Cash-Lock Comparison of Portfolio Insurance Strategies

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This version: May 31, 2008

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Abstract

Portfolio insurance strategies are designed to achieve a minimum level of wealth while at the same time participating in upward moving markets. The strategies all have in common that the fraction of wealth which is invested in a risky asset is increasing in the asset price increments. Therefore, in downward moving markets, the asset exposure is reduced. In the strict sense, a cash–lock describes the event that the asset exposure drops to zero and stays there. Since a cash–lock at an early time prohibits any participation in recovering markets, the cash–lock cage is considered as a major problem with respect to long investment horizons. We analyze a generalized form of a cash–lock by focusing on the probability that the investment quote recovers from small values. It turns out that, even in the case that the dynamic versions of option based strategies and proportional portfolio insurance strategies coincide in their expected return, they give rise to a very different cash–lock behavior. In addition, we point out that, for comparability reasons, it is necessary to distinguish between strategies where an investment quote over one is admissible and such with borrowing constraints.

Keywords: CPPI, OBPI, portfolio insurance, cash–lock, borrowing constraints, path–dependence.

JEL: G11, G12

1. Introduction

Financial strategies which are designed to limit downside risk and at the same time to profit from rising markets are summarized in the class of portfolio insurance strategies. Among others, Grossman and Villa (1989) and Basak (2002) define a portfolio insurance trading strategy as a strategy which guarantees a minimum level of wealth at a specified time horizon, but also participates in the potential gains of a reference portfolio. The most prominent examples of dynamic versions are the constant proportion portfolio insurance (CPPI) strategies and option–based portfolio insurance (OBPI) strategies with synthetic puts.[1] Here, synthetic is understood in the sense of a trading strategy in basic (traded) assets which creates the put. In a complete financial market model, there exists a perfect hedge, i.e. a self–financing and duplicating strategy. In contrast, the introduction of market incompleteness impedes the concept of perfect hedging.[2]

In general, the optimality of an investment strategy depends on the risk profile of the investor. In order to determine the optimal rule, one has to solve for the strategy which maximizes the expected utility. Thus, portfolio insurers can be modeled by utility maximizers where the maximization problem is given under the additional constraint that the value of the strategy is above a specified wealth level.[3] Mostly, the solution of the maximization problem is given by the unconstrained problem including a put option. Obviously, this is in the spirit of the OPBI method. At this point, we also refer to Cox and Huang (1998) and El Karoui, Jeanblanc and Lacoste (2005). In these papers it is shown that, under market completeness and various utility functions, the optimal portfolio resembles an option on a power of the underlying asset. However, it is well known that in a complete market, the classic CPPI strategies are path–independent if no borrowing constraint are posed. In particular, the portfolio value can be represented in terms of the floor plus a power claim. This implies at least some similarities of both methods.

In summary, there is the basic question which of the two methods, OBPI or CPPI is favorable. While the CPPI approach is appealing in its simplicity and easiness to customize the strategies to the preferences of the investor, a major drawback is seen in the

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[1]Option–based portfolio insurance (OBPI) with synthetic puts is introduced in Leland and Rubinstein (1976), constant proportion portfolio insurance (CPPI) in Black and Jones (1987). For the basic procedure of the CPPI see also Merton (1971).

[2]One possible solution is given by quantile and efficient hedging, c.f. Föllmer and Leukert (1999) and Föllmer and Leukert (2000).

[3]Without postulating completeness, we refer to the works of Cox and Huang (1989), Brennan and Schwartz (1989), Grossman and Villa (1989), Grossman and Zhou (1993, 1996), Basak (1995), Cvitanic and Karatzas (1995, 1999), Browne (1999), Tepla (2000, 2001) and El Karoui et al. (2005).

so–called cash–lock cage. Recall that portfolio insurance strategies are designed to achieve a minimum level of wealth while at the same time participating in upward moving markets. The strategies all have in common that the fraction of wealth which is invested in a risky asset is increasing in the asset price increments. Therefore, in downward moving markets, the asset exposure is reduced. In the strict sense, a cash–lock describes the event that the asset exposure drops to zero and stays there. Since a cash–lock at an early time prohibits any participation in recovering markets, the cash–lock cage is considered as a major problem with respect to long investment horizons. The classic CPPI principle implies that, once the investment quote drops to zero, it stays there with probability one. However, the cash–lock probability crucially depends on the borrowing constraints which are posed. In the following paper, we analyze a generalized form of a cash–lock by focusing on the probability that the investment quote recovers from small values. It turns out that, even in the case that the dynamic versions of option based strategies and proportional portfolio insurance strategies coincide in their expected return, they give rise to a very different cash–lock behavior. In addition, we point out that, for comparability reasons, it is necessary to distinguish between strategies where an investment quote over one is admissible and such with borrowing constraints. The whole analysis is carried out in a simple Black–Scholes–type framework. There are a few comments necessary concerning this approach. Obviously, a more realistic picture of cash–lock probabilities can only be gained within a more general model setup. However, the simple model setup used here is suitable in order to understand the differences in the cash–lock behavior of the two approaches, OBPI and CPPI.

With respect to the literature, we restrict ourselves to the strand focusing on stylized strategies, i.e. OBPI, CPPI or a comparison of both. Without postulating completeness, we refer to the following ones:[4] The properties of continuous–time CPPI strategies are studied extensively, c.f. Bookstaber and Langsam (2000) or Black and Perold (1992). A comparison of OBPI and CPPI (in continuous time) is given in Bertrand and Prigent (2002a). An empirical investigation of both methods can, for example, be found in Do (2002) who simulates the performance of these strategies using Australian data. In Cesari and Cremonini (2003), there is an extensive simulation comparison of popular dynamic strategies of asset allocation, including CPPI strategies. The literature also deals with the effects of jump processes, stochastic volatility models and extreme value approaches on the CPPI method, c.f. Bertrand and Prigent (2002b), Bertrand and Prigent (2003). More recently, Cont and Tankov (2007) and Balder et al. (2008) analyze the risk–profil

[4]Basically, one can distinguish between three different strands focusing on investment Strategies with guarantees, optimal investment decision based on utility and optimal insurance contract design.

and gap risk of CPPI strategies.[5] In contrast to Cont and Tankov (2007) who place themselves in a jump diffusion model setup, Balder et al. (2008) introduce the gap risk by the introduction of trading restrictions.

The outline of the paper is as follows. In Sec. 2, we discuss the investment decision and the construction of portfolio insurance strategies. Either, the constructions is based on the investment quote in terms of the fraction of wealth which is to be invested in the risky asset. Alternatively, the strategy can be defined via the number of assets. W.l.o.g. all strategies under consideration are self–financing. In addition, we we define a generalized cash-lock event. In Sec. 3, we focus on the distribution and cash–lock probabilities which are associated with traditional portfolio insurance strategies. While Sec. 3 considers the case where no exogenous restriction on the investment quote is posed, the effects of borrowing constraints are analyzed in Sec. 4. An illustration and comparison is given in Sec. 5. In particular, we focus on the difference of the cash–lock behavior of CPPI and OBPI with borrowing constraints. Sec. 6 concludes the paper.

2. Investment quote vs number of risky assets

Throughout the following, it is enough to consider the question which part of the wealth can be invested in a risky asset and which part has to be placed in a risk–free one. We consider two investment possibilities: a risky asset S and a riskless bond B (which grows with constant interest rate r). The evolution of the risky asset S, a stock or benchmark index, is given by a stochastic differential equation which, in the simplest case, is a geometric Brownian motion. It is worth mentioning that some of the following strategies (their construction, respectively) do and some do not depend on the specific assumptions on the stochastic process which is generating the asset prices.[6] The first case means that, in order to determine the number of assets and bonds which are to be bought or sold it is enough to observe the asset prices while it is not necessary to know the stochastic process which generates them. This is in spirit of the CPPI approach. In contrast, the option based approach with synthetic put (call, respectively) is an example of a strategy construction which explicitly depends on the model assumption.[7]

[5]If the CPPI method is applied in continuous time, the CPPI strategies provide a value above a floor level unless the price dynamic of the risky asset permits jumps. The risk of violating the floor protection is called gap risk. In practice, it is caused by liquidity constraints and price jumps.

[6]Obviously, the performance of the strategies always depends on the data generating process.

[7]For example, in the Black–Scholes model, the volatility of the underlying is one crucial input for the strategy.

First, we define the general strategy definition in terms of the fraction of wealth invested in the risky asset and in terms of the number of assets. After this, we give a generalized definition of the cash–lock probability.

2.1. **Dynamic trading strategies.** Along the lines of the literature on portfolio insurance, a continuous–time investment strategy or saving plan for the interval $[0, T]$ can be represented by a predictable process $(\pi_t)_{0\leq t\leq T}$. π_t denotes the fraction of the portfolio value at time t which is invested in the risky asset S. In the following, we will also refer to π as the investment quote. Notice that it is w.l.o.g. convenient to restrict oneself to strategies which are self–financing, i.e. strategies where money is neither injected nor withdrawn during the trading period $]0, T[$. Thus, the amount which is invested at date t in the riskless bond B is given in terms of the fraction $1 - \pi_t$. Let $V = (V_t)_{0\leq t\leq T}$ denote the portfolio value process which is associated with the strategy π, it follows immediately that $V_t(\pi)$ is the solution of

$$dV_t(\pi) = V_t \left(\pi_t \frac{dS_t}{S_t} + (1 - \pi_t) \frac{dB_t}{B_t} \right), \text{ where } V_0 = x. \tag{1}$$

Alternatively, a trading strategy can be specified in terms of the numbers Φ of assets which are held, i.e. $\Phi = (\phi^0, \phi^1)$ where ϕ^0_t denotes the number of bonds and ϕ^1_t the number of the risky asset. In consequence, one obtains $V_t(\Phi) = \phi^0_t B_t + \phi^1_t S_t$ where for a self–financing strategy ϕ it holds that $V_t(\phi)$ is the solution of

$$dV_t(\phi) = \phi^0_t \, dB_t + \phi^1_t \, dS_t, \text{ where } V_0 = x. \tag{2}$$

Notice that the second terminology is conventionally used in the literature about arbitrage–free pricing and hedging of contingent claims. Without introducing sources of market incompleteness, both *approaches* will give the same result. In this case, we obviously have $\phi^0_t = \frac{(1-\pi)V_t}{S_t}$ and $\phi^1_t = \frac{\pi V_t}{B_t}$. However, there are a few comments necessary. As indicated above, traditionally a specification of the trading strategy in terms of the investment quote is closely linked to proportional portfolio insurance (PPI) while a specification of the total shares held is used in the option based approach (OBPI). In a complete financial market setting, both approaches can be translated, i.e. a dynamic OBPI strategy can also be expressed as a PPI, c.f. Bertrand and Prigent (2002a). In contrast, market incompleteness and/or model risk give rise to different results. Another example is given by trading restrictions in the sense of discrete–time trading. Here, a discrete–time version can only be specified in terms of the numbers Φ as there is no trading possible between two trading dates. While the number of shares must therefore be constant on each trading interval, the investment quote will obviously change if there are movements in the asset price.

2.2. **Cash–Lock probability.** The following section aims at a formal description of the investment quote π. Recall that the basic decision which the investor faces is the division of her wealth between the risky asset and the risk–free one. In this sense, it is interesting to measure the gains which are caused by improvements of the asset price. This is easy in the case that we have a path–independent strategy.[8]

Definition 2.1 (Participation rate, level). *The sensitivity of the (terminal) strategy value with respect to a change in the (terminal) asset price is called participation rate, i.e.*

$$p^{\Phi}(S) = \frac{\partial}{\partial S} V_T^{\Phi}(S) \ . \tag{3}$$

Let L^{Φ} denote the minimal asset price such that the participation rate larger than zero, i.e.

$$L^{\Phi} = \inf \left\{ S \geq 0 | p^{\Phi}(S) > 0 \right\} . \tag{4}$$

In particular, we call L^{Φ} the participation level.

Notice that the above definition is suitable with respect to the portfolio insurance strategies which are characterized by the participation, if any, in upward moving markets.[9] However, more things are to be considered in the case of path–dependent strategies where the payoff (the terminal value of the strategy) does not exclusively depend on the terminal asset price but on the whole price path. One might argue that the path dependence causes an additional source of uncertainty.[10] On the other hand, one might argue against the path–independency along the following lines. Recall that a path–independent strategy implies that the t–value does only depend on the asset price increment $\frac{S_t}{S_0}$. Thus, in the case of a sudden drop in the asset price at t there is no lock in of prior gains possible. However, the following concept of a cash–lock measure does not depend on the path–dependency (path–independency) of the strategy under consideration.

Definition 2.2 (Cash–lock probability). *At t ($t \in [0, T[$), a strategy is called α–cash-dominant iff it holds $\pi_t \leq \alpha$ ($\alpha \in [0, 1]$). For the time interval $[t, \tau]$ ($0 \leq t \leq \tau \leq T$), the α–β Cash–lock probability $P_{t,\tau}^{CL_{\alpha,\beta}}$ denotes the probability that the investment quote at τ is less that β given that it is equal to α at t, i.e.*

$$P_{t,\tau}^{CL_{\alpha,\beta}} := P[\pi_\tau^V \leq \beta | \pi_t = \alpha]$$

[8] Notice that a strategy can be path–independent with respect to one particular model, but fails to be path–independent with respect to another model.

[9] In particular, the strategies do not allow for short positions in the asset. Here the *opposite* definition of the participation level is convenient.

[10] An easy example is given by the stop–loss strategy where a barrier condition implies either a participation rate of one or zero.

In the next section we determine the risk profile of classic portfolio insurance strategies without posing borrowing constraints. After this, we consider the effects of borrowing constraints which automatically introduce a path–dependency.

3. Cash–Lock probabilities and expected values

As a benchmark case, we consider the Black–Scholes model, i.e. the dynamics of the risky asset S are given by the stochastic differential equation (SDE)

$$d\,S_t = S_t\,(\mu\,dt + \sigma\,dW_t)\,, \quad S_0 = s, \tag{5}$$

where $W = (W_t)_{0\leq t\leq T}$ denotes a standard Brownian motion with respect to the *real world* measure P. μ and σ are constants and we assume that $\mu > r \geq 0$ and $\sigma > 0$. The following subsections are all structured analogously. First, we define the strategy under consideration in terms of their their associated numbers of bonds and assets $\phi = (\phi^0, \phi^1)$. Then we summarize the t–value of the strategy, its distribution and finally the cash–lock probability.

3.1. **Stop–loss strategy (SL).** Consider the stop–loss strategy (SL) where

$$\begin{aligned} G_t &:= e^{-r(T-t)}G, \\ \tau &:= \inf\{t \geq 0 : V_t^{SL} = G_t\}. \end{aligned}$$

With respect to the number of bonds and assets which are held in the portfolio, the SL–strategy is then defined by

$$\phi_t^{SL,0} = 1_{\{\tau\leq t\}}\frac{G_\tau}{B_\tau} = 1_{\{\tau\leq t\}}G, \quad \phi_t^{SL,1} = 1_{\{\tau>t\}}\frac{V_0}{S_0} \tag{6}$$

such that the t–value V_t^{SL} which is implied by the SL–strategy is

$$V_t^{\mathrm{SL}} = V_0\frac{S_t}{S_0}1_{\{\tau>t\}} + G_t 1_{\{\tau\leq t\}}. \tag{7}$$

In particular, notice that the SL–strategy is path–dependent in the sense that its value is not exclusively specified in terms of the current asset price but does depend on the whole price path until t. Concerning the distribution of the value, one obtains

Proposition 3.1 (Distribution and density associated with SL). *If the asset price dynamics are given by Equation (5), then it holds*

$$
\begin{aligned}
P[V_t^{SL} \leq w] &= N\left(\frac{\ln \frac{w}{V_0} - (\mu - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right) \\
&\quad + \left(\frac{V_0}{G_0}\right)^{1-2\frac{\mu-r}{\sigma^2}} N\left(\frac{2\ln G_0 - \ln(wV_0) + (\mu - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right) \text{ for } w \geq G_t \quad (8)\\
P[V_t^{SL} = G_t] &= N\left(\frac{\ln \frac{G_0}{V_0} - (\mu - r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right) \\
&\quad + \left(\frac{V_0}{G_0}\right)^{1-2\frac{\mu-r}{\sigma^2}} N\left(\frac{\ln \frac{G_0}{V_0} + (\mu - r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right) \quad (9)\\
P[V_t^{SL} \leq w] &= 0 \text{ for } w < G_t. \quad (10)
\end{aligned}
$$

In particular, for $w > G_t$ it holds

$$
\begin{aligned}
P[V_t^{SL} \in dw] &= \frac{1}{w\sigma\sqrt{t}}\left(\phi\left(\frac{\ln \frac{w}{V_0} - (\mu - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right)\right. \\
&\quad \left. - \left(\frac{V_0}{G_0}\right)^{1-2\frac{\mu-r}{\sigma^2}} \phi\left(\frac{2\ln G_0 - \ln(wV_0) + (\mu - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right)\right) dw. \quad (11)
\end{aligned}
$$

Proof: The proof is given in Appendix A.

It is worth mentioning to recall that the SL–strategy is very aggressive in the sense that it either implies an investment quote π^{SL} of one or zero. Therefore, it is enough to summarize the cash–lock probability with respect to the cases $\pi_0^{\mathrm{SL}} = 1$ and $\pi_0^{\mathrm{SL}} = 0$.

Proposition 3.2 (Cash–lock probability of SL). *Let $\beta \geq 0$, then it holds*

$$
\begin{aligned}
P\left[\pi_t^{SL} \leq \beta \,\middle|\, \pi_0^1 = 0\right] &= 1 \\
\text{and } P\left[\pi_t^{SL} \leq \beta \,\middle|\, \pi_0^1 = 1\right] &= \begin{cases} 1 & \beta \geq 1 \\ P[V_t^{SL} = G_t] & \beta < 1 \end{cases}.
\end{aligned}
$$

Proof: The proof follows immediately with Proposition 3.1.

Concerning the expected value, we have the following proposition.

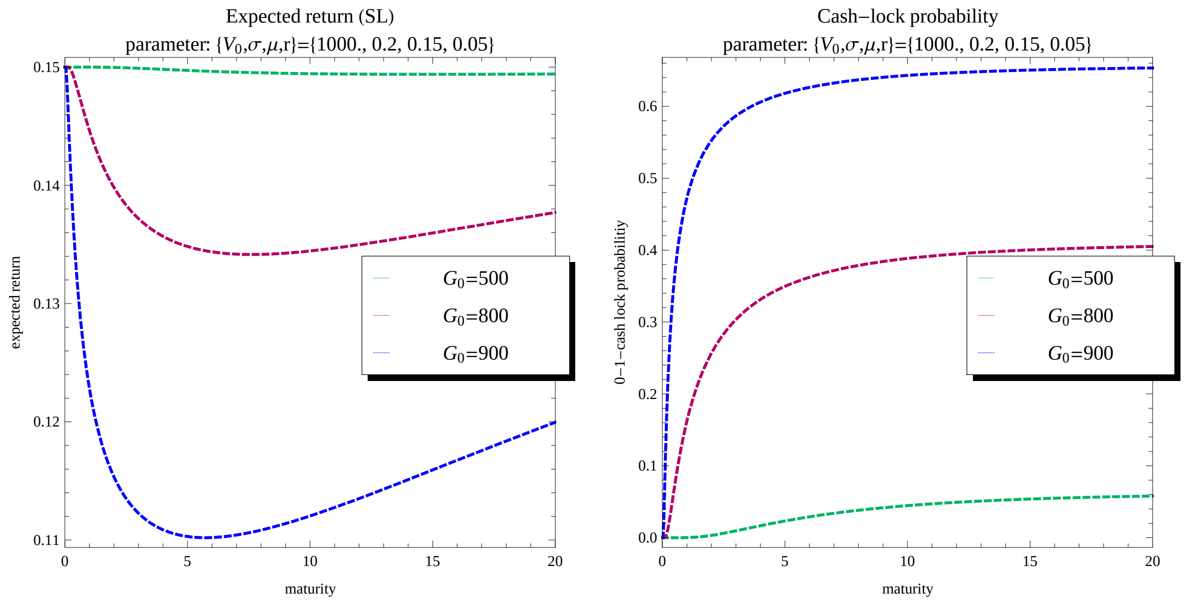

FIGURE 1. Expected return and cash lock probability

PROPOSITION 3.3 (Expected value of SL). *The expected value of the stop–loss strategy is given by*

$$E[V_t^{SL}]$$
$$= G_0 e^{rt} N\left(\frac{\ln\frac{G_0}{V_0} - (\mu - r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right) + V_0 e^{rt}\left(\frac{G_0}{V_0}\right)^{2\frac{\mu-r}{\sigma^2}} N\left(\frac{\ln\frac{G_0}{V_0} + (\mu - r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right)$$
$$-G_0 e^{\mu t}\left(\frac{G_0}{V_0}\right)^{2\frac{\mu-r}{\sigma^2}} N\left(\frac{\ln\frac{G_0}{V_0} + (\mu - r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right) + V_0 e^{\mu t} N\left(\frac{\ln\frac{V_0}{G_0} + (\mu - r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right). \quad (12)$$

PROOF: The proof follows immediately with Proposition 3.1.

3.2. **Generalized OBPI (GO).** The following results refer to the dynamic version of a generalized OBPI strategy (GO) in the Black-Scholes model setup. If the asset price dynamics are given by Equation (1), then the GO–strategy $\phi^{\mathrm{GO}} = (\phi^{\mathrm{GO},0}, \phi^{\mathrm{GO},1})$ is defined by[11]

$$\phi_t^{\mathrm{GO},0} = G_t\left(1 - \frac{\alpha}{\beta}\mathcal{N}(d_-(t, S_t))\right) \quad \text{and} \quad \phi_t^{\mathrm{GO},1} = \alpha\mathcal{N}(d_+(t, S_t))$$

where

$$d_\pm(t, S_t) := \frac{\ln\frac{\beta S_t}{G_T} + (r \pm \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}.$$

[11] It is worth mentioning that, in contrast to the SL–strategy and the following PI–strategies, the dynamic version of the OBPI itself depends on the assumptions on the asset price dynamics.

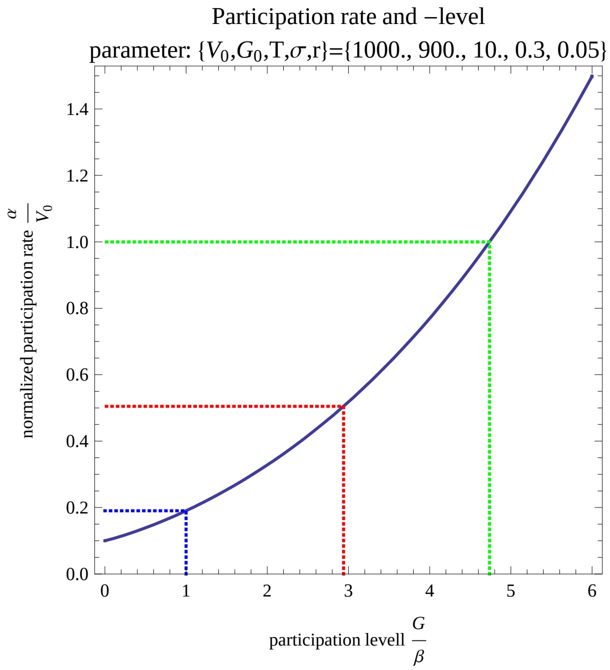

FIGURE 2. Participation rate for varying participation levels.

It is straightforward to show that the t–value of the GO strategy is given by[12]

$$V_t^{\mathrm{GO}} = G_t + \alpha\mathcal{N}(d_+(t,S_t)) - G_t\frac{\alpha}{\beta}\mathcal{N}(d_-(t,S_t)) \text{ for } t < T \tag{13}$$

$$\text{and } V_T(\phi^{\mathrm{GO}}) = V_T^{\mathrm{GO}} = G_T + \alpha\left[S_T - \frac{G_T}{\beta}\right]^+. \tag{14}$$

The above implies that for an exogenously given initial investment V_0, the participation rate α and the participation level $\frac{G_T}{\beta}$ must satisfy the initial condition

$$\frac{V_0 - G_0}{\alpha} = N(d_+(0,S_0)) - \frac{G_0}{\beta}N(d_-(0,S_0)).$$

LEMMA 3.4 (Participation rate). *Let*

$$\alpha^*(\beta) := \frac{V_0 - G_0}{= N(d_+(0,S_0)) - \frac{G_0}{\beta}N(d_-(0,S_0))},$$

then it holds $\frac{\partial\alpha^*}{\partial\beta} < 0$.

PROOF: The proof follows immediately by differentiation. In particular, α^* is inversely related to the price of a call–option with strike $K = \frac{G_T}{\beta}$ where the call price is decreasing in the strike K.

[12]Notice that the strategy $\Phi^{\mathrm{BS}} = (\phi^{\mathrm{GO},0} - G_t, \phi^{\mathrm{GO},1})$ is the Black Scholes hedge for a call–option with maturity T and strike $K = \frac{G}{\beta}$.

Notice that the above proposition states that a higher participation rate α can only be achieved by an increase in the participation level $\frac{G_T}{\beta}$ (a reduction in β, respectively). This effect is illustrated in Fig. 3.2.

Proposition 3.5 (Distribution and density associated with GO).

$$P[V_T^{GO} \leq v] = \mathcal{N}\left(\frac{\ln\left(\frac{v-G_T}{\alpha}+\frac{G_T}{\beta}\right)-(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \text{ for } v \geq G_T \tag{15}$$

$$P[V_T^{GO} = G_T] = \mathcal{N}\left(\frac{\ln\frac{G_T}{\beta}-(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \tag{16}$$

$$P[V_T^{GO} \leq v] = 0 \text{ for } v < G_T. \tag{17}$$

In particular, for $v > G_T$ it holds

$$P[V_T^{GO} \in dv] = \frac{1}{\left(v-\frac{\beta-\alpha}{\beta}G_T\right)\sigma\sqrt{T}}\phi\left(\frac{\ln\left(\frac{v-G_T}{\alpha}+\frac{G_T}{\beta}\right)-(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right).$$

Proposition 3.6 (Cash–lock probability of GO). *For the γ_1-γ_2 cash-lock probability with respect the time points t_1, t_2 of a GO strategy with time to maturity T it holds*

$$P[\pi_{t_2}^{GO} < \gamma_2 | \pi_{t_1}^{GO} = \gamma_1] = \mathcal{N}\left(\frac{\ln\frac{\mathbf{s}(t_1,\gamma_1)}{\mathbf{s}(t_2,\gamma_2)}-(\mu-\frac{1}{2}\sigma^2)(t_2-t_1)}{\sigma\sqrt{t_2-t_1}}\right).$$

where $\mathbf{s}(t,\gamma)$ denotes the inverse mapping of $\pi_t^1(S_t = s)$, i.e. $\mathbf{s}(t,\gamma) = s \Leftrightarrow \pi_t^1(S_t = s) = \gamma$.

Proof: The proof follows immediately from the assumption that the asset price increments are independent and identically distributed and

$$P[\pi_{t_2}^1 < \gamma_2 | \pi_{t_1}^1 = \gamma_1] = P[S_{t_2} < \mathbf{s}(t_2,\gamma_2) | S_{t_1} < \mathbf{s}(t_1,\gamma_1)].$$

Proposition 3.7 (Expected value of GO). *The expected value of the dynamic option based portfolio strategy is given by*

$$E[V_T^{GO}] = G_T\left(1-\frac{\alpha}{\beta}\mathcal{N}\left(\tilde{d}_-\right)\right)+\alpha e^{\mu T}\mathcal{N}\left(\tilde{d}_+\right)$$

where

$$\tilde{d}_\pm := \frac{\ln\frac{\beta S_t}{G_T}+(\mu\pm\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

3.3. **Classic CPPI (CP).** The following results refer to the classic CPPI version (CP) with constant multiplier m and guarantee $G_t = G_0 \exp\{rt\}$. Notice that the classic version of the CPPI strategy relies on a t–guarantee (also called floor) which is growing with the risk–free rate r. However, to specify the strategy, it is not necessary to define a fixed time horizon T. The numbers of assets and bonds are given by

$$\phi_t^{0,\mathrm{CP}} = \frac{V_t^{\mathrm{CP}} - mC_t^{\mathrm{CP}}}{B_t}, \quad \phi_t^{1,\mathrm{CP}} = \frac{mC_t^{\mathrm{CP}}}{S_t} \tag{18}$$

$$\text{where } C_t^{\mathrm{CP}} := V_t^{\mathrm{CP}} - G_t. \tag{19}$$

Proposition 3.8 (Dynamics of cushion). *If the asset price dynamics are lognormal as described by Equation (5), the cushion process* $\left(C_t^{CP}\right)_{0\le t\le T}$ *of a classic CPPI is lognormal, too. It holds*

$$dC_t^{CP} = C_t^{CP}\left((r + m(\mu - r))\, dt + \sigma m\, dW_t\right).$$

Proof: The proof follows immediately with $C_t^{\mathrm{CP}} := V_t^{\mathrm{CP}} - G_t$ and Equation (5) and (1).

Proposition 3.9 (Value and expected value of CP). *The t–value of the a simple CPPI with parameter m and G is*

$$V_t^{CP} = G_t + C_0^{CP} e^{rt} \left(\frac{S_t}{S_0} e^{-(r+\frac{m-1}{2}\sigma^2)t}\right)^m. \tag{20}$$

In particular, the expected value is

$$E\left[V_t^{CP}\right] = G_t + (V_0^{CP} - G_0)\exp\left\{(r + m(\mu - r))\, t\right\} \tag{21}$$

Proof: For example, c.f. Black and Perold (1992) and Bertrand and Prigent (2002a).

Proposition 3.10. *For $\alpha, \beta \geq 0$, the α-β-cash-lock-probability is given by*

$$P\left[\pi_T^{CP} \le \beta \,\middle|\, \pi_t^{CP} = \alpha\right] = 1 \text{ for } \alpha = 0, \tag{22}$$

$$P\left[\pi_T^{CP} \le \beta \,\middle|\, \pi_t^{CP} = \alpha\right] = N\left(\frac{\frac{1}{m}\ln\frac{\beta(m-\alpha)}{\alpha(m-\beta)} - (\mu - r - \frac{m}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \text{ for } \alpha > 0. \tag{23}$$

Proof: The proof is dedicated to the appendix, cf. Appendix B.

3.4. **PPI with constant floor (CFP).** The CFP–strategy can be interpreted as a modified version of the classic CPPI principle: The asset exposure equals a constant multiple times the cushion and the cushion is given by the difference of the portfolio value and the floor. However, while the classic version is based on a floor which is growing according to the risk–free rate r, the CFP–strategy is based on a constant floor G. In particular, the strategy is defined by

$$\phi_t^{0,\text{CFP}} = \frac{V_t^{\text{CFP}} - mC_t^{\text{CFP}}}{B_t}, \quad \phi_t^{1,\text{CP}} = \frac{mC_t^{\text{CFP}}}{S_t} \tag{24}$$

$$\text{where } C_t^{\text{CFP}} := V_t^{\text{CFP}} - G. \tag{25}$$

Proposition 3.11 (Dynamics of cushion). *If the asset price dynamics are lognormal as described by Equation (5), the cushion process* $\left(C_t^{CFP}\right)_{0\le t\le T}$ *of a CFP is lognormal, too. It holds*

$$C_t^{CFP} = e^{(m\mu-(m-1)r-\frac{1}{2}m^2\sigma^2)t+m\sigma W_t}\left(C_0^{CFP} + r\,G\int_0^t e^{-(m\mu-(m-1)r-\frac{1}{2}m^2\sigma^2)s-m\sigma W_s}\,ds\right) \tag{26}$$

$$= C_0^{CFP}e^{rt}\left(\frac{S_t}{S_0}e^{-(r+\frac{m-1}{2}\sigma^2)t}\right)^m + r\,G\int_0^t e^{r(t-s)}\left(\frac{S_t}{S_s}e^{-(r+\frac{m-1}{2}\sigma^2)(t-s)}\right)^m ds \tag{27}$$

Proof: The proof is given in the appendix, cf. Appendix C.

Corollary 3.12 (Expected value of CFP). *For* $m\mu \neq (m-1)r$*, the expected value of a CFP–strategy with parameters* m *and* G *is given by*

$$E\left[V_t^{CFP}\right] = G + C_0^{CFP}e^{(m\mu-(m-1)r)t} + \frac{rG}{m\mu-(m-1)r}\left(e^{(m\mu-(m-1)r)t} - 1\right). \tag{28}$$

Proof: The above result follows immediately with $V_t^{\text{CFP}} = G + C_t^{\text{CFP}}$ and Proposition 3.11.

4. Effects of borrowing constraints

Before we compare the performance of the portfolio insurance strategies under consideration, we consider the effects of borrowing constraints. To our opinion, it seems to be *unfair* to compare strategies which allow for an investment quote which is higher than one with strategies which are constructed such that the investment quote is less or equal to one. Before we illustrate the performance of the constrained and unconstrained strategies, we consider the effects of borrowing constraints in this section.

4.1. **Stop–loss strategy (SL).** Since the investment quote of the SL–strategy is either zero or one, there are no effects of borrowing constraints.

4.2. **Generalized OBPI (GO) with borrowing constraints, classic OBPI (OB).** In the case of borrowing constraints, not all strategies belonging to the class of generalized option based portfolio insurance strategies, c.f. definition of the GO–strategies in subsection 3.2, are admissible. In consequence, there is a restriction on the strategy parameters α and β.[13]

LEMMA 4.1 (Borrowing constraints). *Restricting the investment quote π^{GO} to the value w ($w \geq 1$), i.e. $\pi^{GO} \leq w$, then a necessary and sufficient condition is given by*

$$(1-w)\alpha\mathcal{N}(d_+(t,S_t))S_t \leq wG_t\left(1-\frac{\alpha}{\beta}\mathcal{N}(d_-(t,S_t))\right) \quad \text{for all } S_t \geq 0.$$

In particular, for $w = 1$, i.e. a maximal investment quote of 1, the condition simplifies to

$$\frac{\alpha}{\beta} \leq 1. \tag{29}$$

PROOF: With $\pi^{\mathrm{GO}} = \frac{\phi_t^{\mathrm{GO},1}S_t}{V_t^{\mathrm{GO}}} \leq w$, Equation (13) it follows immediately

$$\begin{aligned} & \pi^{\mathrm{GO}} \leq 1 \\ \Longleftrightarrow \quad & (1-w)\alpha\mathcal{N}(d_+(t,S_t))S_t \leq wG_t\left(1-\frac{\alpha}{\beta}\mathcal{N}(d_-(t,S_t))\right) \text{ for all } S_t \geq 0. \end{aligned}$$

LEMMA 4.2 (Classic OBPI). *For $\alpha = \beta$, it holds $\pi^{GO} \leq 0$. In particular, we call a generalized OBPI strategy (GO) strategy with $\alpha = \beta$ classic OBPI (CO).*

PROOF: The above lemma is an immediate consequence of Lemma 4.1.

In particular, the classic option based portfolio insurance strategy CO is a meaningful version of the generalized one (GO).

4.3. **Capped CPPI (CCP).** Posing borrowing constraints on the classic CPPI strategy straightforwardly results in the capped CPPI (CCP).

DEFINITION 4.3. *The capped CPPI strategy (CCP) is defined by*

$$\begin{aligned} \phi_t^{0,CCP} &= \frac{V_t^{CCP} - mC_t^{CCP}}{B_t}, \quad \phi_t^{1,CCP} = \frac{\max\left(\omega V_t^{CCP}, mC_t^{CCP}\right)}{S_t} \\ \text{where } C_t^{CCP} &:= V_t^{CCP} - G_t. \end{aligned}$$

Again, $G_t = G_0e^{rt}$, m ($m \geq 2$) is a constant and w ($w \geq 1$) denotes the restriction on the investment quote.

PROPOSITION 4.4 (Value and cushion of CCP). *Let*

$$dX_t = \Theta(X_t)dt + dW_t$$

[13] Recall that α gives the participation rate and $\frac{G_T}{\beta}$ gives the participation level of the strategy, cf. definition 2.1.

where

$$\Theta(x) = \begin{cases} \frac{\mu-r}{\sigma} - \frac{1}{2}m\sigma & x \leq 0 \\ \frac{\mu-r}{\sigma} - \frac{1}{2}\sigma & x > 0 \end{cases} \tag{30}$$

and

$$X_0 = \begin{cases} \frac{1}{\sigma}\ln\frac{(m-1)V_0}{mG_0} & mC_0 \geq V_0 \\ \frac{1}{m\sigma}\ln\frac{(m-1)C_0}{G_0} & mC_0 < V_0 \end{cases} . \tag{31}$$

If the asset price dynamics are lognormal as described by Equation (5), the value process $\left(V_t^{CCP}\right)_{0\leq t\leq T}$ *and cushion process* $\left(C_t^{CCP}\right)_{0\leq t\leq T}$ *are given by*

$$V_t = G_t \begin{cases} \frac{m}{m-1}e^{\sigma X_t} & X_t \geq 0 \\ \left(1 + \frac{1}{m-1}e^{\sigma m X_t}\right) & X_t < 0 \end{cases} \tag{32}$$

and

$$C_t = G_t \begin{cases} \left(\frac{m}{m-1}e^{\sigma X_t} - 1\right) & X_t \geq 0 \\ \frac{1}{m-1}e^{m\sigma X_t} & X_t < 0 \end{cases} .$$

Proof: The proof is dedicated to the appendix.

Proposition 4.5 (Density of CCP). *Let X_t be defined as in Proposition 4.4, then it holds*

$$P\left[V_t^{CCP} \in dv\right] = \begin{cases} \frac{1}{\sigma v}\, p\left(\frac{\ln\frac{(m-1)v}{mG_t}}{\sigma}\right) dv & v \geq \frac{m}{m-1}G_t \\ \frac{1}{\sigma m(v-G_t)}\, p\left(\frac{\ln\frac{(m-1)(v-G_t)}{G_t}}{\sigma m}\right) dv & v < \frac{m}{m-1}G_t \end{cases}$$

where $p(x) := P_{X_0}(X_t \in dx)$.

Proof: Cf. Appendix. D .

Proposition 4.6 (Distribution of CCP). *For $m(V_0 - G_0) \geq V_0$ it holds*

$$P[V_T^{CCP} \leq v] = \mathcal{L}_{\lambda,T}^{(-1)} \begin{cases} 0 & v \leq G_T \\ \frac{K_1(\lambda)}{K_2(\lambda)}\left(\frac{(m-1)(v-G_T)}{G_T}\right)^{\frac{K_2(\lambda)}{\sigma m}} & G_T < v \leq \frac{m\,G_T}{m-1} \\ \frac{K_3(\lambda)}{K_4(\lambda)}\left(\frac{(m-1)v}{mG_T}\right)^{\frac{K_4(\lambda)}{\sigma}} + \frac{K_5(\lambda)}{K_6(\lambda)}\left(\frac{(m-1)v}{mG_T}\right)^{\frac{K_6(\lambda)}{\sigma}} & \frac{m\,G_T}{m-1} < v \leq V_0e^{rT} \\ \frac{1}{\lambda} + \frac{K_5(\lambda)+e^{2x_0\sqrt{\theta_0^2+2\lambda}}K_3(\lambda)}{K_6(\lambda)}\left(\frac{(m-1)v}{mG_T}\right)^{\frac{K_6(\lambda)}{\sigma}} & V_0e^{rT} < v \end{cases}$$

and for $m(V_0 - G_0) < V_0$ we have

$$P[V_T^{CCP} \leq v] =$$

$$\mathcal{L}^{(-1)}_{\lambda,T}\begin{cases} 0 & v \leq G_T \\ -\frac{\tilde{K}_5(\lambda)+e^{-2x_0\sqrt{\theta_1^2+2\lambda}}\tilde{K}_3(\lambda)}{\tilde{K}_6(\lambda)}\left(\frac{(m-1)(v-G_T)}{G_T}\right)^{-\frac{\tilde{K}_6(\lambda)}{\sigma m}} & G_T < v \leq V_0 e^{rT} \\ \frac{1}{\lambda} - \frac{\tilde{K}_3(\lambda)}{\tilde{K}_4(\lambda)}\left(\frac{(m-1)(v-G_T)}{G_T}\right)^{-\frac{\tilde{K}_4(\lambda)}{\sigma m}} - \frac{\tilde{K}_5(\lambda)}{\tilde{K}_6(\lambda)}\left(\frac{(m-1)(v-G_T)}{G_T}\right)^{-\frac{\tilde{K}_6(\lambda)}{\sigma m}} & V_0 e^{rT} < v \leq \frac{mG_T}{m-1} \\ \frac{1}{\lambda} - \frac{\tilde{K}_1(\lambda)}{\tilde{K}_2(\lambda)}\left(\frac{(m-1)v}{mG_T}\right)^{-\frac{\tilde{K}_2(\lambda)}{\sigma}} & \frac{mG_T}{m-1} < v \end{cases}$$

where $\mathcal{L}^{(-1)}_{\lambda,T}$ denotes the inverse Laplace transform. With respect to the functions $K_i(\lambda)$ and $\tilde{K}_i$ we refer to the appendix, cf. Theorem E.1.

Proof: Cf. Appendix E.1.

Notice that the cash–lock probability follows immediately with the distribution given in Proposition 4.6. Finally, consider the moments of the CCP.

Proposition 4.7 (Moments of CCP). *For $m(V_0 - G_0) \geq V_0$, it holds*

$$E[(V^{CCP})^n_T] = (e^{rT})^n \mathcal{L}^{(-1)}_{\lambda,T}\left\{\frac{V_0^n}{\lambda - n(\mu - r) - \frac{1}{2}n(n-1)\sigma^2} + G_0^n\left(\sum_{i=0}^{n}\binom{n}{i}\frac{K_1(\lambda)\left(\frac{1}{m-1}\right)^i}{i\sigma m + K_2(\lambda)} - \left(\frac{m}{m-1}\right)^n\left(\frac{K_3(\lambda)}{n\sigma + K_4(\lambda)} + \frac{K_5(\lambda)}{n\sigma + K_6(\lambda)}\right)\right)\right\}$$

For $m(V_0 - G_0) < V_0$, we have

$$E[(V^{CCP})^n_T] = (e^{rT})^n \mathcal{L}^{(-1)}_{\lambda,T}\left\{\sum_{i=0}^{n}\binom{n}{i}\frac{G_0^{n-i}(V_0 - G_0)^i}{\lambda - im(\mu - r) - \frac{1}{2}i(i-1)(m\sigma)^2} + G_0^n\left(\sum_{i=0}^{n}\binom{n}{i}\left(\frac{\tilde{K}_3(\lambda)\left(\frac{1}{m-1}\right)^i}{i\sigma m - \tilde{K}_4(\lambda)} + \frac{\tilde{K}_5(\lambda)\left(\frac{1}{m-1}\right)^i}{i\sigma m - \tilde{K}_6(\lambda)}\right) - \left(\frac{m}{m-1}\right)^n\frac{\tilde{K}_1(\lambda)}{n\sigma - \tilde{K}_2}\right)\right\}$$

Proof: The proof is given in Appendix E.2.

Intuitively, it is clear that the investment quote of the CCP–version of a CP strategy is almost surely less than the one of the underlying CP–strategy. Assuming $\mu > r$, this immediately explains that the CCP–strategy is due to a lower expected return and, at the same time, gives a lower variance. The effects of borrowing constraints are summarized in Table 1 which describes the distribution of the final values of classic CPPI and the version which results from the different borrowing constraints w in terms of the expected value, the standard deviation, the skewness and kurtosis. The initial investment is $V_0 = 1000$,

m	ω	expected value		standarddev.		skewness		kurtosis	
		CP	CCP	CP	CCP	CP	CCP	CP	CCP
1	1	1121	1121.20	68.4	68.4	0.89	0.89	4.44	4.44
3	1	1157	1154.20	280	241	4.16	2.05	44.82	8.71
3	2	1157	1155.50	280	267	4.16	2.97	44.82	16.46
3	2.99	1157	1156.82	280	280	4.16	4.16	44.82	44.82
5	1	1198	1167.81	793	305	23.73	1.59	3949	6.01
5	2	1198	1137.58	793	321	23.73	2.05	3949	7.74
5	2.5	1198	1145.63	793	357	23.73	2.32	3949	8.97
5	4.99	1198	1197.80	793	793	23.73	22.93	3949	2576
10	1	1329	1172.20	24298	325	$163 \cdot 10^3$	1.35	$79 \cdot 10^{12}$	5.06
10	2	1329	1082.65	24298	310	$163 \cdot 10^3$	1.88	$79 \cdot 10^{12}$	6.73
10	5	1329	1059.49	24298	416	$163 \cdot 10^3$	2.95	$79 \cdot 10^{12}$	11.83
10	9.99	1329	1307.01	24298	7650	$163 \cdot 10^3$	82.49	$79 \cdot 10^{12}$	9058
20	1	1780	1172.18	$7.9 \cdot 10^9$	329	$7.0 \cdot 10^{20}$	1.29	$3,.9 \cdot 10^{55}$	4.88
SL		1171.54		330		1.28		4.85	

Table 1. Moments of classic CPPI (CP) and its capped version (CCP) with respect to different borrowing constraints w and multiplier is m. The parameter constellation is summarized by $V_0 = 1000$, $G_0 = 800$, $\mu = 0.085\%$, $r = 0.05$, $\sigma = 0.20$ and $T = 2$.

the initial guarantee $G_0 = 800$. The Black–Scholes parameters are set to $\mu = 0.085\%$, $r = 0.05$ and $\sigma = 0.20$. The time horizon is given by $T = 2$ years.

5. Comparison of capped CPPI (CCP) and classic OBPI (CO)

According to the previous sections, it is interesting to compare the performance of the capped CPPI with the one of a classic OBPI. Unless stated otherwise, we consider an initial investment of $V_0 = 1000$, an initial guarantee of $G_0 = 800$ and a multiplier of $m = 5$. With respect to the asset price dynamics, we assume that the volatility is $\sigma = 0.2$ and $\mu = 0.15$. The risk–free interest rate is given by $r = 0.05$.

Fig. 3 illustrates the density of the terminal value for capped CPPI (CCP), classic OBPI (CO) and stop–loss (SL). The time horizon is 2 years (10 years, respectively).

Fig. 4 shows the expected return of strategies under borrowing constraint for varying times to maturity.

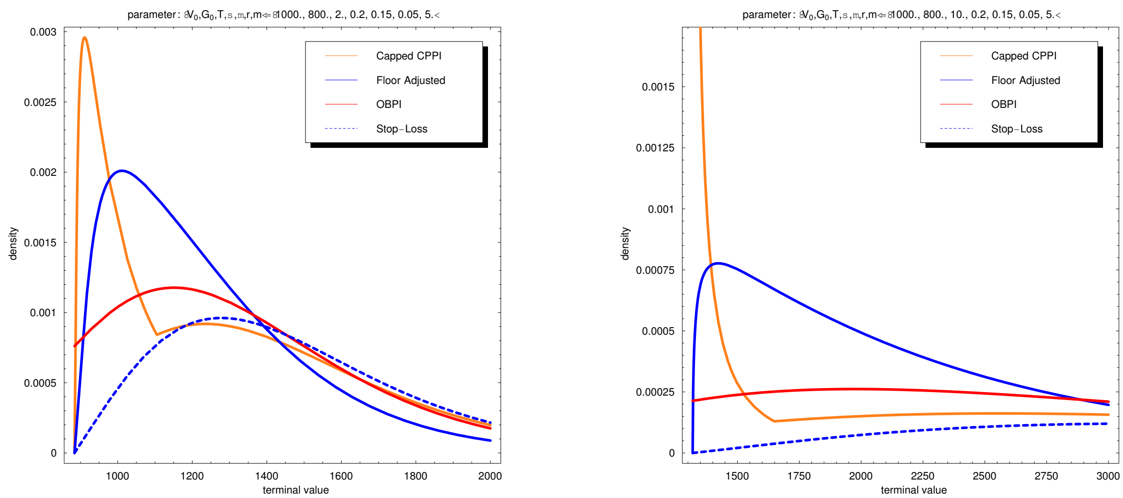

FIGURE 3. Density of the terminal value for capped CPPI (CCP), classic OBPI (CO) and stop–loss (SL). The time horizon is 2 years (left figure) and ten years (right figure).

Finally, Fig. 5 illustrates the different cash–lock behavior.

6. CONCLUSION

Recently, there is a growing literature concerning the topic of portfolio insurance. The revival of OBPI and CPPI strategies is caused by the growing popularity of guarantees which is in particular observable with respect to retail products. In falling markets, the basic protection principle affords a reduction of the fraction of wealth which is invested in risky assets. On the other hand, there is the demand towards upside participation in rising markets such that a high participation level is wanted.

One problem which is associated with dynamic versions of portfolio insurance strategies is caused by the so–called gap risk, i.e. the risk that the guarantee is not honored with probability one. This risk is due to various sources of market incompleteness. For

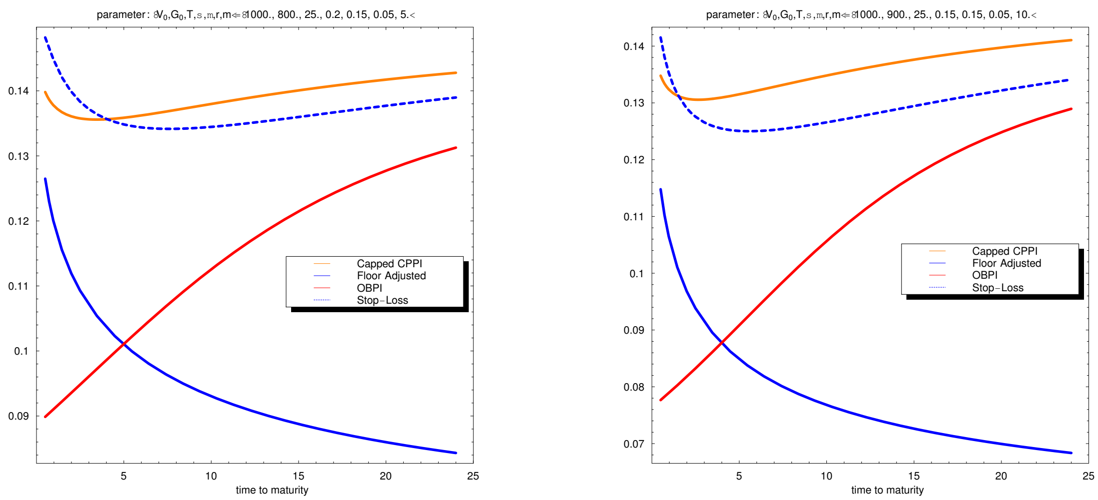

FIGURE 4. Expected return of strategies under borrowing constraint for varying times to maturity.

example, trading restrictions, price jumps impeded the concept of perfect hedging as well as the portfolio protection feature of a CPPI strategy.

However, this paper aims at a different problem, the cash–lock cage. In the strict sense, a cash–lock describes the event that the asset exposure drops to zero and stays there. Since a cash–lock at an early time prohibits any participation in recovering markets, the cash–lock cage is considered as a major problem with respect to long investment horizons.

We give a formal analysis of a generalized cash–lock measure, i.e. we focus on the probability that the investment quote recovers from small values. It turns out that, even in the case that the dynamic versions of option based strategies and proportional portfolio insurance strategies coincide in their expected return, the strategies give rise to a very different cash–lock behavior. In addition, we point out that, for comparability reasons, it is necessary to distinguish between strategies where an investment quote over one is admissible and such with borrowing constraints.

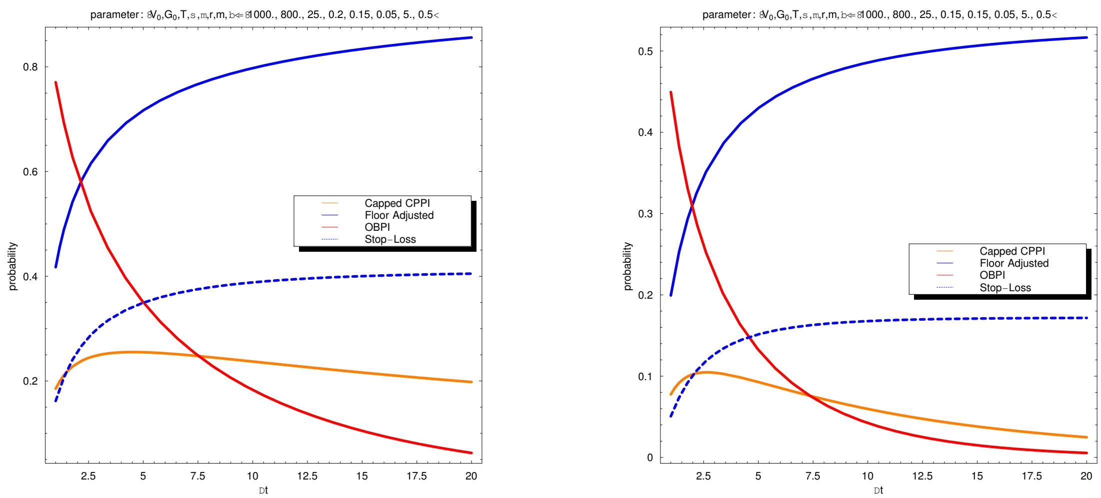

FIGURE 5. ($\beta = 0.5$) Cash–lock probabilities of strategies under borrowing constraint.

APPENDIX A. PROOF OF PROPOSITION 3.1

Notice that

$$P[V_t^{SL} \le w] = P[V_t^{SL} = G_t, \tau \le t] + P[V_t^{SL} \le w, \tau > t]. \tag{33}$$

Observe that the hitting event can be expressed as

$$\begin{aligned} \{\tau \le t\} &= \{\min_{0 \le s \le t} V_s^{SL}/G_s \le 1\} \\ &= \left\{ \min_{0 \le s \le t} \left\{ W_s + \frac{\mu - r - \frac{1}{2}\sigma^2}{\sigma} s \right\} \le \frac{1}{\sigma} \ln \frac{G_0}{V_0} \right\} \end{aligned}$$

Let $(M_t)_{t \ge 0} := \max_{0 \le s \le t} W_s$ and $(m_t)_{t \ge 0} := \min_{0 \le s \le t} W_s$ where W denotes a standard Brownian motion. Accordingly, we define $(M_t^\nu)_{t \ge 0} := \max_{0 \le s \le t} W_s^\nu$ and $(m_t^\nu)_{t \ge 0} := \min_{0 \le s \le t} W_s^\nu$ where W denotes a Brownian motion with drift ν. In particular, we have

$$\{\tau \le t\} \quad = \quad \left\{ m_t^\nu \le \frac{1}{\sigma} \ln \frac{G_0}{V_0} \right\} \quad \text{where } \nu := \frac{\mu - r - \frac{1}{2}\sigma^2}{\sigma}$$

Using well known results gives[14]

$$\begin{aligned} P[V_t^{SL} = G_t] &= P[\tau \le t] = P\left[m_t^{\nu} \le \frac{\ln \frac{G_0}{V_0}}{\sigma}\right] \\ &= P\left[M_t^{-\nu} \ge \frac{\ln \frac{V_0}{G_0}}{\sigma}\right] \qquad (34) \\ &= N\left(\frac{\ln \frac{G_0}{V_0} - (\mu - r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right) \\ &\quad + \left(\frac{V_0}{G_0}\right)^{1-2\frac{\mu-r}{\sigma^2}} N\left(\frac{\ln \frac{G_0}{V_0} + (\mu - r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right) \qquad (35) \end{aligned}$$

and

$$\begin{aligned} P[V_t^{SL} \le w, \tau > t] &= P\left[W_t^{\nu} \le \frac{1}{\sigma}\left(\ln \frac{w}{V_0} - rt\right), m_t^{\nu} \ge \frac{1}{\sigma} \ln \frac{G_0}{V_0}\right] \\ &= P\left[M_t^{-\nu} \le \frac{\ln \frac{V_0}{G_0}}{\sigma}\right] - P\left[W_t^{-\nu} \le \frac{\ln \frac{V_0}{w} + rt}{\sigma}, M_t^{-\nu} \le \frac{\ln \frac{V_0}{G_0}}{\sigma}\right] \end{aligned}$$

With Equation (34), Equation (33) yields

$$P[V_t^{SL} \le w] = 1 - P\left[W_t^{-\nu} \le \frac{\ln \frac{V_0}{w} + rt}{\sigma}, M_t^{-\nu} \le \frac{\ln \frac{V_0}{G_0}}{\sigma}\right]$$

$$= N\left(\frac{\ln \frac{w}{V_0} - (\mu - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right) + \left(\frac{V_0}{G_0}\right)^{1-2\frac{\mu-r}{\sigma^2}} N\left(\frac{2\ln G_0 - \ln(wV_0) + (\mu - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right).$$

Finally, one obtains the density by differentiating Equations (35) and (8).

Appendix B. proof of Proposition 3.10

The strategy definition implies that the investment quote equals $\pi_t^1 = \frac{mC_t}{V_t}$. Therefore,

$$\begin{aligned} P\left[\pi_T^1 \le \beta \,|\, \pi_t^1 = \alpha\right] &= P\left[(m-\beta)C_T \le \beta G_T \,|\, (m-\alpha)C_t = \alpha G_t\right] \\ &= P\left[C_t \left(e^{(\mu - r - \frac{m}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)}\right)^m \le \frac{\beta}{m-\beta} G_t \,\middle|\, C_t = \frac{\alpha}{m-\alpha} G_t\right] \\ &= P\left[\frac{\alpha}{m-\alpha}\left(e^{(\mu - r - \frac{m}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)}\right)^m \le \frac{\beta}{m-\beta}\right] \\ &= P\left[W_T - W_t \le \frac{1}{\sigma}\left(\frac{1}{m} \ln \frac{\beta(m-\alpha)}{\alpha(m-\beta)} - (\mu - r - \frac{m}{2}\sigma^2)(T-t)\right)\right]. \end{aligned}$$

[14]In particular it holds $P[M_t \le b] = \mathcal{N}\left(\frac{b}{\sqrt{t}}\right) - \mathcal{N}\left(\frac{-b}{\sqrt{t}}\right)$, c.f. ...

Appendix C. proof of Proposition 3.11

Notice that

$$\begin{aligned} dC_t^{\text{CFP}} &= dV_t^{\text{CFP}} \\ &= V_t^{\text{CFP}}(\frac{mC_t^{\text{CFP}}}{V_t^{\text{CFP}}}\frac{dS_t}{S_t} + \left(1 - \frac{mC_t^{\text{CFP}}}{V_t^{\text{CFP}}}\right)\frac{dB_t}{B_t} \\ &= C_t^{\text{CFP}}\left(m\frac{dS_t}{S_t} + \left(\frac{G}{C_t^{\text{CFP}}} - (m-1)\right)\frac{dB_t}{B_t}\right). \end{aligned}$$

With Equation (5) and $\frac{dB_t}{B_t} = r\,dt$ it follows

$$\begin{aligned} dC_t^{\text{CFP}} &= (m\mu - (m-1)r)\left(\frac{rG}{m\mu - (m-1)r} + C_t^{\text{CFP}}\right)dt + C_t^{\text{CFP}} m\sigma dW_t && (36) \\ &= A\left(\frac{B}{A} + C_t^{\text{CFP}}\right)dt + \varsigma C_t^{\text{CFP}} dW_t && (37) \end{aligned}$$

where $A := m\mu - (m-1)r$ and $B := rG$. Along the lines of Kloeden and Platen (1999, Kapitel 4), the above can be classified as an inhonogenous linear stochastic differential equation. Multiplying Equation (37) with $e^{-(A-\frac{1}{2}\varsigma^2)t-\varsigma W_t}$ yields

$$\begin{aligned} d\left(C_t^{\text{CFP}} e^{-(A-\frac{1}{2}\varsigma^2)t-\varsigma W_t}\right) &= \Big[dC_t^{\text{CFP}} - (A - \frac{1}{2}\varsigma^2)C_t^{\text{CFP}}\,dt - \varsigma C_t^{\text{CFP}}\,dW_t \\ &\quad + \frac{1}{2}\varsigma^2 C_t^{\text{CFP}}\,d\langle W\rangle_t - \varsigma\,d\langle W, C\rangle_t\Big]\exp\left(-(A - \frac{1}{2}\varsigma^2)t - \varsigma W_t\right) \\ &= \exp\left(-(A - \frac{1}{2}\varsigma^2)t - \varsigma W_t\right)B\,dt \quad . \end{aligned}$$

Notice that the right hand side of the above Equation does not depend on C_t^{CFP}.

Appendix D. Proof of Proposition 4.5

Let $f(z)$ denote the density function of the random variable Z and let $h : \mathbb{R} \to \mathbb{R}$ denote a monotonously increasing function with inverse h^{-1}. Then, a well known is given by

$$P\left[h(Z) \in dz\right] = \left(h^{-1}\right)'(z)\, f(h^{-1}(z)).$$

An application of the above with respect to the value process according to Equation (32) and the random variable X_t gives in the case that $X_t > 0$

$$h(x) = \frac{mG_t}{m-1}e^{\sigma x}, \quad h^{-1}(x) = \frac{\ln\frac{(m-1)x}{mG_t}}{\sigma}, \quad \left(h^{-1}\right)'(x) = \frac{1}{\sigma x}.$$

For $X_t \leq 0$, we haven

$$h(x) = G_t\left(1 + \frac{1}{m-1}e^{m\sigma x}\right), \quad h^{-1}(x) = \frac{\ln\frac{(m-1)(x-G_t)}{G_t}}{m\sigma}, \quad \left(h^{-1}\right)'(x) = \frac{1}{m\sigma(x - G_t)}.$$

The rest of the proof is straightforward.

Appendix E. Proof of Propositions 4.6 and 4.7

Theorem E.1 (Benes et al. (1980)). *Let* $\overline{p}(\lambda, x_0, z, \theta_0, \theta_1)$ *denote the Laplace-transform of* $P_{x_0}(X_t \in dz)$ *with respect to* t*, i.e.*

$$\overline{p}(\lambda, x_0, z, \theta_0, \theta_1) = \int_0^\infty e^{-\lambda t} P_{x_0}(X_t \in dz)dt$$

Dann ist $\overline{p}(\lambda, x_0, z, \theta_0, \theta_1)$*. Then, for* $x_0 \geq 0$ *it holds*

$$\overline{p}(\lambda, x_0, z, \theta_0, \theta_1) = \begin{cases} K_1(\lambda)e^{zK_2(\lambda)} & z < 0 < x_0 \\ K_3(\lambda)e^{zK_4(\lambda)} + K_5(\lambda)e^{zK_6(\lambda)} & 0 < z < x_0 \\ (K_5(\lambda) + e^{2x_0\sqrt{\theta_0^2+2\lambda}}K_3(\lambda))e^{zK_6(\lambda)} & fr\ 0 < x_0 < z \end{cases} . \tag{38}$$

For $x_0 < 0$ *it holds* $\overline{p}(\lambda, x_0, z, \theta_0, \theta_1) = \overline{p}(\lambda, -x_0, -z, -\theta_1, -\theta_0)$*. The fiunctions* K_i *are given as follows*

$$\begin{aligned}
K_1(\lambda) &= \frac{2e^{-x_0(\theta_0+\sqrt{\theta_0^2+2\lambda})}}{\theta_0 - \theta_1 + \sqrt{\theta_0^2+2\lambda} + \sqrt{\theta_1^2+2\lambda}} = \frac{2e^{-x_0K_4(\lambda)}}{K_4(\lambda) - K_8(\lambda)} \\
K_2(\lambda) &= \theta_1 + \sqrt{\theta_1^2+2\lambda} \\
K_3(\lambda) &= \frac{e^{-x_0(\theta_0+\sqrt{\theta_0^2+2\lambda})}}{\sqrt{\theta_0^2+2\lambda}} = \frac{2e^{-x_0K_4(\lambda)}}{K_4(\lambda) - K_6(\lambda)} \\
K_4(\lambda) &= \theta_0 + \sqrt{\theta_0^2+2\lambda} \\
K_5(\lambda) &= \frac{e^{-x_0(\theta_0+\sqrt{\theta_0^2+2\lambda})}}{\sqrt{\theta_0^2+2\lambda}} \frac{\theta_1 - \theta_0 + \sqrt{\theta_0^2+2\lambda} - \sqrt{\theta_1^2+2\lambda}}{\theta_0 - \theta_1 + \sqrt{\theta_0^2+2\lambda} + \sqrt{\theta_1^2+2\lambda}} \\
&= \frac{2(K_8(\lambda) - K_6(\lambda))e^{-x_0K_4(\lambda)}}{(K_4(\lambda) - K_6(\lambda))(K_4(\lambda) - K_8(\lambda))} \\
K_6(\lambda) &= \theta_0 - \sqrt{\theta_0^2+2\lambda} \\
K_8(\lambda) &= \theta_1 - \sqrt{\theta_1^2+2\lambda}
\end{aligned}$$

For $x_0 < 0$*, the associated* K_i *are denoted by* $\tilde{K}_i$*.*

With respect to the following proofs, it is useful to notice that the determination of the inverse Laplace–transform can be achieved by the solution of the Bromwich-integrals

$$\mathcal{L}^{(-1)}_{\lambda,t}\{F(\lambda)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\gamma t}F(\lambda)d\lambda$$

where $\gamma > \max_k \mathbb{R}(z_k)$ and z_k denotes the singulaity of F.[15]

E.1. **Proof of 4.6.** First, consider the case $mC_0 > V_0 \Leftrightarrow X_0 > 0$. Together with Theorem E.1 and 4.4 it follows that for $v \leq \frac{mG_t}{m-1}$

$$P[V_t \leq v] = P\left[G_t\left(1 + \frac{1}{m-1}e^{\sigma m X_t}\right) \leq v\right] = P\left[X_t \leq \frac{1}{\sigma m}\ln\frac{(v)(v-G_t)}{G_t}\right]$$
$$= \mathcal{L}^{(-1)}_{\lambda,t}\left\{\int_{-\infty}^{\frac{1}{\sigma m}\ln\frac{(m-1)(v-G_t)}{G_t}} K_1 e^{xK_2}dx\right\} = \mathcal{L}^{(-1)}_{\lambda,t}\left\{\frac{K_1}{K_2}\left(\frac{(m-1)(v-G_t)}{G_t}\right)^{\frac{K_2}{\sigma m}}\right\}.$$

For $\frac{m}{m-1}G_t < v \leq V_0 e^{rt} \Leftrightarrow 0 < x \leq x_0$ it holds

$$P[\frac{m}{m-1}G_t < V_t \leq v] = P\left[0 < X_t \leq \frac{1}{\sigma}\ln\frac{(m-1)v}{mG_t}\right]$$
$$= \mathcal{L}^{(-1)}_{\lambda,t}\left\{\int_0^{\frac{1}{\sigma}\ln\frac{(m-1)v}{mG_t}} K_3 e^{xK_4} + K_5 e^{xK_6}dx\right\}$$
$$= \mathcal{L}^{(-1)}_{\lambda,t}\left\{\frac{K_3}{K_4}\left(\frac{(m-1)v}{mG_t}\right)^{\frac{K_4}{\sigma}} + \frac{K_5}{K_6}\left(\frac{(m-1)v}{mG_t}\right)^{\frac{K_6}{\sigma}} - \frac{K_3}{K_4} - \frac{K_5}{K_6}\right\}.$$

Notice that $\frac{K_1}{K_2} - \frac{K_3}{K_4} - \frac{K_5}{K_6} = 0$. In particular, for $\frac{m}{m-1}G_t < v \leq V_0 e^{rt}$ it follows

$$P[V_t \leq v] = \mathcal{L}^{(-1)}_{\lambda,t}\left\{\frac{K_3}{K_4}\left(\frac{(m-1)v}{mG_t}\right)^{\frac{K_4}{\sigma}} + \frac{K_5}{K_6}\left(\frac{(m-1)v}{mG_t}\right)^{\frac{K_6}{\sigma}}\right\}.$$

Finally, consider the case $v > V_0 e^{rt} \Leftrightarrow x > x_0$. Here,

$$\begin{aligned} P[V_0 e^{rt} < V_t \leq v] &= P\left[x_0 < X_t \leq \frac{1}{\sigma}\ln\frac{(m-1)v}{mG_t}\right] \\ &= \mathcal{L}^{(-1)}_{\lambda,t}\left\{\int_{x_0}^{\frac{1}{\sigma}\ln\frac{(m-1)v}{mG_t}} \left(K_5 + e^{2x_0\sqrt{\theta_0^2+2\lambda}}K_3\right)e^{xK_6}dx\right\} \\ &= \mathcal{L}^{(-1)}_{\lambda,t}\left\{\frac{K_5 + e^{2x_0\sqrt{\theta_0^2+2\lambda}}K_3}{K_6}\left(\left(\frac{(m-1)v}{mG_t}\right)^{\frac{K_6}{\sigma}} - e^{x_0K_6}\right)\right\}. \end{aligned}$$

Using $\frac{K_3}{K_4}e^{x_0K_4} - \frac{K_3}{K_6}e^{2x_0\sqrt{\theta_0^2+2\lambda}+x_0K_6} = \frac{1}{\lambda}$ gives

$$P[V_t \leq v] = \frac{1}{\lambda} + \frac{K_5 + e^{2x_0\sqrt{\theta_0^2+2\lambda}}K_3}{K_6}\left(\frac{(m-1)v}{mG_t}\right)^{\frac{K_6}{\sigma}}.$$

[15] In particular, this implies that, for a suitable λ, the order of integration can be changed. In the following, this applies since $K_2(\lambda)$ and $K_4(\lambda)$ are, without limit, monotonously increasing and $K_6(\lambda)$ and $K_8(\lambda)$ are, without limit, monotonously decreasing in λ. Therefor, the Bromwich-integral is well defined.In the following, we omit the respresentation of the functional dependence between K_i and λ. In particular, $K_i(\lambda)$ is simply denoted by K_i.

Similar reasoning gives for $mC_0 < V_0$

$$
\begin{aligned}
v < V_0 e^{rt}: \quad & P[V_t \le v] = \mathcal{L}^{(-1)}_{\lambda,t}\left\{ \frac{-\tilde{K}_5 + e^{-2x_0\sqrt{\theta_1^2+2\lambda}}\tilde{K}_3}{\tilde{K}_6}\left(\frac{(m-1)(v-G_t)}{G_t}\right)^{-\frac{\tilde{K}_6}{\sigma m}}\right\} \\
v \in [V_0 e^{rt}, \tfrac{m}{m-1}G_t]: \quad & P[V_0 e^{rt} < V_t \le v] = \mathcal{L}^{(-1)}_{\lambda,t}\left\{ \frac{\tilde{K}_3}{\tilde{K}_4}\left(e^{-x_0\tilde{K}_4} - \left(\frac{(m-1)(v-G_t)}{G_t}\right)^{-\frac{\tilde{K}_4}{\sigma m}}\right)\right. \\
& \left. + \frac{\tilde{K}_5}{\tilde{K}_6}\left(e^{-x_0\tilde{K}_6} - \left(\frac{(m-1)(v-G_t)}{G_t}\right)^{-\frac{\tilde{K}_6}{\sigma m}}\right)\right\} \\
v > \tfrac{m}{m-1}G_t: \quad & P[\tfrac{m}{m-1}G_t < V_t \le v] = \mathcal{L}^{(-1)}_{\lambda,t}\left\{ \frac{\tilde{K}_1}{\tilde{K}_2}\left(1 - \left(\frac{(m-1)v}{mG_t}\right)^{-\frac{\tilde{K}_2}{\sigma}}\right)\right\}.
\end{aligned}
$$

Summing up gives the resulz.

E.2. **Proof of Proposition 4.7.** Let $mC_0 \ge V_0$. Proposition 4.5 and Theorem E.1 imply

$$
E[V_T^n] = \underbrace{\int_{G_T}^{\frac{m}{m-1}G_T} v^n P[V_t \in dv]}_{=:\mathbf{A}} + \underbrace{\int_{\frac{m}{m-1}G_T}^{V_0 e^{rT}} v^n P[V_t \in dv]}_{=:\mathbf{B}} + \underbrace{\int_{V_0 e^{rT}}^{\infty} v^n P[V_t \in dv]}_{=:\mathbf{C}}
$$

By inserting the density function and changing the order of integration, $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ can be determined as an inverse Laplace–transform. In particular, it holds

$$
\begin{aligned}
\mathbf{A} &= \mathcal{L}^{(-1)}_{\lambda,t}\left\{ \int_0^{\frac{G_T}{m-1}} (v+G_T)^n \frac{K_1}{\sigma m v}\left(\frac{m-1}{G_T}v\right)^{\frac{K_2}{\sigma m}} dv \right\} \\
&= \mathcal{L}^{(-1)}_{\lambda,t}\left\{ \sum_{i=0}^n \binom{n}{i} G_T^{n-i}\frac{K_1}{\sigma m}\left(\frac{m-1}{G_T}\right)^{\frac{K_2}{\sigma m}} \int_0^{\frac{G_T}{m-1}} v^{\frac{K_2+i\sigma m}{\sigma m}-1} dv \right\} \\
&= \mathcal{L}^{(-1)}_{\lambda,t}\left\{ \sum_{i=0}^n \binom{n}{i} G_T^{n-i}\frac{K_1}{\sigma m}\left(\frac{m-1}{G_T}\right)^{\frac{K_2}{\sigma m}} \frac{\sigma m}{K_2+i\sigma m}\left(\frac{G_T}{m-1}\right)^{i+\frac{K_2}{\sigma m}} \right\} \\
&= \mathcal{L}^{(-1)}_{\lambda,t}\left\{ \sum_{i=0}^n \binom{n}{i} G_T^{n}\frac{K_1}{K_2+i\sigma m}\left(\frac{1}{m-1}\right)^{i} \right\}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{B} &= \mathcal{L}^{(-1)}_{\lambda,t}\left\{ \int_{\frac{m}{m-1}G_T}^{V_0 e^{rT}} v^n \left(\frac{K_3}{\sigma v}\left(\frac{m-1}{mG_T}v\right)^{\frac{K_4}{\sigma}} + \frac{K_5}{\sigma v}\left(\frac{m-1}{mG_T}v\right)^{\frac{K_4}{\sigma}}\right) dv \right\} \\
&= \mathcal{L}^{(-1)}_{\lambda,t}\left\{ \frac{K_3}{K_4+\sigma n}\left(\frac{m-1}{mG_T}\right)^{\frac{K_4}{\sigma}} v^{n+\frac{K_4}{\sigma}}\Big|_{v=\frac{m}{m-1}G_T}^{V_0 e^{rT}} + \frac{K_5}{K_6+\sigma n}\left(\frac{m-1}{mG_T}\right)^{\frac{K_6}{\sigma}} v^{n+\frac{K_6}{\sigma}}\Big|_{v=\frac{m}{m-1}G_T}^{V_0 e^{rT}} \right\}
\end{aligned}
$$

$$\mathbf{C} = \mathcal{L}^{(-1)}_{\lambda,t}\left\{\int_{V_0e^{rT}}^{\infty} v^n \frac{K_5 + K_3 e^{2x_0\sqrt{\theta_0^2+2\lambda}}}{\sigma v}\left(\frac{m-1}{mG_T}v\right)^{\frac{K_6}{\sigma}} dv\right\}$$
$$= \mathcal{L}^{(-1)}_{\lambda,t}\left\{-\frac{K_5 + K_3 e^{2x_0\sqrt{\theta_0^2+2\lambda}}}{K_6+\sigma n}\left(\frac{m-1}{mG_T}v\right)^{\frac{K_6}{\sigma}}\left(V_0e^{rT}\right)^{n+\frac{K_6}{\sigma}}\right\} \quad .$$

Analogously to the proof of Proposition 4.6, cf. E.1, using the relation between the different K_i gives the result for $mC_0 \geq V_0$, i.e. 4.6

$$\mathbf{B}+\mathbf{C} = -\left(\frac{m}{m-1}G_T\right)^n\left(\frac{K_3}{K_4+\sigma n}+\frac{K_5}{K_6+\sigma n}\right)$$
$$+ K_3(V_0e^{rT})^n\left(\frac{\left(\frac{m-1}{mG_0}V_0\right)^{\frac{K_4}{\sigma}}}{K_4+\sigma n} - \frac{e^{2x_0\sqrt{\theta_0^2+2\lambda}}\left(\frac{m-1}{mG_0}V_0\right)^{\frac{K_6}{\sigma}}}{K_6+\sigma n}\right)$$
$$= -\left(\frac{m}{m-1}G_T\right)^n\left(\frac{K_3}{K_4+\sigma n}+\frac{K_5}{K_6+\sigma n}\right) - \frac{2(V_0e^{rT})^n}{(K_4+\sigma n)(K_6+\sigma n)}$$
$$= -\left(\frac{m}{m-1}G_T\right)^n\left(\frac{K_3}{K_4+\sigma n}+\frac{K_5}{K_6+\sigma n}\right) + \frac{(V_0e^{rT})^n}{\lambda-(\mu-r)n-\frac{1}{2}n(n-1)\sigma^2}.$$

In the case that $mC_0 < V_0$, it follows

$$E[V_T^n] = \underbrace{\int_{G_T}^{V_0e^{rT}} v^nP[V_t\in dv]}_{=:\tilde{\mathbf{A}}} + \underbrace{\int_{V_0e^{rT}}^{\frac{m}{m-1}G_T} v^nP[V_t\in dv]}_{=:\tilde{\mathbf{B}}} + \underbrace{\int_{\frac{m}{m-1}G_T}^{\infty} v^nP[V_t\in dv]}_{=:\tilde{\mathbf{C}}}$$

where

$$\tilde{\mathbf{A}} = \mathcal{L}^{(-1)}_{\lambda,t}\left\{\sum_{i=0}^{n}\binom{n}{i}G_T^{n-i}\frac{\tilde{K}_5+\tilde{K}_3e^{-2x_0\sqrt{\theta_1^2+2\lambda}}}{i\sigma m-\tilde{K}_6}\left(\frac{m-1}{G_T}\right)^{-\frac{\tilde{K}_6}{\sigma m}}\left(V_0e^{rT}-G_T\right)^{i-\frac{\tilde{K}_6}{\sigma m}}\right\}$$

$$\tilde{\mathbf{B}} = \mathcal{L}^{(-1)}_{\lambda,t}\left\{\sum_{i=0}^{n}\binom{n}{i}G_T^{n-i}\left(\frac{\tilde{K}_3\left(\frac{m-1}{G_T}\right)^{-\frac{\tilde{K}_4}{\sigma m}}}{i\sigma m-\tilde{K}_4}v^{i-\frac{\tilde{K}_4}{\sigma m}}+\frac{\tilde{K}_5\left(\frac{m-1}{G_T}\right)^{-\frac{\tilde{K}_6}{\sigma m}}}{i\sigma m-\tilde{K}_6}v^{i-\frac{\tilde{K}_6}{\sigma m}}\right)\Bigg|_{v=e^{rt}(V_0-G_0)}^{\frac{1}{m-1}G_T}\right\}$$

$$\tilde{\mathbf{C}} = \mathcal{L}^{(-1)}_{\lambda,t}\left\{-\frac{\tilde{K}_1}{n\sigma-\tilde{K}_2}G_T^n\left(\frac{m}{m-1}\right)^n\right\}$$

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