RISK PREFERENCE BASED OPTION PRICING IN A FRACTIONAL BROWNIAN MARKET

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1 Introduction

Fractional Brownian motion as a model of a self-similar process with stationary increments was originally introduced by Mandelbrot and van Ness (1968), who also suggested its usage in financial models in order to easily capture long-range dependencies or persistence.

After the success of riskneutral valuation in the Markovian models of Black, Scholes and Merton, it was hoped to extend the famous option pricing formula and make it usable in a fractional context. In the course of the 90s however, it turned out, that arbitrage-free pricing in the fractional market model based on pathwise integration should not be possible (see Rogers (1997) or Shiryayev (1998)).

The research interest in this field was re-encouraged by new insights in stochastic analysis using a definition of integration with respect to fractional Brownian motion based on the Wick product. In the last years many of the useful tools applied in the classical Markovian case could be translated to the fractional, Wick-calculus based world, like a fractional Itô theorem, a fractional Girsanov theorem or a fractional Clark-Ocone formula, to name only the most important results (for a detailed survey see Bender (2003)). As a consequence, efforts on deriving no-arbitrage based valuation methods have been reinforced and several arbitrage-free models have been proposed.

However, Delbaen and Schachermayer (1994) proved for the continuous case, that irrespective of the choice of integration theory a weak form of arbitrage called free lunch with vanishing risk can only be excluded if and only if the underlying stock price process S is a semimartingale. It is though easy to verify that, due to their persistent character, processes driven by fractional Brownian motion are not semimartingales. For a motivating access to this topic, see the discussion of the discrete framework of Sottinen (2001). Moreover, Cheridito (2003) constructs explicite arbitrage strategies in a fractional Black-Scholes market.

Actually, the above statement of Delbaen and Schachermayer (1994) holds true as long as the definitions of the fundamental concepts as arbitrage, self-financing properties and admissibility remain unchanged. Hence, concepts have been proposed to overcome the existing difficulties by modification of the underlying definitions, among them the approaches due to Hu and Øksendal (2003) and Elliot and van der Hoek (2003). They extended the idea of Wick calculus beyond integration theory and changed the definitions of the portfolio value and/or the property of being self-financing, incorporating the Wick product. As Bjørk and Hult (2005) showed recently, these concepts lead to some problems concerning economic interpretation.

Cheridito (2003) proposes a different modification of the framework: He shows, that – when postulating the existence of an arbitrarily small minimal amount of time that must lie between two consecutive transactions – all kinds of arbitrage opportunities can be excluded. But, while the assumption of non-continuous trading strategies doesn't seem to be too restrictive when thinking of real markets, it entails one problem: Though excluding arbitrage, no arbitrage option pricing approaches continue to fail, as now the possibility of a continuous adjustment of the replicating portfolio is no longer given.

In this paper we link the modified framework of Cheridito (2003) – which by absence of arbitrage makes sure that the financial model in general and option pricing in particular make sense – with a switch-over to a preference based pricing approach. This introduction of risk preferences allows us to renounce continuous tradability.

The advantages of a transition to a preference based pricing approach will turn out to be the following: The use of conditional expectation in its traditional sense will make it possible to point out the problems arising in valuation models when dealing with path-dependent processes. Moreover, advances in stochastic analysis will be used to plausibly illustrate the features of fractional Brownian motion and to make fractional option pricing comparable to the classical Brownian model. Especially, the consequences of the existence of long-range-dependence on option pricing should be clarified.

The rest of the paper is organized as follows: After giving a short review about some important results with respect to fractional Brownian motion in section 2, we'll go into details concerning conditionality of distributional forecasts, in particular, we will recall and interpret the results of Gripenberg and Norros (1996). Section 3 will be devoted to this. In the sequel, we'll focus in section 4 on a risk preference based option pricing approach exemplified by the assumption of risk-neutral market participants. The derived pricing formulae will be interpreted in order to underline the necessity of capturing memory in models using fractional Brownian motion. Moreover, we'll examine the effect of the Hurst parameter on the option price deriving its partial derivative with respect to H. The main results will be summarized in the conclusion at the end of the paper.

2 The setup of the fractional Brownian market

We use the definition of fractional Brownian motion via its original presentation as a moving average of Brownian increments. For $0 < H < 1$, fractional Brownian motion $\{B_t^H, t \in \mathbb{R}\}$ is the stochastic process defined by:

$$
\begin{aligned}
B_0^H(\omega) &= 0 \quad \forall \omega \in \Omega \\
B_t^H(\omega) &= c_H \left[\int_{\mathbb{R}} \left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} \right) dB_s(\omega) \right]
\end{aligned}
$$

where $\{B_s, s \in \mathbb{R}\}$ is a two-sided Brownian motion, H is the so-called Hurst parameter and

$$
c_H = \sqrt{\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(\frac{1}{2}+H)\Gamma(2-2H)}}
$$

is a normalizing constant. Note that for $t > 0$, B_t^H can be rewritten by

$$B_t^H = c_H \left[\int_{-\infty}^{0} \left((t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}} \right) dB_s + \int_0^t (t-s)^{H-\frac{1}{2}} \, dB_s \right]$$

Obviously, for $H = \frac{1}{2}$, $B_t^{\frac{1}{2}}$ coincides with classical Brownian motion. On the other hand, the cases $0 < H < \frac{1}{2}$ and $\frac{1}{2} < H < 1$ can be identified with the occurence of anti-persistence and persistence respectively. To account for the latter phenomenon, regard a fractional increment

$$\begin{aligned} \Delta B^H(t) &= B_{t+\Delta t}^H - B_t^H \\ &= c_H \int_t^{t+\Delta t} (t + \Delta t - s)^{H-\frac{1}{2}} \, dB_s \\ &\quad + c_H \int_{-\infty}^{t} \left[(t+\Delta t - s)^{H-\frac{1}{2}} - (t-s)^{H-\frac{1}{2}} \right] dB_s \end{aligned}$$

As can be seen, in the case $\frac{1}{2} < H < 1$, a fractional Brownian increment positively depends on all historical increments of its generating Brownian motion, where recent changes have a greater influence than older ones. Throughout this paper we'll focus on this persistent case, however, drawing from time to time comparisons to the classical Brownian theory.

This kind of memory of the process can also be illustrated using the covariance properties of fractional Brownian motion. It is easy to verify (see Mandelbrot/ van Ness (1968)) that B_t^H is the unique Gaussian process satisfying

$$\begin{aligned} E(B_t^H) &= 0 \quad \forall\, t \in \mathbb{R} \\ E(B_t^H B_s^H) &= \frac{1}{2} \left[|t|^{2H} + |s|^{2H} - |t-s|^{2H} \right] \quad \forall\, t, s \in \mathbb{R}. \end{aligned}$$

Again, in the limit case $H = \frac{1}{2}$, the moment properties of classical Brownian motion can be obtained. For $H > \frac{1}{2}$, define the sequence

$$r_n = E\left(B_1^H (B_{n+1}^H - B_n^H) \right)$$

As easily follows from the covariance property, we observe that $\sum_{n=1}^{\infty} r_n = \infty$, which justifies the use of the term long-range dependence.

Based on the definition of fractional Brownian motion, we look at a fractional Brownian market consisting of a riskless asset or bond $A(t)$ with dynamics

$$dA(t) = rA(t)dt \tag{1}$$

as well as of a risky asset or stock $S(t)$ with dynamics

$$dS(t) = \mu S(t)dt + \sigma S(t) dB_t^H. \tag{2}$$

The process satisfying the latter equation is called geometric fractional Brownian motion. The parameters r and σ are assumed to be constant, symbolizing the interest rate and the volatility respectively. The drift parameter

μ may be varying over time, but has to satisfy the condition of integrability. The mathematical interpretation of equation (2) depends on the assumed integration theory, by name pathwise integration or Wick-based integration respectively. Throughout this paper we'll focus on the latter concept.

Based on the Wick product, Duncan (2000) introduced a fractional Itô theorem using Malliavin calculus. Bender (2003) pointed out the limitations of the derived results and generalized the theorem by using a concept called S-transform. In the special case needed for our purposes, the result reads as follows (see Bender (2003), Theorem 2.6.5):

Theorem 1 *Let S_t be a geometric fractional Brownian motion as above. Let $F(t, S_t)$ be once continuously differentiable with respect to t and twice with respect to S_t. Under certain regularity conditions it holds:*

$$\begin{aligned} F(T, S_T) = F(t, S_t) &+ \int_t^T \frac{\partial}{\partial s} F(s, S_s)\, ds + \int_t^T \frac{\partial}{\partial x} F(s, S_s) \mu_s S_s\, ds \\ &+ \sigma \int_t^T \frac{\partial}{\partial x} F(s, S_s) S_s\, dB_s^H + H\sigma^2 \int_t^T s^{2H-1} \frac{\partial^2}{\partial x^2} F(s, S_s) S_s^2\, ds \end{aligned} \tag{3}$$

For the limit $H \to \frac{1}{2}$ the well-known Itô formula can be obtained. We'll need a version of this theorem in section 4, slightly modified to the case of a conditional stochastic process.

3 The conditional distribution of fractional Brownian motion

3.1 Prediction based on an infinite knowledge about the past

In this section we focus on the distribution of fractional Brownian motion given all information concerning the history of the path. Specially we regard $E[B_T^H | \mathfrak{F}_t^H], T > t$, where $\mathfrak{F}_t^H = \sigma(B_s^H, s \leq t)$ is the σ-field generated by all $B_s^H, s \leq t$. In the first instance $E[B_T^H | \mathfrak{F}_t^H], T > t$ is a random variable, a coarsening of B_T^H, yielding in each case the expected value over all $\omega \in \Omega$ having the same path on $(-\infty, t]$. Knowing this kind of equivalence class $[\omega_1]_t = \{\omega \in \Omega | B_s^H(\omega) = B_s^H(\omega_1), \forall s \in (-\infty, t]\}$ from the observance of the past, as we will see, the distribution of future realizations will again be normal. Furthermore, we'll be able to specify the distribution by use of the available information yielding an adjustment of the expected value as well as a variance reduction. As a first step, the following theorem gives a representation formula for conditional expectation.

Theorem 2 *Let $B_s^H, s \in \mathbb{R}$ be a fractional Brownian motion with $\frac{1}{2} < H < 1$. For each $T > t > 0$, the conditional expectation of B_T^H based on $\mathfrak{F}_t^H$ can be represented by:*

$$\hat{B}_{T,t}^H = E[B_T^H | \mathfrak{F}_t^H] = B_t^H + \int_{-\infty}^t g(T - t, s - t) dB_s^H \tag{4}$$

where

$$g(v,w) = \frac{\sin(\pi(H-\frac{1}{2}))}{\pi}(-w)^{-H+\frac{1}{2}}\int_0^v \frac{x^{H-\frac{1}{2}}}{x-w}dx$$
$$= \frac{\sin(\pi(H-\frac{1}{2}))}{\pi}\left(\frac{1}{H-\frac{1}{2}}(\frac{-w}{v})^{-H+\frac{1}{2}} - \beta_{v/(v-w)}\left(H-\frac{1}{2},\frac{3}{2}-H\right)\right)$$

and $\beta_{\cdot}(\cdot,\cdot)$ is the incomplete Beta function.

The result is due to Nuzman and Poor (2000) and is an extension of the result of Gripenberg and Norros (1996) who proved the theorem for the case $t = 0$. Note that for technical reasons we translated the formula of Nuzman and Poor (2000) to the original notation of Gripenberg and Norros (1996). The proof uses both the self-similarity and the Gaussian character of fractional Brownian motion.

For prediction purposes we are interested in the conditional distribution of B_T^H within its equivalence class resulting of the observation of the historical path. Let ω_1 be a representative of this equivalence class. We state the following theorem:

Theorem 3 *The conditional distribution of B_T^H based on the observation $[\omega_1]_t$ is normal with the following moments:*

$$E[B_T^H|\mathfrak{F}_t^H](\omega_1) = B_t^H + \int_{-\infty}^t g(T-t, s-t)dB_s^H(\omega_1) := B_t^H + \hat{\mu}_{T,t} \quad (5)$$

$$Var\left[B_T^H|\mathfrak{F}_t^H\right](\omega_1) = E\left[(B_T^H - \hat{B}_{T,t}^H)^2|\mathfrak{F}_t^H\right](\omega_1) = \rho_H(T-t)^{2H} := \hat{\sigma}_{T,t}^2 \quad (6)$$

with

$$\rho_H = \frac{\sin(\pi(H-\frac{1}{2}))}{\pi(H-\frac{1}{2})}\frac{\Gamma(\frac{3}{2}-H)^2}{\Gamma(2-2H)} \quad (7)$$

For the proof, see Appendix A.

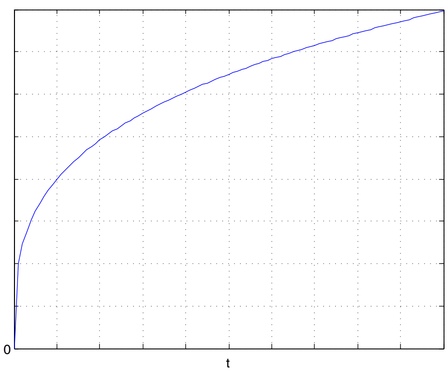

Figure 1 The concavity of $\hat{\mu}_{T,t}$

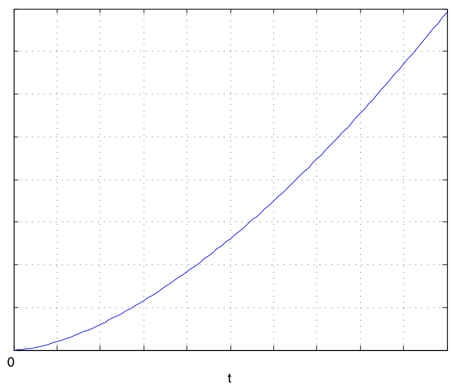

Figure 2 The convexity of $\hat{\sigma}_{T,t}^2$

The dependence of the first moment on the forecasting horizon $\tau = T - t$ is qualitatively of order $\tau^{H-\frac{1}{2}}$ and therefore implies concavity, whereas the relation between τ and $\hat{\sigma}^2_{T,t}$ apparently is of order τ^{2H} which yields a convex curve (see Figure 1 and Figure 2).

Figure 3 shows the shape of ρ_H for $\frac{1}{2} < H < 1$. Obviously, the factor is between 0 and 1, confirming the narrowing of conditional variance mentioned above. Note that as H tends to $\frac{1}{2}$, ρ_H tends to 1 as well as $g(T-t, u-t)$ and therefore $\hat{\mu}_{T,t}$ equals zero, yielding $N(B^H_t, T-t)$ as limit distribution. So, again the limit of the fractional case coincides with the results of the Markovian case where conditional equals unconditional distribution and the present value is the best forecast of the future. On the other hand, as H tends to 1, ρ_H nears zero, suggesting a deterministic process in the limit of perfect dependence.

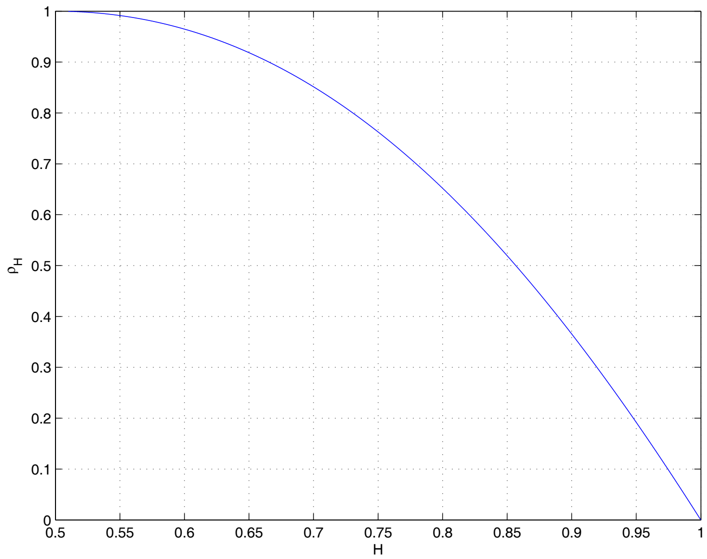

Figure 3 Shape of the narrowing factor ρ_H

We also point out, that, whereas the conditional variance only depends on H, the conditional mean is really path-dependent and has to be calculated by means of equation (5) which actually means evaluating the past. However, it seems to be quite difficult to make observations of an infinite past. In the next section we focus on a finite observation interval.

3.2 Prediction based on a partial knowledge about the past

For practical purposes it is desirable to make predictions that are based on only a part of the past and to go back only to a finite point of time $t - a$, that is we restrict ourselves to a finite observation interval of length a and regard the distribution of B_T^H conditional on $\mathfrak{F}_{t,a}^H = \sigma(B_s^H, t-a \leq s \leq t)$ which is the σ-field generated by all $B_s^H, t-a \leq s \leq t$.

We state the following theorem concerning this kind of conditional expectation, denoted by $\hat{B}_{T,t,a}^H$:

Theorem 4 *Let $B_s^H, s \in \mathbb{R}$ be a fractional Brownian motion with $\frac{1}{2} < H < 1$. For all $T, t, a > 0$, the conditional expectation of B_T^H based on $\mathfrak{F}_{t,a}^H$ can be represented as follows:*

$$\hat{B}_{T,t,a}^H = E[B_T^H | \mathfrak{F}_{t,a}^H] = \int_{t-a}^{t} g_a(T-t, s-t) dB_s^H \tag{8}$$

where

$$g_a(u,v) = \frac{\sin(\pi(H-\frac{1}{2}))}{\pi}(-v)^{-H+\frac{1}{2}}(a+v)^{-H+\frac{1}{2}} \int_0^u \frac{x^{H-\frac{1}{2}}(x+a)^{H-\frac{1}{2}}}{x-v} dx$$

Again, we can derive statements concerning conditional distribution of fractional Brownian motion, this time based on limited knowledge about the past, which is expressed by the restriction to the equivalence class $[\omega_1]_t^a = \{\omega \in \Omega | B_s^H(\omega) = B_s^H(\omega_1), \forall t-a \leq s \leq t\}$:

Theorem 5 *The conditional distribution of B_T^H based on the observation $[\omega_1]_t^a$ is normal with the following moments:*

$$\hat{\mu}_{T,t,a} = E[B_T^H | \mathfrak{F}_{t,a}^H](\omega_1) = \int_{t-a}^{t} g_a(T-t, s-t) dB_s^H(\omega_1) \tag{9}$$

$$\begin{aligned} \hat{\sigma}_{T,t,a}^2 &= Var\left[B_T^H | \mathfrak{F}_{t,a}^H\right](\omega_1) := E\left[(B_T^H - \hat{B}_{T,t,a}^H)^2 | \mathfrak{F}_{t,a}^H\right](\omega_1) \\ &= (T-t)^{2H}(1-\rho_{H,a}) \end{aligned} \tag{10}$$

with

$$\rho_{H,a} := 1 - H \int_0^{\frac{a}{T-t}} g_{\frac{a}{T-t}}(1,-s)\left((1+s)^{2H-1} - s^{2H-1}\right) ds$$

The proof of theorem 4, can be seen in Nuzman and Poor (2000), however in a different notation, as we used again a representation referring to that of Gripenberg and Norros (1996), who derived the result for $t = 0$. The argumentation of the proof of theorem 5 is equivalent to the case of infinite historical information and can be omitted at this point.

It's worth noting that Gripenberg and Norros (1996) showed that as soon as the observation interval becomes as large as the interval that should be predicted, $\rho_{H,a}$ tends to ρ_H or $\hat{\sigma}_{T,t,a}^2$ to $\hat{\sigma}_{T,t}^2$ respectively. So, concerning the variance, a limited historical observation interval is justified, whereas the influence of additionally observed historical increments on the conditional mean won't vanish, yet is decreasing.

4 Risk preference based option pricing in a fractional Brownian market

4.1 Fractional European option prices

In this section we look again at the fractional Brownian market presented in section 2. In the sequel we are further interested in the price at time t of a European call on S with maturity T and strike K.

As mentioned above, the existence of a minimal amount of time lying between two consecutive transactions, takes it toll in regard to the feasibility of pricing approaches based on no-arbitrage arguments with a continuously adjusted replicating portfolio. Therefore it seems to be natural to focus on preference based equilibrium pricing approaches. We do this in a very simple but all the more illustrative way, assuming risk-neutral investors, yet possessing and using information about the past. We hence regard the discounted conditional expected value of a contingent claim based on the observation of $[\omega_1]_t$:

$$C_{T,H}(t) = e^{-r(T-t)} E\left[max(S_T - K)|\mathfrak{F}_t^H\right]$$

The calculation is an analogon to the case of Brownian motion, however using the respective tools of fractional calculus. First we want to consider the conditional distribution of S_T given $[\omega_1]_t = \{\omega \in \Omega | B_s^H(\omega) = B_s^H(\omega_1), \forall s \in (-\infty, t]\}$. For that purpose we introduce the notation of the conditional process $\tilde{S}_s = S_s|[\omega_1]_t$, that is we restrict the process to a part of the probability space $(\Omega, \mathfrak{A}, P)$, namely to the space generated by the equivalence class $[\omega_1]_t$, which is $([\omega_1]_t, \sigma([\omega_1]_t), \tilde{P})$. The probability measure $\tilde{P}$ of course equals the conditional probability $\hat{P}$ so that for any process X the accordance of $\tilde{E}(\tilde{X}_T)$ and $E[X_T|\mathfrak{F}_t^H](\omega_1)$ immediately follows. We further look at the dynamics of $\ln(\tilde{S}_T)$, applying a conditional version of the fractional Itô theorem 1:

Theorem 6 *For $s > t$ let $\tilde{S}_s$ be the conditional process of geometric fractional Brownian motion as above. For $F(s, \tilde{S}_s)$ once continuously differentiable with respect to s and twice with respect to $\tilde{S}_s$ we obtain under certain regularity conditions:*

$$\begin{aligned} F(T, \tilde{S}_T) = F(t, \tilde{S}_t) &+ \int_t^T \frac{\partial}{\partial s} F(s, \tilde{S}_s)\, ds \\ &+ \int_t^T \mu(s) \frac{\partial}{\partial x} F(s, \tilde{S}_s) \tilde{S}_s\, ds + \sigma \int_t^T \frac{\partial}{\partial x} F(s, \tilde{S}_s) \tilde{S}_s\, d\tilde{B}_s^H \\ &+ \rho_H H \sigma^2 \int_t^T (s-t)^{2H-1} \frac{\partial^2}{\partial x^2} F(s, \tilde{S}_s) \tilde{S}_s^2\, ds \end{aligned}$$

For the proof, see the Appendix B. With $F(s, \tilde{S}_s) = \ln \tilde{S}_s$ we get

$$\ln\left(\tilde{S}_T\right) = \ln \tilde{S}_t + \int_t^T \mu(s)\, ds - \frac{1}{2}\rho_H \sigma^2 (T-t)^{2H} + \sigma(\tilde{B}_T^H - \tilde{B}_t^H)$$

The first three terms being deterministic at time t, we obtain the distribution of $\ln\left(\tilde{S}_T\right)$ by means of the foregoing considerations and application of theorem 3. We deduce that the logarithm of the conditional process $\tilde{S}_T$ is normally distributed with the following moments:

$$\begin{aligned} m &= \tilde{E}\left(\ln\left(\tilde{S}_T\right)\right) = E\left[\ln\left(\tilde{S}_T\right)|\mathfrak{F}_t^H\right](\omega_1) \qquad (11) \\ &= \ln S_t + \int_t^T \mu(s)\,ds - \frac{1}{2}\rho_H\sigma^2(T-t)^{2H} + \sigma\hat{\mu}_{T,t} \\ v &= \tilde{E}\left(\ln(\tilde{S}_T) - m\right)^2 = E\left[(\ln\left(\tilde{S}_T\right) - m)^2|\mathfrak{F}_t^H\right](\omega_1) \qquad (12) \\ &= \rho_H\sigma^2(T-t)^{2H} \end{aligned}$$

where $\hat{\mu}_{T,t}$ and ρ_H are as in section 3.

From now on, the necessary steps for the derivation of the pricing formulae are well-known. We assert that, $\ln(\tilde{S}_T)$ being $N(m,v)$ distributed on $([\omega_1]_t, \sigma([\omega_1]_t), \tilde{P})$, $\tilde{S}_T$ must be log-normally distributed thereon with moments

$$\begin{aligned} M &= \exp(m + \frac{1}{2}v) = S_t e^{\int_t^T \mu(s)\,ds + \sigma\hat{\mu}_{T,t}} \\ V &= \exp(2m+2v) - \exp(2m+v) = S_t^2 e^{2\int_t^T \mu(s)\,ds}\left(e^{\rho_H\sigma^2(T-t)^{2H}} - 1\right) \end{aligned}$$

For equilibrium reasons, a risk-neutral investor should be indifferent between buying the stock and holding the amount S_t of the riskless asset. That is, expectations must be equal, or more formally

$$\begin{aligned} E(\tilde{S}_T|\mathfrak{F}_t^H) &= E(S_t e^{r(T-t)}) \qquad \text{or} \\ S_t e^{\int_t^T \mu(s)\,ds + \sigma\hat{\mu}_{T,t}} &= S_t e^{r(T-t)}. \end{aligned}$$

This leads to

$$\int_t^T \mu(s)\,ds = r(T-t) - \sigma\hat{\mu}_{T,t}. \qquad (13)$$

The latter equation can be interpreted in the following way: The expected return of the stock can be split up into a deterministic part $\int_t^T \mu(s)\,ds$ and one that is due to the stochastics of fractional Brownian motion, which is the historically induced shift of the distribution $\sigma\hat{\mu}_{T,t}$. For instance, a positive historical trend results in a distributional upward shift, that is an increased mean for the stochastic part of geometric fractional Brownian motion. But, as we assumed the interest rate r to be constant over time, in equilibrium, this effect will be compensated by a converse adjustment of the deterministic part of the stock process. So the sum of $\int_t^T \mu(s)\,ds$ and $\sigma\hat{\mu}_{T,t}$ must always equal the riskless interest rate.

In combination with equations (11) and (12) we obtain

$$m = \ln S_t + r(T-t) - \frac{1}{2}\rho_H \sigma^2 (T-t)^{2H} \tag{14}$$
$$v = \rho_H \sigma^2 (T-t)^{2H}. \tag{15}$$

Note that as $H \to \frac{1}{2}$ the limits of these moments are

$$m = \ln S_t + (r - \frac{1}{2}\sigma^2)(T-t)$$
$$v = \sigma^2 (T-t)$$

So, as expected, in the Brownian case, the conditional distribution coincides with the unconditional one.

The associated density of the conditional process $\tilde{S}_T$ – which naturally is the conditional density of S_T based on the observation $[\omega_1]_t$– is as follows:

$$f(x)|_{[\omega_1]_t} = \frac{1}{x\sqrt{2\pi v}} e^{-\frac{1}{2}\frac{(lnx-m)^2}{v}} \mathrm{I}_{[x>0]}$$

The well-known calculations lead to the following presentation for the price of the European call:

$$\begin{aligned} C_{T,H}(t) &= e^{-r(T-t)} E\left[max(S_T - K)|\mathfrak{F}_t^H\right] \\ &= S_t e^{m+\frac{1}{2}v - r(T-t)} \Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \end{aligned}$$

where

$$d_1^H = \frac{m + v - \ln K}{\sqrt{v}}$$
$$d_2^H = \frac{m - \ln K}{\sqrt{v}} = d_1 - \sqrt{v}$$

Inserting the terms for m and v of equations (14) and (15) we obtain the pricing formula for the fractional European call:

Theorem 7 *The price of a fractional European call with strike K and maturity T valued by a risk-neutral investor is given by the following formula:*

$$C_{T,H}(t) = S_t \Phi(d_1^H) - Ke^{-r(T-t)}\Phi(d_2^H) \tag{16}$$

where

$$d_1^H = \frac{\ln(\frac{S_t}{K}) + r(T-t) + \frac{1}{2}\rho_H \sigma^2 (T-t)^{2H}}{\sqrt{\rho_H}\sigma(T-t)^H}$$
$$d_2^H = \frac{\ln(\frac{S_0}{K}) + r(T-t) - \frac{1}{2}\rho_H \sigma^2 (T-t)^{2H}}{\sqrt{\rho_H}\sigma(T-t)^H} = d_1^H - \sqrt{\rho_H}\sigma(T-t)^H$$

Following the same arguments as in the derivation of theorem 7, we receive the price of the appropriate European put:

$$P_{T,H}(t) = Ke^{-r(T-t)}\Phi(-d_2^H) - S_t\Phi(-d_1^H) \tag{17}$$

Again, consider the limit as $H \to \frac{1}{2}$, where the familiar risk-neutral valuation formulae are obtained.

We take a first look at the values of the fractional European call option for different Hurst parameters H. Apparently in the case displayed in Figure 4, an increase of dependence comes along with a decrease of the option value. But that is only half the truth as will be shown in the following subsection.

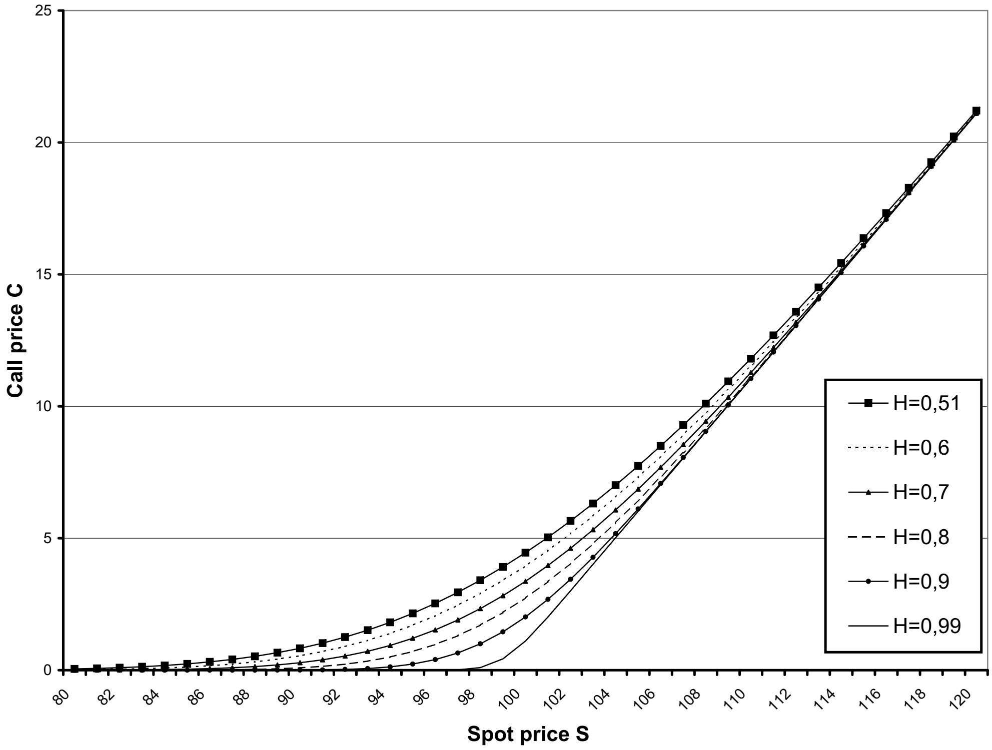

Figure 4 Price of the fractional European call with varying Hurst parameter H (chosen parameters: $r = 0.02$, $K = 100$, $\sigma = 0.2$, $T - t = 0.25$)

4.2 The fractional Greeks

As we showed in the preceding section, in the course of our simplified analysis assuming risk-neutral investors, the equilibrium condition rules out the influence of the conditional mean on the fractional call price, that is we can focus on the variance effects. Table 1 gives an overview of the partial derivatives of the call price formula, the so-called fractional Greeks.

Table 1 The fractional Greeks

$\Delta_H = \frac{\partial C_H}{\partial S}$	$\Phi(d_1^H)$
$\Gamma_H = \frac{\partial^2 C_H}{\partial S^2}$	$\frac{\varphi(d_1^H)}{S_t\sqrt{\rho_H}\sigma(T-t)^H}$
$\Theta_H = \frac{\partial C_H}{\partial t}$	$-H\frac{S_t\varphi(d_1^H)\sqrt{\rho_H}\sigma}{(T-t)^{1-H}} - rKe^{-r(T-t)}\Phi(d_2^H)$
$\varrho_H = \frac{\partial C_H}{\partial r}$	$KTe^{-r(T-t)}\Phi(d_2^H) - (T-t)S_t\Phi(d_1^H)$
$\Lambda_H = \frac{\partial C_H}{\partial \sigma}$	$S_t\varphi(d_1^H)\sqrt{\rho_H}(T-t)^H$

The proof of the formulae is straightforward. We underline that as $H \to \frac{1}{2}$, $(T-t)^H$ becomes $\sqrt{T-t}$, ρ_H tends to 1 and d_1^H becomes d_1 and the well-known parameters of the Markovian case are obtained. So again, the fractional solution in the limit also yields the results of classical Brownian theory.

By means of these partial derivatives it is furthermore possible to illustrate that also a special case of the Feynman- Kac formula can be translated to the fractional context. At time t, the price $V(t, S_t)$ of a derivative – conditional mean of a payoff function $p(T, S_T)$ discounted under risk-neutrality–

$$V(t, S_t) = e^{-r(T-t)}E\left[p(T, S_T)|\mathfrak{F}_t^H\right](\omega_1)$$

is the solution of the partial differential equation

$$rS_tV_S(t, S_t) + V_t(t, S_t) + H\rho_H\sigma^2S_t^2(T-t)^{2H-1}V_{SS}(t, S_t) - rV(t, S_t) = 0$$

The proof is similar to the classical case, using the conditional version of the fractional Itô Theorem. Insertion of the derived partial derivatives and the formula of the call price confirms the validity of equation (16).

The preceding results confirm the high degree of transferability of the classical concepts into the fractional framework. However, an aspect of additional interest arises from the consideration of the partial derivative with respect to the Hurst parameter H, which will be denoted by η. To get an ex ante idea of what we examine, recall that the Hurst parameter indicates the process-immanent level of persistence. While $H = \frac{1}{2}$ ensures independent increments and hence a Markovian process, larger values of H exhibit a certain extent of dependence. The question is, in which manner such an increase of dependence influences the price of the fractional call.

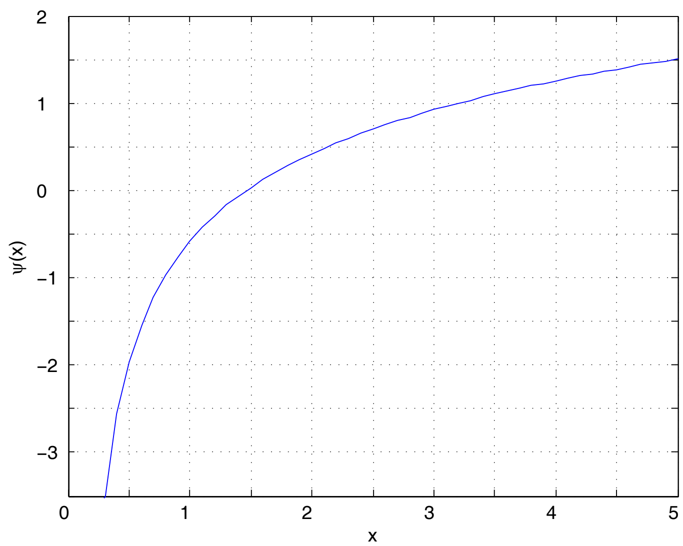

Figure 5 The digamma function on $\mathbb{R}_+$

We thus differentiate equation (16) with respect to H and get

$$\begin{aligned}\eta = \frac{\partial C}{\partial H} &= S_t\varphi(d_1^H)\frac{\partial d_1^H}{\partial H} - Ke^{-r(T-t)}\varphi(d_2^H)\frac{\partial d_2^H}{\partial H} \\ &= S_t\varphi(d_1^H)\frac{\partial\left(\sqrt{\rho_H}\sigma(T-t)^H\right)}{\partial H} \\ &= S_t\varphi(d_1^H)\frac{\partial\sqrt{v}}{\partial H}\end{aligned} \tag{18}$$

We further look at $\frac{\partial v}{\partial H}$ and obtain

$$\frac{\partial v}{\partial H} = \rho_H\sigma^2(T-t)^{2H}\left(\psi_0(1-H) - \psi_0(H+\frac{1}{2}) + 2\ln 2 + 2\ln(T-t)\right) \tag{19}$$

where ψ_0 denotes the digamma function. For the proof of the latter equality see Appendix C.

Note that the digamma function $\psi_0(x)$ for $x > 0$ is strictly monotonic increasing but concave, the negative axis of ordinates being vertical asymptote as x tends to zero (see Figure 5).

Therefore the difference $\psi_0(1-H) - \psi_0(H+\frac{1}{2})$ is strictly monotonic decreasing for $\frac{1}{2} < H < 1$ and its maximum is received for $H \to \frac{1}{2}$. In this case we get

$$\begin{aligned}\lim_{H\to\frac{1}{2}}\left[\psi_0(1-H) - \psi_0(H+\frac{1}{2})\right] &= \psi_0(\frac{1}{2}) - \psi_0(1) \\ &= -\gamma - (2\ln 2 + \gamma) = -2\ln 2\end{aligned}$$

where γ denotes the Euler-Mascheroni constant.

Summarizing we can state the following theorem, denoting by τ the time to maturity, that is $\tau = T - t$.

Theorem 8 *The partial derivative of the fractional call price C with respect to the Hurst parameter H is given by*

$$\eta = S_t \varphi(d_1^H) \sqrt{\rho_H} \sigma (T-t)^H \frac{\left(\psi_0(1-H) - \psi_0(H+\frac{1}{2}) + 2\ln 2 + 2\ln(T-t)\right)}{2}$$

and has the following properties:

1. *For a fix $\tau \leq 1$, it holds:*
$$\frac{\partial C}{\partial H}(H) \quad < 0 \quad \forall\, \frac{1}{2} < H < 1$$
2. *For a fix $\tau > 1$, there exists a critical Hurst parameter $\frac{1}{2} < \bar{H} < 1$, so that:*
$$\begin{aligned} &\frac{\partial C}{\partial H}(\bar{H}) = 0 \\ &\frac{\partial C}{\partial H}(H) > 0 \quad \forall\, \frac{1}{2} < H < \bar{H} \\ &\frac{\partial C}{\partial H}(H) < 0 \quad \forall\, \bar{H} < H < 1 \end{aligned}$$

The results are immediate consequences of the preceding observations as well as of the properties of the natural logarithm. In order to be able to explain this phenomenon we recall that according to equation (19) the main effect arises from the product $\rho_H(\tau)^{2H}$, which is the variance v of the normal distribution of the conditional logarithmic stock price. But, with increasing H, the factors of v generate converse effects. The factor ρ_H concentrates the distribution – what we from now on call narrowing effect –, whereas the higher exponent of τ for $\tau > 1$ tends to enlarge the variance – which is further referred to as the power effect. The resulting effect thus depends on the scale of τ. For small τ, which means nearby distributional forecasts, both effects have a variance-reducing character so the call price decreases. On the other hand for $\tau > 1$, starting from the classical case $H = \frac{1}{2}$, the call price increases with higher level of persistence due to the power effect, but only up to the critical parameter $\bar{H}$, where this effect is fully compensated by the narrowing effect caused by ρ_H. With a further increase of H this confining character of ρ_H overbalances the power effect and the call price decreases.

Figure 6 illustrates these characteristics graphically, showing the relation between the Hurst parameter H and the call price for a fix initial price S_t. A brief look at the limit of the call price as H tends to 1 provides another fact that confirms our intuition with regards to fractional Brownian motion. With an increasing Hurst parameter, we obtain an increasing level of dependence, that is, the future price of the underlying becomes less volatile or uncertain. In the limit, we distinguish between two cases. For $S > e^{-r(T-t)}K$,

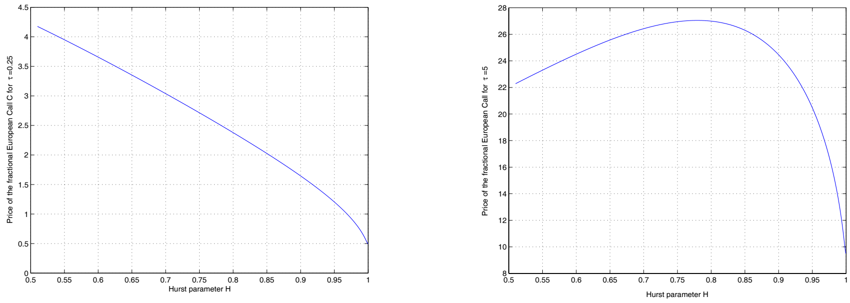

Figure 6 Maturity effect on the relation between Price of the fractional European call and Hurst parameter H (chosen parameters: $r = 0.02$, $S = 100$, $K = 100$, $\sigma = 0.2$).

d_1^H and d_2^H tend to infinity and for the call price we actually receive the difference between the initial stock price and the discounted strike price. On the other hand, if we have $S < e^{-r(T-t)}K$, d_1^H and d_2^H tend to $-\infty$, and the call price tends to zero. So in the case of perfect dependence, either the contracts value is zero right from the beginning or we get a simple forward contract under certainty.

5 Conclusion

The nature of fractional Brownian motion, especially its non-martingale property, doesn't allow for no arbitrage pricing methods within the common framework. Albeit restricting trading strategies to be non-continuous ensures absence of arbitrage, this non-continuity of trading strategies still rules out the common arbitrage pricing approach. In this paper we suggest a preference based pricing approach which allows us to renounce continuous tradability. This approach makes it reasonable and necessary to evaluate the historical information from the path of the stock price process.

The derived formulae draw their attractiveness from the fact, that the fractional pricing model includes the traditional Markovian case, so that the existing parallels enhance the understanding of fractional option pricing. Moreover the analysis of the partial derivative with respect to the Hurst parameter made it possible to point out the fractional particularities of the formulae. By name, these are the variance-based narrowing and power effects, which accord with the economic intuition concerning the phenomenon of persistence.

Appendix A: Proof of Theorem 3

The normality of the conditional distribution is an immediate consequence of the Gaussian character of the process B_t^H. It is well known that Gaussian processes like multivariate normal distributions assure the normality of all kinds of conditional densities. Intuitively, the mean of the conditional distribution should be defined by $\int_{\omega\in[\omega_1]_t} B_T^H(\omega)d\hat{P}(\omega)$ where $\hat{P}(\omega) = \frac{P(\omega)}{P([\omega_1]_t)}$ is the conditional probability of ω. The characterization of the conditional mean given in theorem 3 then easily follows from theorem 2 and the fact that the conditional expectation by definition satisfies:

$$\int_{\omega\in[\omega_1]_t} B_T^H(\omega)dP(\omega) = \int_{\omega\in[\omega_1]_t} \hat{B}_T^H(\omega)dP(\omega)$$

as $[\omega_1]_t \in \mathfrak{F}_t^H$. $\hat{B}_T^H$ being constant on $[\omega_1]_t$ we can rewrite this by

$$\int_{\omega\in[\omega_1]_t} B_T^H(\omega)dP(\omega) = \hat{B}_T^H(\omega_1)P([\omega_1]_t)$$

or

$$\hat{B}_T^H(\omega_1) = \int_{\omega\in[\omega_1]_t} B_T^H(\omega)d\left(\frac{P(\omega)}{P([\omega_1]_t)}\right) = \int_{\omega\in[\omega_1]_t} B_T^H(\omega)d\hat{P}(\omega)$$

Respectively, the conditional variance should be defined by

$$\hat{\sigma}_{T,t}^2 = \int_{\omega\in[\omega_1]_t} \left[B_T^H(\omega) - \hat{B}_T^H(\omega)\right]^2 d\hat{P}(\omega)$$

which can be rewritten – applying the same argument as above – by

$$\hat{\sigma}_{T,t}^2 = E\left[(B_T^H - \hat{B}_T^H(\omega_1))^2|\mathfrak{F}_t^H\right](\omega_1)$$

But $\hat{B}_T^H$ is the orthogonal projection of B_T^H on the span of $\{B_s^H, s \le t\}$. So the coprojection $(B_T^H - \hat{B}_T^H)$ or $((B_T^H - B_t^H) - \hat{\mu}_{T,t})$ respectively as well as the squared terms are orthogonal to and therefore independent of $\{B_s^H, s \le t\}$, so that the conditional expectation $E\left[(B_T^H - \hat{B}_T^H(\omega_1))^2|\mathfrak{F}_t^H\right]$ is non-random. Hence we can omit the argument ω_1 in the sequel, add expectation operators and write:

$$\begin{aligned}
\hat{\sigma}_{T,t}^2 &= E\left[(B_T^H - \hat{B}_T^H)^2|\mathfrak{F}_t^H\right] = E\left(E\left[((B_T^H - B_t^H) - \hat{\mu}_{T,t})^2|\mathfrak{F}_t^H\right]\right) \\
&= E\left(E\left[(B_T^H - B_t^H)^2|\mathfrak{F}_t^H\right] - 2E\left[(B_T^H - B_t^H)\hat{\mu}_{T,t}|\mathfrak{F}_t^H\right] + E\left[\hat{\mu}_{T,t}^2|\mathfrak{F}_t^H\right]\right) \\
&= E(B_T^H - B_t^H)^2 - 2E(\hat{\mu}_{T,t})^2 + E(\hat{\mu}_{T,t})^2 = E(B_T^H - B_t^H)^2 - E(\hat{\mu}_{T,t})^2
\end{aligned}$$

We now look at

$$\begin{aligned}
E(\hat{\mu}_{T,t})^2 &= E\left(\int_{-\infty}^t g\left((T-t),(s-t)\right) dB_s^H\right)^2 \\
&= \int_{-\infty}^t \int_{-\infty}^t g\left((T-t),(v-t)\right) g\left((T-t),(w-t)\right) \phi_H(v,w)dvdw \\
&= \int_0^\infty \int_0^\infty g\left((T-t),(-x)\right) g\left((T-t),(-y)\right) \phi_H(x,y)dxdy \\
&= (T-t)^{2H}(1-\rho_H),
\end{aligned}$$

where $\phi_H(a,b) = H(2H-1)|a-b|^{2H-2}$ and where we used Proposition 2.2 of Gripenberg and Norros (1996) and then substituted $x = t_0 - v$ and $y = t_0 - w$. The correctness of the last equality is carried out in the proof of Corollary 3.2 of Gripenberg and Norros (1996) where we refer to for more details.
With that and

$$\begin{aligned} E((B_T^H - B_t^H)^2) &= E(B_T^H)^2 - 2E(B_T^H B_t^H) + E(B_t^H)^2 \\ &= T^{2H} - (T^{2H} + t^{2H} - (T-t)^{2H}) + t^{2H} = (T-t)^{2H} \end{aligned}$$

we get

$$\hat{\sigma}_{T,t}^2 = (T-t)^{2H} - (T-t)^{2H}(1-\rho_H) = \rho_H (T-t)^{2H}$$

which completes the proof.

Appendix B: A conditional version of the fractional Itô Theorem

We sketch the derivation of theorem 6 modifying the proof of Bender (2003) for the unconditional case. For $\frac{1}{2} < H < 1$, the Riemann-Liouville fractional integrals are defined by

$$\begin{aligned} I_-^{H-\frac{1}{2}} f(x) &= \frac{1}{\Gamma(H-\frac{1}{2})} \int_x^\infty f(s)(s-x)^{H-\frac{1}{2}}\, ds \\ I_+^{H-\frac{1}{2}} f(x) &= \frac{1}{\Gamma(H-\frac{1}{2})} \int_{-\infty}^x f(s)(x-s)^{H-\frac{1}{2}}\, ds. \end{aligned}$$

The operators $M_\pm^H$ are defined by

$$\begin{aligned} & M_\pm^H f = K_H I_\pm^{H-\frac{1}{2}} f \\ \text{where} \qquad & K_H = \Gamma(H+\frac{1}{2}) \sqrt{\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}}. \end{aligned}$$

The fractional Girsanov formula reads as follows:

Theorem 9 *Let $\frac{1}{2} < H < 1$ and B_s^H be a fractional Brownian motion with respect to the measure P. Furthermore let Q_f be the measure with $\frac{dQ_f}{dP} = \exp(\int_{\mathbb{R}} f(u)\, dB_u - \frac{1}{2}\int_{\mathbb{R}} f(u)^2\, du)$, where B_u is the generating Brownian motion. Then $\breve{B}_s^H$, defined via*

$$\breve{B}_s^H = B_s^H - \int_0^s M_+^H f(u)\, du$$

is a fractional Brownian motion with respect to Q_f.

Using this formula and according to section 4, we obtain the distribution of the $[\omega_1]_t$ - restricted process $\tilde{B}_T^H$ with respect to Q_f to be normal with mean $\breve{m}_{T,t} + \int_0^t M_+^H f(s)\,ds$ and variance $\rho_H(T-t)^{2H}$, where $\breve{m}_{T,t} = \breve{B}_t^H + \int_{-\infty}^t g(T-t, s-t) d\breve{B}_s^H(\omega_1)$ is the conditional mean of the fractional Brownian motion $\breve{B}_s^H$ abd $\breve{B}_s$ is the generating Brownian motion of $\breve{B}_s^H$. Knowing this, we can replace the moments of the unconditional case by those of the conditional case and successively modify theorems 1.2.8, 2.6.3 and 2.6.5 of Bender (2003). In particular we can take theorem 2.6.5 and replace the unconditional variance term $|M_-^H(1_{[0,s]})\sigma|_0^2$ – which as expected for a constant σ equals $\sigma^2 s^{2H}$ – by the conditional variance $\rho_H \sigma^2 (s-t)^{2H}$. We obtain theorem 6.

Appendix C: The partial derivative $\frac{\partial v_T}{\partial H}$

Recall that $v_T = \sigma^2 \rho_H (T-t)^{2H}$. We first look at $\frac{\partial \rho_H}{\partial H}$ and differentiate the nominator $n(H) = \sin(\pi(H-\frac{1}{2}))\Gamma(\frac{3}{2}-H)^2$ and the denominator $d(H) = \pi(H-\frac{1}{2})\Gamma(2-2H)$ separately. For that purpose, note that $\Gamma'(x) = \Gamma(x)\psi_0(x)$ where ψ_0 denotes the digamma function. We get

$$
\begin{aligned}
\frac{\partial n}{\partial H} &= \pi \cos(\pi(H-\frac{1}{2}))(\Gamma(\frac{3}{2}-H))^2 \\
&\quad - \sin(\pi(H-\frac{1}{2}))2\Gamma(\frac{3}{2}-H)\Gamma(\frac{3}{2}-H)\psi_0(\frac{3}{2}-H) \\
&= (\Gamma(\frac{3}{2}-H))^2 \sin(\pi(H-\frac{1}{2})) \left[\pi \cot(\pi(H-\frac{1}{2})) - 2\psi_0(\frac{3}{2}-H)\right] \\
\frac{\partial d}{\partial H} &= \pi\Gamma(2-2H) - 2\pi(H-\frac{1}{2})\Gamma(2-2H)\psi_0(2-2H) \\
&= \pi\Gamma(2-2H)\left[1-(2H-1)\psi_0(2-2H)\right]
\end{aligned}
$$

Using the quotient rule, we obtain

$$
\frac{\partial \rho_H}{\partial H} = \rho_H \left[\pi \cot(\pi(H-\frac{1}{2})) - 2\psi_0(\frac{3}{2}-H) - \frac{1}{H-\frac{1}{2}} + 2\psi_0(2-2H)\right]
$$

We further make use of the following properties of the digamma function (see Abramowitz and Stegun (1972), section 6.3):

$$
\begin{aligned}
\pi \cot(\pi x) &= \psi_0(1-x) - \psi_0(x) \\
\psi_0(x+1) &= \psi_0(x) + \frac{1}{x} \\
\psi_0(2x) &= \frac{1}{2}\left(\psi_0(x) + \psi_0(x+\frac{1}{2}) + 2\ln 2\right)
\end{aligned}
$$

Thus we can write

$$\begin{aligned}\frac{\partial\rho_H}{\partial H} &= \rho_H\left(\psi_0(\frac{3}{2}-H)-\psi_0(H-\frac{1}{2})-2\psi_0(\frac{3}{2}-H)\right.\\ &\qquad\left.+\psi_0(1-H)+\psi_0(\frac{3}{2}-H)+2\ln 2\right)\\ &= \rho_H\left(\psi_0(1-H)-\psi_0(H-\frac{1}{2})-\frac{1}{H-\frac{1}{2}}+2\ln 2\right)\\ &= \rho_H\left(\psi_0(1-H)-\psi_0(H+\frac{1}{2})+2\ln 2\right).\end{aligned}$$

Finally we can calculate $\frac{\partial v_T}{\partial H}$:

$$\begin{aligned}\frac{\partial v_T}{\partial H} &= \frac{\partial\sigma^2\rho_H(T-t)^{2H}}{\partial H}\\ &= \sigma^2\left(\rho_H(\psi_0(1-H)-\psi_0(H+\frac{1}{2})\right.\\ &\qquad\left.+2\ln 2)(T-t)^{2H}+\rho_H 2\ln(T-t)(T-t)^{2H}\right)\\ &= \rho_H\sigma^2(T-t)^{2H}\left(\psi_0(1-H)-\psi_0(H+\frac{1}{2})+2\ln 2+2\ln(T-t)\right)\end{aligned}$$

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