Portfolio Selection: How to Integrate Complex Constraints

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Abstract

For the standard Mean-Variance model for portfolio selection with linear constraints, there are several algorithms that can efficiently compute both a single point on the Pareto front and even the whole front. Unfortunately, commonly used constraints (e.g. cardinality constraints or buy-in thresholds) result in the optimization problem to become intractable by standard algorithms. In this paper, two paradigms to deal with this kind of constraint are presented and their advantages and disadvantages are highlighted.

keywords: portfolio selection, complex constraints, mixed integer quadratic programming, metaheuristic, constraint handling

1 Introduction

Mean-Variance optimization is probably the most popular approach to portfolio selection. It was introduced more than 50 years ago in the pioneering work by Markowitz [1, 2]. The basic assumptions in his theory are a rational investor with either multivariate normally distributed asset returns or, in the case of arbitrary returns, a quadratic utility function [3]. If these assumptions are valid, Markowitz has shown that the optimal portfolio for the investor lies on the Mean-Variance *Efficient Frontier*. (He defined a portfolio of financial assets as efficient, if and only if for any given expected return, there is no other portfolio with lower variance, and for any given variance there is no other portfolio with higher expected return. The Efficient Frontier consists of all efficient portfolios.)
Computationally efficient algorithms that calculate a single point or even the whole Efficient Frontier have been discussed in a large number of publications. (For an introduction see e.g. [4], [5], and [6].)
Unfortunately, some difficulties crop up when this approach is used in mutual fund management. One of the main problems is caused by the obligation of the fund managers to comply with both contractual constraints defined by the prospectus of their particular fund and legal constraints laid down in the respective laws of the country where the shares of the fund are offered. Some – albeit not too many – of these rules can lead to constraints that can't be easily expressed in a way that permits standard algorithms to handle them. Three examples for such constraints are the so called *cardinality constraints*, *buy-in thresholds*, and a rule especially important for German mutual funds, the *5-10-40-constraint*.
Cardinality constraints normally are rules that determine the number of securities which are allowed to be part of the portfolio.
Buy-in thresholds set a lower limit on all assets that are part of the portfolio.
The 5-10-40-constraint is based on §60(1) of the German investment law [7]. It defines an upper limit for each individual asset and for the sum of all "heavyweights" in the portfolio.
Taking into account these more complex constraints, two quite different approaches can be taken to find solutions for the portfolio selection problem. The first one uses a suitable mixed integer solver, the second one uses metaheuristics to compute the solutions.

This paper is organized as follows:
In Section 2 we formulate the standard model for portfolio selection and present several categories of constraints that − when included − make the optimization problems difficult to handle. Section 3 describes the mixed integer and the metaheuristic approach to solve these problems. The advantages and disadvantages of these algorithms are discussed in Section 4. Section 5 concludes with a short summary.

2 Mean-variance Model and Extensions

2.1 The Standard Model

The standard Mean-Variance (M-V) model can be described as a bicriterial optimization problem:

SMVM

$$\min V(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} \quad (1a)$$
$$\max E(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{\mu} \quad (1b)$$
$$\boldsymbol{x}^T \boldsymbol{e} = 1 \quad (1c)$$
$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \quad (1d)$$
$$\boldsymbol{x} \geq \boldsymbol{0} \quad (1e)$$

where the element x_i of the vector $\boldsymbol{x}$ denotes the fraction of the budget invested in asset i. $\boldsymbol{Q}$ is the covariance matrix, $\boldsymbol{e}$ represents the unit vector, $\boldsymbol{\mu}$ is the vector of expected returns of all assets. An this formulation of the M-V-model it is assumed that all additional constraints beside the budget constraint (1c) are expressed as inequalities (1d). $\boldsymbol{A}$ denotes the matrix of inequality coefficients with $\boldsymbol{b}$ as the respective right hand sides. Additionally, we assume that short sales are not allowed (1e).
There exist several different formulations for the linear constraints of the optimization problem which can be easily transformed into each other without any loss of information ([6] pp. 24-27).
Depending on the intention of the optimization, there are two classical approaches to solve this problem:

1. If the focus is put on a single point on the Efficient Frontier, we can assume the expected return E of the portfolio is fixed. This removes one optimization criterion and adds an additional constraint. The resulting optimization problem is – as the covariance matrix is positive semidefinite – a convex quadratic programming problem (QP):

   MVQP

   $$\min V(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} \quad (2a)$$
   $$\boldsymbol{x}^T \boldsymbol{\mu} = E \quad (2b)$$
   $$\boldsymbol{x}^T \boldsymbol{e} = 1 \quad (2c)$$
   $$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \quad (2d)$$
   $$\boldsymbol{x} \geq \boldsymbol{0} \quad (2e)$$

   The solution can easily be computed using a QP-solver from any of the more advanced optimization software packages. A list of suitable programs and libraries can be found at [8].

Such a solution, however, represents only one point on the Efficient Frontier. If necessary, an approximation of the complete Pareto front can be computed by increasing (or decreasing) E iteratively.

2. If it is desired to calculate the complete Efficient Frontier, the obvious choice is an active set algorithm for parametric quadratic programming (see e.g. [9]), whose most known variant in the field of portfolio selection is the Critical Line Algorithm by Markowitz [2, 6, 10].

   Starting from one point on the Efficient Frontier, the algorithm computes a sequence of so called *corner portfolios* $\boldsymbol{x_1}, \ldots, \boldsymbol{x_m}$. These corner portfolios define the complete Efficient Frontier. If $\boldsymbol{x_i}$ and $\boldsymbol{x_{i+1}}$ are adjacent corner portfolios with expected returns E_i and E_{i+1}, $E_i \leq E_{i+1}$, then for every E_λ with $E_\lambda = \lambda E_i + (1-\lambda)E_{i+1}$, $\lambda \in [0,1]$ the optimal portfolio $\boldsymbol{x_\lambda}$ is calculated as $\boldsymbol{x_\lambda} = \lambda \boldsymbol{x_i} + (1-\lambda)\boldsymbol{x_{i+1}}$.

   For an efficient implementation of the Critical Line Algorithm see [5]

2.2 Extensions: Complex Constraints

This section is focused on three types of constraints: buy-in thresholds, cardinality constraints, and the 5-10-40-constraint. Portfolio selection problems that contain any of the three types can't be solved with conventional means (i.e. QP-solvers or the Critical Line Algorithm). Minimum transaction lots are another type of "complex" constraint often mentioned in the related articles (see e.g. [11, 12, 13, 14]). They are not considered any further, however, as several managers of mutual funds have stated in interviews that minimum transaction lots are not relevant in their daily work.

Buy-In Thresholds

Buy-in thresholds prevent assets with small weights from being included in the portfolio. Assets either are above a lower bound l, or they are not part of the portfolio at all. The main reason for such a constraint is to reduce transaction costs.
As shown in [12], if there are N possible investment alternatives, the buy-in threshold constraint can be formulated using binary variables $\rho_i,\ i = 1 \ldots N$ in the following way :

$$l\rho_i \leq x_i \leq u\rho_i \qquad \rho_i \in \{0,1\} \qquad i = 1 \ldots N$$

(u denotes the upper bound for each individual asset.)
Unfortunately, binary variables can't be handled by either a normal QP-solver or the Critical Line Algorithm. Binary variables are inherently nonconvex, and convexity of the search space and the objective function(s) is a prerequisite for both algorithms. But even if it is possible to model the problem without binary variables, the search space is nonconvex:

Proof: Assume the portfolios $\boldsymbol{x}$ and $\boldsymbol{y}$ are valid, and that variable $x_i = 0$, and variable $y_i = l$. Obviously any convex combination of $\boldsymbol{x}$, $\boldsymbol{y}$, apart from $\boldsymbol{x}$ and $\boldsymbol{y}$ themselves, is not valid.

Cardinality Constraints

Investors and fund managers often wish to control the number of assets in the fund they own/manage:
The fund manager might – due to monitoring reasons, or in order to reduce transaction costs – set an upper limit on the number of securities in a portfolio.
The investor, however, might prefer a well diversified portfolio, and therefore sets a minimum number of assets which the mutual fund must contain. Horniman et al. pointed out in [12], that such a constraint is intrinsically linked with buy-in thresholds: on the one hand, a high threshold limits the number of assets the portfolio can contain, and on the other hand, if there is no threshold at all, any x_i can be set to a very small value instead of being zero, meaning that the cardinality constraint doesn't effectively influence the solution.
The cardinality constrained portfolio selection problem with buy-in threshold and fixed return E can now be formulated as:

QP-CARD-TH

$$\min V(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} \tag{3a}$$

$$\boldsymbol{x}^T \boldsymbol{\mu} = E \tag{3b}$$

$$\boldsymbol{x}^T \boldsymbol{e} = 1 \tag{3c}$$

$$l\rho_i \leq x_i \leq u\rho_i \qquad \rho_i \in \{0, 1\} \qquad i = 1 \ldots N \tag{3d}$$

$$\sum_{i=1}^{n} \rho_i \ (\geq, \leq) \ N \tag{3e}$$

(3e) describes either an upper bound or a lower bound on the number of assets in the portfolio. To simplify the model, the additional linear constraints – beside the upper and lower bounds on the assets – have been removed.
The nonconvexity of the search space prevents the classical methods from being applicable here as well.

The 5-10-40-Constraint

§60(1) of the German investment law [7] states, roughly translated, that securities of the same issuer are allowed to amount to up to 5% of the net asset value of the mutual fund. They are allowed to amount to 10%, however, if the total of all of these assets is less than 40% of the net asset value. This constraint is especially interesting because it is the only one

based on German law that can't be incorporated into the standard model in the form of linear constraints.
Using the vector $\boldsymbol{\rho}$ consisting of binary variables $\rho_i,\ i = 1, \ldots, N$, the M-V optimization problem with 5-10-40-constraint and fixed return E can be formulated as follows:

QP-5-10-40

$$\min V(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} \tag{4a}$$
$$\boldsymbol{x}^T \boldsymbol{\mu} = E \tag{4b}$$
$$\boldsymbol{x}^T \boldsymbol{e} = 1 \tag{4c}$$
$$\boldsymbol{\rho}^T \boldsymbol{x} \leq 0.4 \tag{4d}$$
$$\boldsymbol{x} - 0.05\boldsymbol{\rho} \leq 0.05\boldsymbol{e} \tag{4e}$$
$$\rho_i \in \{0, 1\}\ \forall i = 1, \ldots, N \tag{4f}$$
$$\boldsymbol{x} \geq \boldsymbol{0} \tag{4g}$$

Obviously, this problem can't be handled by either a QP-solver or the Critical Line Algorithm. The search space is nonconvex here as well.
Proof: Assume that the portfolios $\boldsymbol{x}$ and $\boldsymbol{y}$ are valid, and that $x_i = 0.1,\ i = 1, \ldots, 4$ and $x_j = 0.05,\ j = 5, \ldots, 16$. Assume further that $y_i = x_i$ for all i except $y_4 = 0.05$ and $x_5 = 0.1$. Any convex combination of $\boldsymbol{x}$ and $\boldsymbol{y}$, apart from $\boldsymbol{x}$ and $\boldsymbol{y}$ themselves, is not valid: the sum of all assets with $x_i > 0.05$ is obviously larger than 40%.

3 Algorithms for Solving Extensions of the M-V-Model

3.1 Mixed Integer Approach

The extended M-V-model with buy-in thresholds and cardinality constraints presented in Section 2 can be categorized as a mixed integer programming problem with a quadratic objective function (MIQP). Unfortunately, due to constraint (4d), problem QP-5-10-40 does not fit into this category. By introducing an additional vector $\boldsymbol{t}$ of size N, the problem can be transformed into an equivalent MIQP-problem:

QP-5-10-40-MIQP

$$\min V(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} \tag{5a}$$
$$\boldsymbol{x}^T \boldsymbol{\mu} = E \qquad \boldsymbol{x}^T \boldsymbol{e} = 1 \tag{5b}$$
$$\boldsymbol{x} - 0.05\boldsymbol{\rho} \leq 0.05\boldsymbol{e} \tag{5c}$$
$$\boldsymbol{t} - 0.1\boldsymbol{\rho} \leq \boldsymbol{0} \qquad \boldsymbol{t} - \boldsymbol{x} \leq \boldsymbol{0} \tag{5d}$$
$$\boldsymbol{x} + 0.1\boldsymbol{\rho} - \boldsymbol{t} \leq 0.1\boldsymbol{e} \qquad \boldsymbol{t}^T \boldsymbol{e} \leq 0.4 \tag{5e}$$
$$\boldsymbol{x}, \boldsymbol{t} \geq \boldsymbol{0} \qquad \rho_i \in \{0, 1\}\ \forall i = 1, \ldots, N \tag{5f}$$

This modification has doubled the number of continuous variables, but the number of discrete variables remains the same.
Any of the available MIQP-solvers can be used to solve the model QP-CARD-TH (with or without cardinality constraint) or the model QP-5-10-40-MIQP. Software packages that contain such a solver can be found at [8], although – compared to the number of available QP-solvers – there are by far fewer programs with the required capability. Depending on the performance of the mixed integer code of the solver and the difficulty of the optimization problem, calculation time may vary strongly when this approach is used. If e.g. a problem with an upper bound on the number of securities is solved, and the cardinality constraint is not binding due to the fact that the number of assets is smaller than the upper bound, the runtime is essentially the same as if there were no cardinality constraint at all, given the MIQP-solver works efficiently. There is, however, no guarantee for getting the optimal portfolio quickly. If only a fixed calculation time is available, and the time runs out, the best valid portfolio calculated so far can be used instead — a normal practice when working with mixed integer solvers.
For further information about the mixed integer approach to the cardinality constrained problem without using an external MIQP-solver, the reader is referred to Bienstock [15]. Bienstock examined computational complexity of the problem, tested a self-developed branch-and-cut algorithm using disjunctive cuts, and discussed some of the implementation problems that occurred.
Jobst, Horniman et al. [12] use a branch-and-bound algorithm to solve the cardinality constrained problem with buy-in thresholds as well as a portfolio tracking problem. As they calculate many points on the efficient frontier, the available time is too short for individually solving each MIQP to optimality. Therefore, they limit the number of nodes for each point of expected return E in the search tree of the branch-and-bound algorithm. To speed up the algorithm further, a previously calculated solution for an adjacent point is used as a warm start solution for the new value of expected return.

3.2 Metaheuristic Approaches

The term *metaheuristic* normally describes an optimization principle that can be applied to not only one problem type, but is widely usably for many different problem categories. Very often, the basic idea behind a metaheuristic is taken from some naturally appearing phenomenon, be it a physical process like Simulated Annealing (SA) or biologically inspired algorithms like Ant Algorithms or Evolutionary Algorithms (EAs).

Related Work on the Use of Metaheuristics for Portfolio Selection

There are several publications discussing the use of metaheuristics to solve portfolio selection problems that have extensions which make the problem intractable with classical means.
Chang et al. [16] used a Genetic Algorithm, Tabu Search, and Simulated Annealing to solve portfolio selection problems where each solution had to contain a predetermined number of assets. Schaerf [17] improved on the work of Chang et al. by testing several neighborhood relations for Tabu Search.
Crama and Schyns [18] applied Simulated Annealing to a portfolio selection problem with cardinality constraints, turnover and trading restrictions.
Derigs and Nickel [19] used Simulated Annealing to solve a variation of Problem QP-5-10-40 with an additional cardinality constraint. In [20] they expanded their previous work, but the focus was put on developping a decision support system for portfolio selection.
A completely different approach was followed in [13, 14]. Streichert et al. did not set the expected return E as constant and then solved a separate optimization problem for each E, but applied a multi-objective EA instead. The big advantage of multi-objective EAs is their capability to produce an approximation of the complete Pareto Front in one run, which saves a lot of time. For further details concerning multi-objective EAs see e.g. [21].

Coding of Solutions and Constraint Handling

Common to all successful applications of metaheuristics in the field of portfolio selection seems to be the use of a vector of continuous variables $\boldsymbol{c} = (c_1, \dots, c_N)^T$ to represent the weights of the individual assets. Often, but not always, an additional vector of binary variables $\boldsymbol{k} = (k_1, \dots, k_N)^T$ is used to indicate if the asset is included in the portfolio at all. The latter vector permits an easy way to handle cardinality constraints.
If we have a portfolio selection problem that contains a cardinality constraint and buy-in thresholds, the decoding of the two vectors to get the actual portfolio could work as follows (see e.g. [16]):

Example 1

1. If $\sum_{i=1}^{N} k_i$ is more or less than the required cardinality, elements of $\boldsymbol{k}$ are set to 0 or 1 accordingly. The selection criterion that determines which elements are changed could be random or follow given rules.

2. The vector $\boldsymbol{c}$ is normalized:

$$\bar{c_i} = \frac{c_i}{\sum_{j \in \Upsilon} c_i} \tag{6}$$

   where Υ is the set of all i with $k_i = 1$.

3. The final weight x_i is calculated:

$$x_i = \begin{cases} u + \bar{c_i}\,(1 - |\Upsilon| u) & \text{if } i \in \Upsilon \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

($|\Upsilon|$ denotes the number of elements in Υ.)

There are three obvious ways to integrate constraints into metaheuristic algorithms:

- Integrating the constraint directly into the coding of the solution, so that obtaining an invalid solution is not possible.

  In example 1, the budget constraint $\sum_{i=1}^{N} x_i = 1$ is always fulfilled:

$$\begin{aligned} \sum_{i=1}^{N} x_i &= \sum_{i\in\Upsilon}(u + \frac{c_i}{\sum_{j\in\Upsilon} c_i}(1 - |\Upsilon|u)) \\ &= |\Upsilon|u + \sum_{i\in\Upsilon}\frac{c_i}{\sum_{j\in\Upsilon} c_i} - \sum_{i\in\Upsilon}\frac{c_i}{\sum_{j\in\Upsilon} c_i}|\Upsilon|u = |\Upsilon|u + 1 - |\Upsilon|u \end{aligned}$$

  In order to guarantee that no infeasible solutions are constructed, the operators that modify existing or create new solutions have to be adapted to an inherently coded constraint. This is not always possible, especially when there are many constraints or the constraints are quite complex.

- Using a repair algorithm if an invalid solution is constructed.

  In example 1, this was done in the first step to guarantee the compliance with the cardinality constraint.

  The problems with using this method are twofold: first, if the repair algorithm is not constructed well, the metaheuristic may be pushed into a suboptimal direction of the search space, an second, if there are many constraints handled in this fashion, it gets quite complicated to construct such a repair algorithm.

- Introducing a penalty term into the objective function for constraints that are not fulfilled.

  One problem with this approach is that the best solutions found by the algorithm don't have to be valid, especially when tho optimal solution is situated at the edge of the search space. Another disadvantage is the difficulty of incorporating a large number of constraints.

An extensive list of references on constraint-handling techniques used with EAs can be found at [22]. [23] is a starting point if the reader is interested in alternative techniques to handle constraints in Evolutionary Algorithms.

4 Comparison of Mixed Integer Solvers and Metaheuristics

The following list describes those advantages and disadvantages of MIQP-solvers and metaheuristics which the authors find the most compelling. The intention of the list is to give the reader a little support in taking the right decision, should it become necessary to select one of the two approaches for a portfolio selection problem.

Advantages of the mixed integer approach:

- If the algorithm terminates, the solution is guaranteed to be optimal.
- Even if the solver does not terminate in the given time, the best feasible solution found so far is often quite good. (The reader is referred to [12] for comparative test results).
- The user can work with several modelling languages (e.g. AMPL, GAMS), which give access to advanced solvers even if he has no previous programming experience.
- The most important constraint category, linear constraints only containing continuous variables, are easily integrated in large numbers.

Disadvantages of the mixed integer approach:

- For large problems, there is no guarantee that the algorithm terminates in an acceptable amount of time.
- If a problem does not fit into one of the well known categories, even with extensive modelling experience, it is often not obvious how to transform the problem into any of those categories. Sometimes, the transformation is not possible at all.
- The Efficient Frontier can only be calculated pointwise; a mixed integer version of the Critical Line Algorithm is not known.

Advantages of metaheuristics for portfolio selection:

- The algorithms work even if the objective function is changed and another risk measure is used.
- Special constraints (e.g. cardinality constraints) are easily integrated.
- Many software libraries that implement metaheuristics are freely available.
- When using a multi-objective EA, the complete Pareto front can be calculated in one algorithm run.

- Metaheuristics are easy to parallelize.

Disadvantages of metaheuristics:

- The performance of the tested metaheuristics was poor in comparison to a mixed integer approach (see [12]).
- More work has to be put into the design of the algorithms than when a mixed integer solver is used.
- There exist many libraries that offer metaheuristics in an adaptable form, but programming experience is required nevertheless.
- Metheuristics require extensive parameter tuning.

5 Conclusion

In this paper, the standard M-V-model for portfolio selection as well as several extensions have been presented. Both the mixed integer quadratic programming approach and the use of metaheuristics to compute an acceptable solution have been discussed, and their respective strong and weak points have been highlighted. An interesting idea for future research could be the (partial) integration of both paradigms.

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