_B^* K)
\]

and the fact that the derivative of the payoff function of the warrantholder \( A \) and \( B \) are zero, we get

\[
m_A^* \frac{m_b^*}{N} = \frac{m_b^*}{N + m_b^*}
\]

implying

\[
m_A^* (N + m_b^*) - m_b^* N = m_A^* m_b^* + N m_B^* = 0.
\]

This is not possible since for \( V_{T-} > \bar{V} \) the exercise policies \( m_A^*, m_B^* \) and \( m_b^* \) are positive. Therefore we must have \( m_A^* < m_b^* + m_B^* \) for all \( V_{T-} \in (\bar{V}_B, \bar{V}_A) \), since \( m_A^* \) and \( m_B^* \) are continuous in \( V_{T-} \) and \( m_A^* < m_b^* + m_B^* \) for \( V_{T-} = \bar{V}_B \).
Proof of Proposition 4:

Proof of part (a): The payoff function of the non-pricetaker $A$ is defined through the equations (2), (8) and (9):

$$\pi_A(m_A, m_{-A}, V_0) = m_A (S_0(V_0, m) - K) + (n_A - m_A)W_0(V_0, m)$$

$$= m_Ae^{-rT}\int_{\mathbb{R}_+} S_T(V_{T-}, m)dQ - m_AK$$

$$+ (n_A - m_A)e^{-rT}\int_{V_T(m)}^{\infty} (S_T(V_{T-}, m) - K) dQ$$

$$= m_Ae^{-rT}\int_{\mathbb{R}_+} (S_T(V_{T-}, m) - K) dQ$$

$$- m_Ae^{-rT}\int_{0}^{\infty} K\left(\frac{V_{T-}}{V_0} - 1\right) dQ$$

$$+ (n_A - m_A)e^{-rT}\int_{V_T(m)}^{\infty} (S_T(V_{T-}, m) - K) dQ$$

$$= -m_Ae^{-rT}\int_{0}^{\infty} K\left(\frac{V_{T-}}{V_0} - 1\right) dQ$$

$$+ m_Ae^{-rT}\int_{0}^{V_T(m)} \left(\frac{1}{N + m}\overline{S}_T(V_{T-}, m) - K\right) dQ$$

$$+ n_Ae^{-rT}\int_{V_T(m)}^{\infty} \left(\frac{1}{N + n}\overline{S}_T(V_{T-}, m) - K\right) dQ.$$

So we get for the marginal payoff of warrantholder $A$:

$$\frac{\partial}{\partial m_A}\pi_A(m_A, m_{-A}, V_0) = -e^{-rT}\int_{0}^{\infty} K\left(\frac{V_{T-}}{V_0} - 1\right) dQ$$

$$+ e^{-rT}\int_{0}^{V_T(m)} \left(\frac{1}{N + m}\overline{S}_T(V_{T-}, m) - K\right) dQ$$

$$+ \frac{m_A}{N + m}e^{-rT}\int_{0}^{V_T(m)} \left(K\frac{V_{T-}}{V_0}\Delta_{r}(V_{T-}, m) - \frac{\overline{S}_T(V_{T-}, m)}{N + m}\right) dQ$$

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The marginal payoff of a pricetaking warrant holder is
\[
\frac{\partial}{\partial m} \pi_A(m_i, m_{-i}, V_0) = S_0(V_0, m) - K - W_0(V_0, m),
\]
which is always negative.

**Proof of part (b):** The derivative of the warrant price is given by
\[
\frac{\partial}{\partial m} W_0(V_0, m) = e^{-rT} \frac{K}{N + n} \int_{V_T(m)}^{\infty} \left( \frac{V_T - V_0}{V_0} - 1 \right) \Delta_T(V_T, m) dQ \geq 0,
\]
\[
\frac{\partial^2}{\partial m^2} W_0(V_0, m) = e^{-rT} \frac{K^2}{N + n} \int_{V_T(m)}^{\infty} \left( \frac{V_T - V_0}{V_0} - 1 \right)^2 \Gamma_T(V_T, m) dQ \geq 0.
\]

The payoff function and derivative of the payoff function with respect to the number of warrants exercised is given by
\[
\pi_A(m_A, m_{-A}, V_0) = m_A \left( S_0(V_0, m) - K - W_0(V_0, m) \right) + n_A W_0(V_0, m)
\]
\[
\frac{\partial}{\partial m_A} \pi_A(m_A, m_{-A}, V_0) = S_0(V_0, m) - K - W_0(V_0, m)
\]
\[
+m_A \frac{\partial}{\partial m} S_0(V_0, m) + (n_A - m_A) \frac{\partial}{\partial m} W_0(V_0, m).
\]
From the condition \( \frac{\partial}{\partial m} \pi_A(m_A^*, m_{A}^*, V_0) = 0 \) we get

\[
S_0(V_0, m^*) - K - W_0(V_0, m^*) = -m_A^* \frac{\partial}{\partial m} S_0(V_0, m^*) - (n_A - m_A^*) \frac{\partial}{\partial m} W_0(V_0, m^*)
\]

and

\[
0 \leq \pi_A(m_A^*, m_{A}^*, V_0) - \pi_A(0, m_{A}^*, V_0) = n_A \left( W_0(V_0, m^*) - W_0(V_0, 0) \right) + m_A^* \left( S_0(V_0, m^*) - K - W_0(V_0, m^*) \right) = n_A \left( W_0(V_0, m^*) - W_0(V_0, 0) \right)
\]

\[
- m_A^* \left( m_A^* \frac{\partial}{\partial m} S_0(V_0, m^*) + (n_A - m_A^*) \frac{\partial}{\partial m} W_0(V_0, m^*) \right).
\]

As the debt and the partial derivative of the debt is given by

\[
D_0(V_0, m) = e^{-rT} \int_0^\infty \frac{V_0 + mK}{V_0} V_T - dQ - e^{-rT} \int_0^\infty \bar{S}_T(V_T^-, m)dQ
\]

\[
- e^{-rT} \int_0^\infty (N + m)KdQ - e^{-rT} \int_0^\infty \bar{S}_T(V_T^-, m) - (n - m)K \right) dQ
\]

\[
\frac{\partial}{\partial m} D_0(V_0, m) = e^{-rT} \int_0^\infty K \left( 1 - \frac{V_T^-}{V_0} \Delta_T(V_T^-, m) \right) dQ
\]

\[
+ e^{-rT} \int_0^\infty K \left( \frac{V_T^-}{V_0} - 1 \right) dQ - (N + n) \frac{\partial}{\partial m} W_0(V_0, m),
\]

a lower bound of the partial derivative of the stock price is given by

\[
\frac{\partial}{\partial m} S_0(V_0, m) = e^{-rT} \int_0^\infty \left( \frac{1}{(N + m)^2} \bar{S}_T(V_T^-, m) - \frac{1}{N + m} \frac{V_T^-}{V_0} K \Delta_T(V_T^-, m) \right) dQ
\]

\[
+ \frac{\partial}{\partial m} W_0(V_0, m)
\]
\[
\geq -e^{-rT} \frac{1}{N+m} \int_0^{\mathcal{V}^{(m)}} K \left( 1 - \frac{V_T}{V_0} \Delta_p(V_T, m^*) \right) dQ
\]
\[+ \frac{1}{N+m} \frac{\partial}{\partial m} W_0(V_0, m) \]
\[= - \frac{1}{N+m} \frac{\partial}{\partial m} D_0(V_0, m) - \frac{n-m}{N+m} \frac{\partial}{\partial m} W_0(V_0, m) \]
\[+ \frac{1}{N+m} \int_0^\infty K \left( \frac{V_T}{V_0} - 1 \right) dQ . \]

We differentiate the equation
\[V_0 = (N+m) S_0(V_0, m) + (n-m) W_0(V_0, m) + D_0(V_0, m) - mK\]
and get with the relation (A6)
\[0 = S_0(V_0, m^*) - K - W_0(V_0, m^*) \]
\[+ (N+m^*) \frac{\partial}{\partial m} S_0(V_0, m^*) + (n-m^*) \frac{\partial}{\partial m} W_0(V_0, m^*) + \frac{\partial}{\partial m} D_0(V_0, m^*) \]
\[= (N+m^*) \frac{\partial}{\partial m} S_0(V_0, m^*) + (n-A-m^*_A) \frac{\partial}{\partial m} W_0(V_0, m^*) + \frac{\partial}{\partial m} D_0(V_0, m^*) . \]

Combining the equations (A8) and (A9) we get
\[
\frac{\partial}{\partial m} S_0(V_0, m) \geq \frac{1}{m_A^*} e^{-rT} \int_0^\infty K \left( \frac{V_T}{V_0} - 1 \right) dQ + \frac{n_A-m_A^*}{m_A^*} \frac{\partial}{\partial m} W_0(V_0, m) \]
\[(A10)\]
and with equation (A7)
\[0 \leq n_A \left( W_0(V_0, m^*) - W_0(V_0, 0) \right) - m_A^* e^{-rT} \int_0^\infty K \left( \frac{V_T}{V_0} - 1 \right) dQ \]
\[= n_A \left( W_0(V_0, m^*) - W_0(V_0, 0) \right) - m_A^* K \left( 1 - e^{-rT} \right) \]
\[
\frac{1}{n_A} K \left( 1 - e^{-rT} \right) \leq \frac{W_0(V_0, m^*) - W_0(V_0, 0)}{m_A^*} \leq \frac{\partial}{\partial m} W_0(V_0, m^*) .
\]
This proves the lower bound. The upper bound can be proved by writing equation (A9) as
\[0 = (N+m^*_A) \frac{\partial}{\partial m} S_0(V_0, m^*) + (n_A-m^*_A) \frac{\partial}{\partial m} W_0(V_0, m^*) + \frac{\partial}{\partial m} D_0(V_0, m^*) .\]
\[
\begin{align*}
\leq & \quad e^{-rT} \int_0^{V_T(m^*)} \left( K - \frac{N + m^*_A}{(N + m^*)} \tilde{S}_T(V_T^-, m^*) \right) dQ \\
& + e^{-rT} \int_0^\infty K \left( \frac{V_T^-}{V_0} - 1 \right) dQ - n_A \frac{\partial}{\partial m} W_0(V_0, m^*)
\end{align*}
\]
resulting in
\[
\frac{\partial}{\partial m} W_0(V_0, m^*) \leq \frac{1}{n_A} e^{-rT} \int_0^\infty K \left( \frac{V_T^-}{V_0} - 1 \right) dQ + \frac{1}{n_A} K Q \left( \{ V_T^- \leq V_T(m^*) \} \right) \\
= \frac{1}{n_A} K \left( 1 - e^{-rT} \right) + \frac{1}{n_A} K Q \left( \{ V_T^- \leq V_T(m^*) \} \right).
\]

\[\square\]

**Proof of Proposition 5:**

The payoff function of non-pricetaker \( A \) with \( n_A \in [0, n] \) is given by
\[
\begin{align*}
\pi_A(m_A, m^-_A, V_0) &= m_A(S_0 - K) + (n_A - m_A) W_0 \\
&= -m_A e^{-rT} \int_0^{V_T(m)} K \left( e^{rT} - 1 \right) dQ \\
&\quad + e^{-rT} \int_0^{V_T(m)} \left( \frac{m_A}{N + m} \tilde{S}_T(V_T^- + e^{rT} mK) - m_A K \right) dQ \\
&\quad + e^{-rT} \int_{V_T(m)}^\infty \left( \frac{n_A}{N + n} \tilde{S}_T(V_T^- + n K + mK(e^{rT} - 1)) - n_A K \right) dQ,
\end{align*}
\]
where \( V_T(m) \) and \( \tilde{V}_T(m) \) solve the equations
\[
\tilde{S}_T(V_T^- + e^{rT} mK) = (N + m) K \quad \text{and} \quad \tilde{S}_T(V_T^- + n K + mK(e^{rT} - 1)) = (N + n) K.
\]
The proof is now straightforward. \[\square\]

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Proof of Proposition 6:

We prove the statement for arbitrary exercise strategies \((m_i)_i \in I\) and prove that there exists a sequential exercise strategy \(m_A \in \{0, n_A\}\) of non-pricetaker \(A \in I\) with 
\[
\pi_A(m_A, m_{-A}, V_0) = \sup_{m_A \in \{0, n_A\}} \pi_A(m_A, m_{-A}, V_0).
\]

Let \(m_A \in [0, n_A]\). For \(m_A = 0\) relation (10) and its partial derivative collapse to
\[
\pi_A(0, m_{-A}, V_0) = e^{-rT} \int_\mathcal{V}_T(m) \left( \frac{n_A}{N+n} S_T(V_{T^-} + (n-m)K) - n_A K \right) dQ,
\]
\[
\frac{\partial}{\partial m_A} \pi_A(0, m_{-A}, V_0) = -e^{-rT} \int_\mathcal{V}_T(m) \left( \frac{n_A}{N+n} K \Delta(V_{T^-} + (n-m)K) \right) dQ.
\]

The payoff function of warrantholder \(A \in I\) is also given by
\[
\pi_A(m_A, m_{-A}, V_0) = m_A \pi'(m_A + m_{-A}) + \pi_A(0, m_A + m_{-A}, V_0)
\]
and the partial derivative of the payoff function of a non-pricetaker reads as
\[
\frac{\partial}{\partial m_A} \pi_A(m_A, m_{-A}, V_0) = \frac{N + m_{-A}}{N + m} \pi'(m) + e^{-rT} \int_\mathcal{V}_T(m) \frac{m_A}{N + m} K dQ
\]
\[-e^{-rT} \int_\mathcal{V}_T(m) \frac{n_A}{N+n} K \Delta(V_{T^-} + (n-m)K) dQ.
\]

From
\[
\pi_A(0, m_{-A}, V_0) - \pi_A(m_A, m_{-A}, V_0)
\]
\[
= \pi_A(0, m_{-A}, V_0) - m_A \pi'(m_A + m_{-A}) - \pi_A(0, m_A + m_{-A}, V_0)
\]
\[
= -m_A \pi'(m_A + m_{-A}) - \int_{m_{-A}}^{m_{A+m_{-A}}} \partial_{m_{-A}} \pi_A(0, x, V_0) dx
\]
\[
= -m_A \pi'(m_A + m_{-A})
\]
\[
+ \int_{m_{-A}}^{m_{A+m_{-A}}} e^{-rT} \frac{n_A}{N+n} \int_{\mathcal{V}_T(m)} K \Delta(V_{T^-} + (n-x)K) dQ dx
\]
\[
\geq -m_A \pi'(m_A + m_{-A})
\]
\[
+ m_A e^{-rT} \frac{n_A}{N+n} \int_{\mathcal{V}_T(m)} K \Delta(V_{T^-} + (n-m)K) dQ = \alpha
\]
follows, that if $\alpha \geq 0$, warrantholder $A$ is better off when exercise no warrant at all. If $\alpha < 0$ it follows from

$$
\frac{\partial}{\partial m_A} \pi_A(m_A, m_{-A}, V_0) > e^{-rT} \frac{m_A}{N + m} \int_{\mathcal{V}_T^m} KdQ
$$

$$
- e^{-rT} \frac{m_A}{N + m} \int_{\mathcal{V}_T^m} \frac{n_A}{N + n} K \Delta(V_T^- + (n - m)K)dQ
$$

$$
> e^{-rT} \frac{m_A}{N + m} \left( \int_{\mathcal{V}_T^m} KdQ - \int_{\mathcal{V}_T^m} \frac{n_A}{N + n} KdQ \right) \geq 0,
$$

that $m_A$ is not an optimal exercise policy if $m_A \neq n_A$. \hfill \Box
References


Crouhy, Michel/ Galai, Dan, 1994, The interaction between the financial and investment decisions of the firm: the case of issuing warrants in a levered firm, Journal of Banking and Finance 18, pp.861-880


Schulz, Uwe G./ Trautmann, Siegfried, 1994, Robustness of option-like warrant valuation, Journal of Banking and Finance 18, pp.841-859


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