Impied Market Price of Weather Risk

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Abstract

Weather influences our daily lives and choices and has an enormous impact on cooperate revenues and earnings. Weather derivatives differ from most derivatives in that the underlying weather cannot be traded and their market is relatively illiquid. This paper implements a pricing methodology for weather derivatives that can increase the precision of measuring weather risk, which is an important issue for financial institutions and energy companies. We applied continuous autoregressive models (CAR) to model the temperature in Berlin and with that to get explicit nature of non-arbitrage prices for temperature derivatives. A clear seasonal variation in the regression residuals of the temperature is observed and the volatility term structure of cumulative average temperature futures presents a Samuelson effect. We infer the implied market price of temperature risk for Berlin futures traded at the Chicago Mercantile Exchange (CME).

Keywords: Weather derivatives, weather risk, weather forecasting, seasonality, continuous autoregressive model, stochastic variance

JEL classification: G19, G29, N26, N56, Q29, Q54

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1 Introduction

Weather influences our daily lives and has an enormous impact on corporate revenues and earnings. The global climate changes the volatility of weather and the occurrence of extreme weather events increases. In particular, disfavoured extreme natural events like earthquakes, hurricanes, long cold winter, heat, drought, freeze, etc. may cause substantial financial losses. The traditional way of protection against unpredictable weather conditions is the insurance, which covers the losses in exchange for the payment of a premium. However, recently one observes an inception of new financial instruments linked to weather conditions: CAT bonds, sidecars and weather derivatives. It is therefore of scientific importance to
study Weather Derivatives (WD), especially when they are used to hedge against weather related exposures.

The key factor in efficient usage of weather derivatives is a reliable valuation procedure. However, due to their specific nature one suffers several difficulties. First, weather derivatives are different from most financial derivatives because the underlying weather cannot be traded. Second, the weather derivatives market is relatively illiquid, i.e. the weather derivatives cannot be cost-efficiently replicated with other weather derivatives. One of the consequences of this is that the valuation of weather derivatives is in spirit and methodology closer to insurance pricing than to derivative pricing (arbitrage pricing).

It is therefore important to concentrate on pricing methodologies that can increase the accuracy of measuring weather risk. A precise validation is a question and is central under new insurance and climate change regulations and for the emergence of a liquid weather derivative market.

The pricing based of weather derivatives attracted the attention of many researchers. ? fitted an Ornstein-Uhlenbeck stochastic process with constant variance to temperature observations at Chicago O’Hare airport and started to investigate future prices on temperature indices. Later ? applied the Ornstein-Uhlenbeck model with a monthly variation in the variance to temperature data of Bromma airport (Stockholm). They applied their model to get prices for different temperature prices. ? modelled temperature in several US cities with a high order autoregressive model. They observed seasonality behaviour in the autocorrelation function (ACF) of the squared residuals. However, they did not price temperature derivatives. ? studied the temperature in Casablanca, Morocco using a mean reverting model with stochastic volatility and a temperature swap was considered. ? calculates an arbitrage free price for different temperature derivatives prices by using the fractional Brownian motion model of ?, which drives the noise in an Ornstein-Uhlenbeck process.

In the temperature derivative market, ? proposed to use a marginal utility technique to price temperature derivatives based on the HDD-index. ? present an optimal design of weather derivatives in an illiquid framework, arguing that the standard risk neutral point of view is not applicable to valuate them. ? and ? apply an extended version of Lucas’ (1978) equilibrium pricing model where direct estimation of weather risk’s market price is avoided. Instead, pricing is based on the stochastic processes of the weather index, an aggregated dividend and an assumption about the utility function of a representative investor. ? used the world stock index as the numeraire to price temperature derivatives. ? and (2007) propose the continuous time autoregressive model with seasonality for the temperature evolution in time and fit this model to data observed in Stockholm, Sweden. They derive future and option prices for contracts on CDD and CAT indices. They also discuss hedging strategies for the options and volatility term structure.
Currently weather derivatives are priced using detrended historical data, \(?.\) The important issues here are the methods of detrending, the choice of models for residuals and model validation. The temperature dynamics are modelled with autoregressive processes (continuous or discrete) with lag \(p\) and seasonal variation. The seasonality is observed watching at the autocorrelation function (ACF) for the (squared) residuals.

In this paper, we applied continuous autoregressive models (CAR) to model the temperature in Berlin and with that to get explicite nature of non-arbitrage prices for temperature derivatives, as \(?\) did for Stockholm Temperature data. In fact, his model works also for Berlin Temperature data. Our paper is structured as follows. In the next section we discuss fundamentals of WD, weather dynamics explained by a continuous autoregressive model (CAR), definitions of temperature derivatives (future and options) and their indices, and the obtained explicit prices of WD. Section 3 is devoted to an application to Berlin temperature data. Section 4 presents the explicit prices of WD for Berlin and its comparison to Chicago Mercantile Exchange (CME) data prices. In section 5, we propose a benchmark model to calibrate the market price of risk for temperature derivatives of Berlin traded at the Chicago Mercantile Exchange (CME). In section 6, similar to Berlin, the assumptions of weather and prices dynamics hold for other German cities. Section 7 concludes.

2 Weather dynamics for Berlin temperature Data

In this section, we compute the weather dynamics for Berlin daily temperature data by means of the model used in \(?\). The temperature data considers daily average temperatures from 1950/1/1-2006/7/24 recorded at the Tempelhof Airport Station. We worked with 20,645 recordings after removing 29 February’s, this with the end of working with 360 days per year.

We check first the presence of a linear trend and investigate the seasonal pattern of the data. The upper panel in Figure ?? shows 57 years of daily average temperatures from Berlin and the fitted seasonal function with trend

\[
\Lambda_t = a_0 + a_1 t + a_2 \sin \left( \frac{2\pi t}{365} - a_3 \right)
\]

where \(a_0 = 91.51, a_1 = 0.00, a_2 = 97.95, a_3 = -180.34\) are significant at 10\% and the mean squared error (RMSE) is equal to 38.20. The estimation is made by means of the least squares minimization. The lower panel in Figure ??, it is more clear to observe the fitness of the seasonal function to the daily average temperatures. We observe low temperatures in the winter and high temperatures in the summer.
After removing the seasonality from the daily average temperatures,

\[ X_t = T_t - \Lambda_t \]

we check whether the process \( X_t \) is a stationary process \( I(0) \). In order to do that, we apply the Augmented Dickey-Fuller test (ADF):

\[
[(1 - L)x = c_1 + c_2 \text{trend} + \tau Ly + \alpha_1 (1 - L)Lx + \ldots + \alpha_p (1 - L)L^p x + u]
\]

where the test statistic for a unit root in a time series \( \tau = -39.81 \), with 1% critical value equal to -2.56. Therefore, we reject the null hypothesis \( H_0 (\tau = 0) \) and hence \( Y_t \) is a stationary process \( I(0) \).

![Graph showing temperature data](image)

Figure 1: Upper Panel: Berlin daily temperature data from 1950/1/1-2006/7/24, weather station: Tempelhof Airport Station. The line in red color denotes the daily average temperatures and the blue line is the fitted seasonal function. Lower Panel: Temperature in Berlin during 1990-2000, weather station: Tempelhof Airport Station. The line in red color denotes the daily average temperatures and the blue line is the fitted seasonal function.

We plot the Partial Autocorrelation Function (PACF) in Figure ?? to study the time behaviour of the residuals of \( Y_t \). Due to its periodic behaviour, temperature
reverts back to its mean over time. This is well explained by the PACF, which suggests that the AR(3) model suggested by? also holds for Berlin Temperature data. In this case,

\[ X_{i+3} = \beta_1 X_{i+2} + \beta_2 X_{i+1} + \beta_3 X_i + \sigma_i \varepsilon_i \]

with \( \hat{\beta}_1 = 0.91, \hat{\beta}_2 = -0.20, \hat{\beta}_3 = 0.07, \hat{\sigma}_i^2 = 510.63, \) the AIC estimator is equal to 6.23 and the BIC estimator equals to 6.23.

![Partial autocorrelation function (PACF)](image)

Figure 2: Partial autocorrelation function (PACF)

The residuals and squared residuals of the Berlin Temperature data, after trend and seasonal component were removed, are plotted in Figure ???. We reject at 1% significance level the null hypothesis \( H_0 \) that the residuals are uncorrelated.

The Autocorrelation function (ACF) of the residuals of AR(3), upper panel in Figure ???, are close to zero and according to Box-Ljung statistic the first few lags are insignificant. But, the ACF for the squared residuals in the lower panel Figure ?? shows a high seasonality pattern. We calibrate this seasonal dependence of variance of residuals of the AR(3) for 57 years with a truncated Fourier function

\[ \sigma_t^2 = c + \sum_{i=1}^{4} c_i \sin \left( \frac{2i\pi t}{365} \right) + \sum_{j=1}^{4} d_j \cos \left( \frac{2j\pi t}{365} \right) \]

Figure ?? shows the daily empirical variance and the fitted squared volatility function for the residuals. Here again, we also get similar results to the study conducted by? to Stockholm temperature data, high variance in winter - earlier summer and low variance in spring - late summer.

After dividing out the seasonal volatility from the regression residuals, we observed closed to normal residuals (zero mean uncorrelated noise). Figure ??
Figure 3: Residuals (upper panel) and Squared residuals (lower panel) of the AR(3) process.
Figure 4: ACF for residuals (upper panel) and squared residuals (lower panel) of the AR(3) process.

Figure 5: Seasonal variance: daily empirical variance (blue line), fitted squared volatility function (red line) at 10% significance level.
Figure 6: ACF for residuals (upper panel) and squared residuals (lower panel) after removing seasonal volatility.
shows that the ACF plot of the residuals remain unchanged and now the ACF plot for squared residuals presents a non-seasonal pattern.

We used the Ljung-Box’s test statistic (Qstat) to check the significance level of the lags of the ACF of residuals with and without seasonal volatility. Table 1 presents the statistics and the corresponding significance levels of the lags.

We plot a histogram of the residuals together with a normal distribution in Figure 7 to verify if residuals become normal distributed. Under the $t$-test with $p$-value equal to 0.96, we accept the null hypothesis $H_0$ of normality and we reject it, with $p$=0.0028, under the Kolmogorov-Smirnov test. The obtained residuals have a skewness equal to -0.0765 and a kurtosis equal to 3.5527.

In order to get explicit derivations of prices dynamics for all different temperature contracts available at the CME, they analyze a stochastic model for the temperature with continuous dynamics. To get that, they showed (by further substituting iteratively in the discrete-time dynamics) that a AR($p$) time series in continuous
time can be written as a continuous time autoregressive model stochastic process CAR(p) with seasonal variance.

For this inconvenience, they use a $X_{1t}$-CAR $(p)$ to model the temperature dynamics:

$$T_t = \Lambda_t + X_{1t}$$ \hspace{1cm} (2)

where $X_q$ is q’th coordinate of vector $X$ with $q = 1, \ldots, p$ and $\Lambda_t$ is a deterministic seasonal function that represents the average temperature. They noticed that the temperature residuals possess a positive, continuous and bounded seasonal variance $\sigma_t$ and stationarity holds when the variance matrix

$$\int_0^t \sigma_t^2(t - s) \exp(A_s) e_p e_p^\top \exp(A_s^\top) ds$$ \hspace{1cm} (3)

converges as $t \to \infty$.

For this convenience, they define a matrix $p \times p$-matrix:

$$A = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \ddots & 0 & \vdots \\
0 & \ldots & \ldots & \ldots & 0 & 0 \\
-\alpha_p & -\alpha_{p-1} & \ldots & 0 & -\alpha_1
\end{pmatrix}$$

in the vectorial Ornstein-Uhlenbeck process $X_t \in \mathbb{R}^p$ for $p \geq 1$ as:

$$dX_t = AX_t dt + e_p \sigma_t dB_t$$ \hspace{1cm} (4)

where $e_k$ denotes the k’th unit vector in $\mathbb{R}^p$ for $k = 1, \ldots, p$, $\sigma_t > 0$ states the temperature volatility, $B_t$ is a Wiener Process and $\alpha_k$ are positive constants.

By applying the multidimensional Itô Formula, the stochastic process $X_t$ has the explicit form

$$X_s = \exp(A_{s-t}) x + \int_t^s \exp(A_{s-u}) e_p \sigma_u dB_u$$ \hspace{1cm} (5)

for $s \geq t \geq 0$ and $X_t = x \in \mathbb{R}^p$.

For the discrete version of the CAR(p) process, they iterate the finite difference approximations of the time dynamics, when $p = 1, 2, 3$. Using $\epsilon_t = B_{t+1} - B_t$, we repeat the exercise:

for $p = 1$, we get that $X_t = X_{1t}$ and $dX_{1t} = -\alpha_1 X_{1t} dt + \sigma_t dB_t$.

for $p = 2$, we have that

$$X_{1(t+2)} \approx (2 - \alpha_1) X_{1(t+1)} + (\alpha_1 - \alpha_2 - 1) X_{1(t)} + \sigma_t (B_{t-1} - B_t)$$
Table 2: Coefficients

<table>
<thead>
<tr>
<th></th>
<th>AR(3)</th>
<th>CAR(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>β₁</td>
<td>0.91</td>
<td>α₁</td>
</tr>
<tr>
<td>β₂</td>
<td>-0.20</td>
<td>α₂</td>
</tr>
<tr>
<td>β₃</td>
<td>0.07</td>
<td>α₃</td>
</tr>
</tbody>
</table>

for \( p = 3 \), the \( A \) matrix is defined as

\[
A = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-\alpha_3 & -\alpha_2 & -\alpha_1
\end{pmatrix}
\]

and

\[
\begin{align*}
X_{1(t+1)} - X_{1(t)} &= X_{1(t)}dt + \sigma_t \epsilon_t \\
X_{2(t+1)} - X_{2(t)} &= X_{3(t)}dt + \sigma_t \epsilon_t \\
X_{3(t+1)} - X_{3(t)} &= -\alpha_3 X_{1(t)}dt - \alpha_2 X_{2(t)}dt - \alpha_1 X_{3(t)}dt + \sigma_t \epsilon_t \\
X_{1(t+2)} - X_{1(t+1)} &= X_{1(t+1)}dt + \sigma_{t+1} \epsilon_{t+1} \\
X_{2(t+2)} - X_{2(t+1)} &= X_{3(t+1)}dt + \sigma_{t+1} \epsilon_{t+1} \\
X_{3(t+2)} - X_{3(t+1)} &= -\alpha_3 X_{1(t+1)}dt - \alpha_2 X_{2(t+1)}dt - \alpha_1 X_{3(t+1)}dt + \sigma_{t+1} \epsilon_{t+1} \\
X_{1(t+3)} - X_{1(t+2)} &= X_{1(t+2)}dt + \sigma_{t+2} \epsilon_{t+2} \\
X_{2(t+3)} - X_{2(t+2)} &= X_{3(t+2)}dt + \sigma_{t+2} \epsilon_{t+2} \\
X_{3(t+3)} - X_{3(t+2)} &= -\alpha_3 X_{1(t+2)}dt - \alpha_2 X_{2(t+2)}dt - \alpha_1 X_{3(t+2)}dt + \sigma_{t+2} \epsilon_{t+2}
\end{align*}
\]

Substituting iteratively in \( X_1 \) dynamics, we get that:

\[
X_{1(t+3)} \approx (3 - \alpha_1) X_{1(t+2)} + (2\alpha_1 - \alpha_2 - 3) X_{1(t+1)} + (-\alpha_1 + \alpha_2 - \alpha_3 + 1) X_{1(t)} + \sigma_t (B_{t-1} - B_t)
\]

which coefficients are equal to the ones obtained for the CAR process in \( ? \).

Once working in continuous time,

From equation \( ?? \), the coefficients of the CAR(3) can be estimated. Table \( ?? \) summarizes the AR(3) and CAR(3) estimated coefficients for 57 years of Berlin Temperature. Also the stationarity condition for the CAR(3) (equation \( ?? \)) is fulfilled, since the eigenvalues of the matrix \( A \) have negative real parts (\( \lambda_1 = -0.2317, \lambda_{2,3} = -0.9291 \pm 0.2934i \)).

3 Weather Derivatives

In the 1990’s WD were developed to hedge against volatility caused by weather. WD are financial contracts, which payments are based on weather-related measurements. They are formally exchanged in the Chicago Mercantile Exchange (CME), where monthly and seasonal temperature future, call and put options
contracts on future prices are traded. The futures and options at CME are cash settled. WDs cover against extreme changes on temperature, rainfall, wind, snow, frost, but do not cover catastrophic events, such as earthquakes or hurricanes.

4 Temperature Indices

Temperature derivatives are written on a temperature index. The most common weather indices on temperature are: Heating Degree Day (HDD), Cooling Degree Day (CDD), Cumulative Averages (CAT), Average of Average Temperature (AAT) and Event Indices (EI),

The HDD measures the temperature over a period $[τ_1, τ_2]$, usually between October to April, and it is defined as:

$$HDD(τ_1, τ_2) = \int_{τ_1}^{τ_2} \max(K - T_u, 0)du$$

where $K$ is the baseline temperature (typically 18°C or 65°F) and $T_u$ is the average temperature on day $u$. Similarly, the CDD measures the temperature over a period $[τ_1, τ_2]$, usually between November and March, and it is defined as:

$$CDD(τ_1, τ_2) = \int_{τ_1}^{τ_2} \max(T_u - K, 0)du$$

The HDD and the CDD index are used to trade future and options in 18 US cities and 9 European cities. The CAT index accounts the accumulated average temperature over a period $[τ_1, τ_2]$ days:

$$CAT(τ_1, τ_2) = \int_{τ_1}^{τ_2} T_u du$$

Since $\max(T_u - k, 0) - \max(K - T_u, 0) = T_u - k$, we get the HDD-CDD parity

$$CDD(τ_1, τ_2) - HDD(τ_1, τ_2) = CAT(τ_1, τ_2) - K$$

Therefore, it is sufficient to analyse only CDD and CAT indices. The AAT measures the "excess" or deficit of temperature i.e. the average of average temperatures over $[τ_1, τ_2]$ days:

$$AAT(τ_1, τ_2) = \frac{1}{τ_1 - τ_2} \int_{τ_1}^{τ_2} T_u du$$

This index is just the average of the CAT and it is relevant for the Pacific Rim consisted of two Japanese cities. The EI considers the number of times a certain meteorological event occurs in the contract period. For example, a frost day is considered when the temperature at 7:00-10:00 local time is less than or equal to -3.5°C. To illustrate this, Table ?? shows the expected number of HDDs, CDDs, CATs and AATs.
Indices

<table>
<thead>
<tr>
<th></th>
<th>Jan</th>
<th>Feb</th>
<th>March</th>
<th>April</th>
<th>May</th>
<th>Jun</th>
<th>Jul</th>
<th>Aug</th>
<th>Sept</th>
<th>Oct</th>
<th>Nov</th>
<th>Dec</th>
</tr>
</thead>
<tbody>
<tr>
<td>CDD</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>28.3</td>
<td>42</td>
<td>71</td>
<td>23.3</td>
<td>24.9</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>HDD</td>
<td>472.8</td>
<td>526.4</td>
<td>471.4</td>
<td>241.1</td>
<td>150.2</td>
<td>71.8</td>
<td>24.8</td>
<td>43.9</td>
<td>73.5</td>
<td>199.5</td>
<td>398.2</td>
<td>525.8</td>
</tr>
<tr>
<td>CAT</td>
<td>103.2</td>
<td>-4.4</td>
<td>104.6</td>
<td>316.9</td>
<td>454.1</td>
<td>528.2</td>
<td>622.2</td>
<td>555.4</td>
<td>509.4</td>
<td>376.5</td>
<td>159.8</td>
<td>50.2</td>
</tr>
<tr>
<td>AAT</td>
<td>3.32</td>
<td>-0.15</td>
<td>3.37</td>
<td>10.56</td>
<td>14.64</td>
<td>17.60</td>
<td>20.07</td>
<td>17.91</td>
<td>16.98</td>
<td>12.14</td>
<td>5.32</td>
<td>1.61</td>
</tr>
</tbody>
</table>

Table 3: Degree day indices for temperature data (2005) Berlin.

5 Temperature Derivatives

In this section, the continuous weather dynamics will allow for explicit derivations of Future/Option price dynamics for all different temperature contracts available at the CME.

Basic principles of Finance tell us that the arbitrage free price of a derivative is given by the present expected payoff from the derivative under the risk neutral probability measure $Q$. The problem with WD is that weather cannot be traded and therefore no-arbitrage models to weather derivatives are impractical, since one cannot replicate a future contract with the underlyng index. In order to derive the future price dynamics and to have a class of pricing measures with the martingale property, introduce a parametrized class of equivalent probabilities $Q^\theta$ via the Girsanov transformation

$$B^\theta_t = B_t - \int_0^t \theta_u du$$ (12)

where $\theta_t$ is a real valued, bounded and piecewise continous function (market price of risk) on $[0, \tau_{\text{max}}]$. By Girsanov theorem, there exist an equivalent probability measure $Q^\theta$ so that $B^\theta_t$ is a Brownian montion for $t \in [0, \tau_{\text{max}}]$. Under $Q^\theta$, equation (??) becomes:

$$dX_t = (AX_t + e_p \sigma_t \theta_t)dt + e_p \sigma dB^\theta_t$$ (13)

with explicit dynamics

$$X_s = \exp (A_{s-t})x + \int_t^s \exp (A_{s-u})e_p \sigma_u \theta_u du + \int_t^s \exp (A_{s-u})e_p \sigma_u dB^\theta_u$$ (14)

for $s \geq t \geq 0$.

When an owner of a future contract enters in a contract, at time $t$, he agrees to pay the price $F_{t,\tau_1,\tau_2}$ at the end of the period $\tau_2$, instead of receiving the payoff from the CAT/HDD/CDD future defined in equations (??, ??, ??). Then the $\mathcal{F}_t$-adapted arbitrage free price of this future under the $Q$ risk neutral probability is equal to:

$$0 = \exp \{-r(\tau_2 - t)\} E^Q [Y - F_{(t,\tau_1,\tau_2)} | \mathcal{F}_t]$$ (15)
where \( r \) is the constant risk-free interest rate. Under the \( Q^\theta \) pricing probability, the price of the future is equal to:

\[
F_{(t, \tau_1, \tau_2)} = E^{Q^\theta}[Y | \mathcal{F}_t]
\]

(16)

5.0.1 CAT Futures/Option

Following equation (??), the risk neutral price of a future based on a CAT index is defined as:

\[
F_{CAT(t, \tau_1, \tau_2)} = E^{Q^\theta} \left[ \int_{\tau_1}^{\tau_2} T_s ds | \mathcal{F}_t \right]
\]

(17)

\( Q^\theta \)-dynamics of \( F_{CAT(t, \tau_1, \tau_2)} \) is

\[
dF_{CAT(t, \tau_1, \tau_2)} = \sigma_t a_{t, \tau_1, \tau_2} e_p dB^\theta_t
\]

with \( a_{t, \tau_1, \tau_2} = e_1^\top A^{-1} \{ \exp(A_{\tau_2-t}) - \exp(A_{\tau_1-t}) \} \) and the \( p \times p \) identity matrix \( (I_p) \). Then the \( Q^\theta \)-dynamics of \( F_{CAT(t, \tau_1, \tau_2)} \) is

\[
dF_{CAT(t, \tau_1, \tau_2)} = \sigma_t a_{t, \tau_1, \tau_2} e_p dB^\theta_t
\]

(18)

Similarly, calculate the value of a CAT call option, with strike \( K \) at exercise time \( T \leq \tau_1 \) and written on a CAT future during the period \( [\tau_1, \tau_2] \) is:

\[
C_{CAT(t, T, \tau_1, \tau_2)} = \exp\left\{ -r(T-t) \right\} \times \left\{ (F_{CAT(t, \tau_1, \tau_2)} - K) \Phi (d(t, T, \tau_1, \tau_2)) + \int_t^T \Sigma^2_{CAT(s, \tau_1, \tau_2)} ds \Phi ' (d(t, T, \tau_1, \tau_2)) \right\}
\]

(19)

where

\[
d(t, T, \tau_1, \tau_2) = \frac{F_{CAT(t, \tau_1, \tau_2)} - K}{\sqrt{\int_t^T \Sigma^2_{CAT(s, \tau_1, \tau_2)} ds}}
\]

and

\[
\Sigma^2_{CAT(s, \tau_1, \tau_2)} = \sigma_t a_{t, \tau_1, \tau_2} e_p
\]

and \( \Phi \) denotes the standard normal cdf.

To replicate the call option with CAT-futures, one should compute the number of CAT-futures held in the portfolio, which is simply computed by the option’s delta:

\[
\Phi(d(t, T, \tau_1, \tau_2)) = \frac{\partial C_{CAT(t, T, \tau_1, \tau_2)}}{\partial F_{CAT(t, \tau_1, \tau_2)}}
\]

(20)
5.0.2 CDD Futures/Options

Analogously, derived explicit CDD future price dynamics. Following equation (??), the risk neutral price of a CDD future is defined as:

$$F_{CDD}(t,\tau_1,\tau_2) = \mathbb{E}^{Q^\theta} \left[ \int_{\tau_1}^{\tau_2} \max(T_u - K, 0) du | \mathcal{F}_t \right]$$

and by inserting the temperature model (equation ??) into equation (??), we have

$$F_{CDD}(t,\tau_1,\tau_2) = \int_{\tau_1}^{\tau_2} \psi \left( \frac{m(t,s,e^\top_1 \exp(A_{s-t})X_t)}{v_{t,s}} \right) ds$$

where

$$m(t,s) = \Lambda_s - c + \int_s^t \sigma_u \theta u e^\top_1 \exp(\Lambda_s - u) e_p du + x, \quad v_{t,s}^2 = \int_t^s \sigma_u^2 \left( e^\top_1 \exp(A_s - t) e_p \right)^2 du,$$

$$\psi(x) = x\Phi(x) + \Phi^\top(x) \text{ with } x = e^\top_1 \exp(A(s-t))X_t$$

and $\Phi$ is the standard normal cdf.

From the martingale property and Itô’s formula, the time dynamics of the CDD-futures prices are defined by

$$dF_{CDD}(t,\tau_1,\tau_2) = \sigma_t \int_{\tau_1}^{\tau_2} \left( e^\top_1 \exp(A_{s-t}) e_p \right) \times \Phi \left( \frac{m(t,s,e^\top_1 \exp(A_{s-t})X_t)}{v_{t,s}} \right) ds dB^\theta_t$$

For the call option written CDD-future, found no analytical solution, but an expression suitable for Monte Carlo simulation. The value of a call option, with strike $K$ at exercise time $T \leq \tau_1$ and written on a CDD future during the period $t \leq T$ is given as

$$C_{CDD}(t,T,\tau_1,\tau_2) = \exp \left\{ -r(T-t) \right\} \mathbb{E} \left[ \max \left( \int_{\tau_1}^{\tau_2} v_{T,s} \right) \psi \left( m(T,s,e^\top_1 \exp(A_{s-t})X_s) \right) ds - K, 0 \right] \right]_{x=X_t}$$

where $Y$ is a standard normal variable and $\Sigma^2_{s,t,T} = \int_t^T \left\{ e^\top_1 \exp(A_{s-u}) e_p \right\}^2 \sigma_u^2 du$.

Moreover, after numerous calculation, the hedging strategy $H_{CDD}(t,\tau_1,\tau_2)$, with $t \leq T$ that replicates de CDD-call with CDD-futures is

$$H_{CDD}(t,\tau_1,\tau_2) = \frac{\sigma_t}{\Sigma^2_{CDD}(s,\tau_1,\tau_2)} \mathbb{E} \left[ \mathbf{1} \left( \int_{\tau_1}^{\tau_2} v_{T,s} \psi \left( \frac{m(T,s,Z)}{v_{T,s}} \right) ds > K \right) \right]_{x=X_t} \times \int_{\tau_1}^{\tau_2} e^\top_1 \exp(A_{s-t}) e_p \Phi \left( \frac{m(T,s,Z)}{v_{T,s}} \right) ds \right]_{x=X_t}$$
where $Z$ is a normal distributed with mean $\mathbf{e}_1^\top \exp (A_{s-t}) \mathbf{x} + \int_t^T \mathbf{e}_1^\top \exp (A_{s-u}) \mathbf{e}_p \sigma_u^2 du$, and variance $\int_t^T \sigma_u^2 (\mathbf{e}_1^\top \exp (A_{s-u}) \mathbf{e}_p)^2 du$, and

$$\Sigma_{CDD(s, \tau_1, \tau_2)}^2 = \sigma_t \mathbf{a}_{\tau_1, \tau_2} \mathbf{e}_p \mathbf{\Phi} \left[ m \{ t, s, \mathbf{e}_1^\top \exp (A_{s-t}) \mathbf{x} \} \right]$$

which is similar to $\Sigma_{CAT(s, \tau_1, \tau_2)}^2$ except by the last term.

### 6 A market price risk model

After computing explicit non-arbitrage prices for temperature derivatives, we propose a benchmark model to calibrate the market risk of price, i.e. the right price among possible arbitrage free prices. We do that by calibrating the market data (CME data) and thereby pin down the price.

First let us consider the price for the CAT future, which can be written $\theta_u$ is a real valued piecewise linear function:

$$\theta(u) = \begin{cases} \theta_1, & u \in (u_1, u_2) \\ \theta_2, & u \in (u_1, u_2) \end{cases}$$

### 7 Conclusion

We apply a higher order continuous-time autoregressive models CAR(3) with seasonal variance for modelling temperature in Berlin for more than 57 years of daily observations.

This paper also analyzes the weather options/future products for Berlin traded at the Chicago Mercantile Exchange (CME), written on different temperatures indices like heating (HDD), cooling degree days (CDD) and cumulative average temperature (CAT) measured over different time periods. We computed future prices for different contracts available at the CME were computed, after calculating the weather dynamics from daily temperature data in Berlin and using methods from mathematical finance.

Similar results hold for other German cities.
References


