Robust Recovery Risk Hedging: Only the First Moment Matters

by

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Credit derivatives are subject to at least two sources of risk: the default time and the recovery payment. This paper examines the impact of modeling the recovery payment on hedging strategies in a reduced-form model as well as a Merton-type model. We show that quadratic hedging approaches do only depend on the *expected* recovery payment at default and not the whole shape of the recovery payment distribution. This justifies assuming a *certain* recovery payment conditional on the default time. Hence, this result allows a simplified modeling of credit risk.

JEL: C10, G13, G24

Key words: Credit Risk, Recovery Risk, Hedging

Contents

1 Introduction

In contrast to a large amount of theoretical and empirical work available on the valuation of credit derivatives (see Bielecki and Rutkowski (2002), Duffie and Singleton (2003), Lando (2004) for reviews), hedging of credit derivatives remains a largely unexplored avenue of research. When valuing and hedging credit derivatives, two quantities are crucial. The first is the *probability of default* (or default intensity, if it exists), and the second is the *default recovery* (or recovery rate) in the event of default. While in traditional models the recovery rate is given exogenously as a known constant at the default time, this rate is stochastic in reality, even conditional on the default time. This uncertainty in the *default recoveries* of both the underlying instrument (e.g., equity) and particularly the credit derivative (e.g., a convertible bond) is perhaps the most important reason why hedges in practise are not self-financing.

The main purpose of this paper is therefore not valuation but hedging credit derivatives in the presence of recovery risk. While in a complete market setting a self-financing hedging strategy derives immediately, it is still somewhat unclear how to hedge credit risk if markets are incomplete. Since in general, the common objective of arbitrageurs in credit derivatives markets is to minimize the variance of the hedging costs, we focus on the *locally risk-minimizing* hedging strategy. Föllmer and Sondermann (1986) pioneered this approach in the special case where the underlying instrument follows a martingale. At each point in time they require that the risk, defined as the expected quadratic hedging costs, is minimized. However, in semimartingale models a risk-minimizing strategy does not always exist. Therefore, Schweizer (1991) introduced a locally risk-minimizing (LRM) hedging strategy and showed that – under certain assumptions – a strategy is *locally* risk-minimizing if the cost process is a martingale which is orthogonal to the martingale part of the underlying instrument process. The LRM-strategy is *mean-self-financing*, that is at each point in time the expected sum of discounted cash infusions or withdrawals until maturity is zero. The value of the hedge portfolio is then the discounted expected terminal payoff of the option under the so-called *minimal equivalent martingale measure.*

We derive LRM-hedging strategies for a reduced-form model as well a structural Merton-type model when there are two hedging instruments: a locally riskless money market account and a risky underlying instrument. The latter model differs from the original Merton-model by assuming positive bankruptcy costs, given as *per-*

centage of the firm value at default. As long as this percentage is a constant, we denote the corresponding recovery rate as *single-stochastic* since the recovery amount depends only on the default event. Otherwise, that is, if the percentage bankruptcy costs are random, we call the corresponding recovery rate *double-stochastic* since the recovery amount depends not only on the default event but also on the realization of another random variable. In this model framework the shares of the firm's common stock serve as the underlying instrument. Corresponding model variants are examined for the reduced-form model framework. In this framework we assume the existence of a tradable zero coupon bond with total loss at default of the firm under consideration.

For both model classes it turns out that the corresponding LRM-strategy is not only mean-self-financing but also self-financing if the modeled default recovery is single-stochastic. That is, as long as the recovery amount is known in the event of default, there exists a self-financing replication strategy for credit derivatives. Moreover, we find that in the more realistic case of double-stochastic default recoveries, the LRM-hedging strategy does only depend on the *expected* recovery amount, not on other characteristics of its distribution. This key result of the paper helps to justify the frequent simplifying assumption that the default recovery is a constant, conditional on the default event, when valuing and hedging credit derivatives.

At first glance this result seems to contradict the result of Grünewald and Trautmann (1996) when deriving LRM-strategies for stock options in the presence of jump risk. In that setting the LRM-strategy depends in addition on the variance of the stock's jump amplitude, or more precisely, the percentage of the total stock variance explained by the jump component. This key difference is due to the fact that in our model default of the firm implies that the underlying instrument's price jumps always to *zero* while in Merton's (1976) jump diffusion setting assumed by Grünewald and Trautmann (1996), the option's underlying stock price jumps to an arbitrary price level.

The paper is organized as follows: Section 2 describes hedging as a sequential regression and illustrates the paper's basic insight. Section 3 looks at locally risk-minimizing hedging policies in a reduced-form model when recovery is single-stochastic and double-stochastic, respectively. Section 4 examines locally risk-minimizing hedging policies in a structural Merton-type model when recovery is single-stochastic and double-stochastic, respectively. Section 5 concludes the paper. All technical proofs are given in Appendix A.

2 Hedging by sequential regression

In incomplete financial markets not every contingent claim is replicable. For this reason a lot of different hedging strategies have been evolved in literature. On one hand there exist hedging approaches searching self-financing strategies which reproduce the derivative at the best. On the other hand there are hedging strategies replicating the derivative exactly at maturity by taking in account additional costs during the trading period. While the first class of hedging strategies optimizes the *hedging error*, to be more precisely the difference between the pay-off of the derivative F_T and the liquidation value of the hedging strategy, the other class minimizes the *hedging costs*. In a discrete time set-up Föllmer and Schweizer (1989) developed a hedging approach of the latter type, the so-called *locally risk-minimizing hedging*. When using two hedging instruments, the underlying asset with price process S and the money market account with price process B, $\mathbf{H} = (h^S, h^B)$ describes the hedging strategy composed of h^S shares in the underlying and h^B shares in the money market account. $\mathrm{V}_t(\mathbf{H}) = h^S_{t+1} S_t + h^B_{t+1} B_t$ denotes the *liquidation value* of the strategy, $\mathcal{G}_t(\mathbf{H}) = \sum_{i=1}^t (h^S_i \Delta S_i + h^B_i \Delta B_i)$ the *cumulated gain* and finally $\mathcal{C}_t(\mathbf{H}) = \mathrm{V}_t(\mathbf{H}) - \mathcal{G}_t(\mathbf{H})$ the cumulated *hedging costs* at time t. To achieve a locally risk-minimizing hedging strategy, Föllmer and Schweizer (1989) solve the following

Problem 1 (Locally risk-minimizing hedging in discrete time)
Search the trading strategy $\mathbf{H}$ *which replicates exactly the derivative* F *at maturity* T *and in addition minimizes the expected quadratic growth of the hedging cost at every point of time:*

$$E_P\left[(\Delta\mathcal{C}_t(\mathbf{H}))^2 \,|\mathcal{F}_{t-1}\right] \to \min \;\textit{ for all } t = 1,\ldots,T \textit{ and } \mathbf{H} \in \mathcal{H} \textit{ with } V_T(\mathbf{H}) = F_T\ .$$

A solution of problem 1 we call *locally risk-minimizing hedging strategy* or *LRM-hedge*[1]. Föllmer and Schweizer (1989) have pointed out that the above problem 1 is a sequential regression task and can be solved by backwards induction: At first we determine h^S_T and h^B_T by identifying the solution of the subproblem

$$\mathrm{E}_P\left[(\Delta\mathcal{C}_t(\mathbf{H}))^2 |\mathcal{F}_{t-1}\right] \to \min \quad \text{for all} \quad h^S_t,\, h^B_t \quad \text{given} \quad \mathrm{V}_t(\mathbf{H}) \tag{1}$$

[1] A LRM-hedge also solves the problem

$$\mathrm{E}_P\left[(\Delta\mathcal{C}_t(\mathbf{H}))^2 \,|\mathcal{F}_{t-1}\right] \to \min \text{ for all } t = 1,\ldots,T \text{ and } \mathbf{H} \in \mathcal{H} \text{ with } \mathrm{V}_T(\mathbf{H}) = F_T\ ,$$

where $\Delta\mathcal{C}_t(\mathbf{H}) = \Delta\mathcal{C}_t(\mathbf{H})/B_t$ denotes the discounted growth of the hedging costs and B_t is the value of the money market account at time t.

at time $t = T$, since $\mathrm{V}_T(\mathbf{H}) = F_T$ is specified. Subsequently, we know $\mathrm{V}_{T-1}(\mathbf{H})$ and we can solve the subproblem (1) at $t = T-1$ and thus obtain h^S_{T-1} (as slope of the regression line) and h^B_{T-1} (as intercept), and so on. Since $\Delta\mathcal{C}_t(\mathbf{H}) = \mathrm{V}_t(\mathbf{H}) - (h^S_t S_t + h^B_t B_t)$ holds, (1) is a linear regression problem, which can be solved by the least square principal. Figure 2 illustrates this idea.

In the following, we show that this relation shows directly that two different types of recovery modeling lead to the same locally risk-minimizing hedge of credit derivatives. The first type of recovery modeling, the so-called *single-stochastic* recovery payment, only depends on the default-time and perhaps the development of the interest rate as illustrated in part (a) of figure 1 for a two period set-up. Thus, the recovery amount is clearly indicated conditional on the default time (and the term structure).

The second type of recovery, the so-called *double-stochastic* recovery, allows in addition to the default time and the term structure other risk factors influencing the recovery payment (see part (b) of Fig. 1). For example these additional factors can characterize the uncertain costs of financial distress or the uncertain time delay of the promised recovery payment. Thus knowing the default time (and the term structure) the recovery payment is not unique determined, there exist different realizations of the recovery amount.

Figure 2 shows, that the locally risk-minimizing hedging strategy of the credit derivative is the same for single- and double-stochastic recovery modeling. Provided, that the expectation of the double-stochastic recovery payment conditional on the default time (and the term structure) coincides with the uniquely determined single-stochastic recovery payment knowing the default time (and the development of the interest rate).

We provide the proof in the following. More precisely we show, that the single-stage regression approach (delivers the LRM-hedge of a defaultable claim assuming double-stochastic recovery) and two-stage procedure (delivers the LRM-hedge of a defaultable claim assuming single-stochastic recovery which coincides at any default time with the expectation of the double-stochastic recovery conditional on the default time) provide the same result. We define the probability $p_i = \sum_j p(\omega_i^j)$, the random variables $\overline{X}_t(\omega_i^j) = X_t(\omega_i^j)$ and $\overline{\mathrm{V}}_t(\mathbf{H})(\omega_i^j)$, which do not depend on the risk factor represented by j, by $\overline{\mathrm{V}}_t(\mathbf{H})(\omega_i^j)p_i = \sum_k \mathrm{V}_t(\mathbf{H})(\omega_i^k)p(\omega_i^k)$ for all j. Thus, we obtain

$$\mathrm{E}_P[\mathrm{V}_t(\mathbf{H})|\mathcal{F}_{t-1}] = \sum_{i,k} p(\omega_i^k)\mathrm{V}_t(\mathbf{H})(\omega_i^k) = \sum_i p_i \overline{\mathrm{V}}_t(\mathbf{H})(\omega_i^j) = \mathrm{E}_P[\overline{\mathrm{V}}_t(\mathbf{H})|\mathcal{F}_{t-1}] \ ,$$

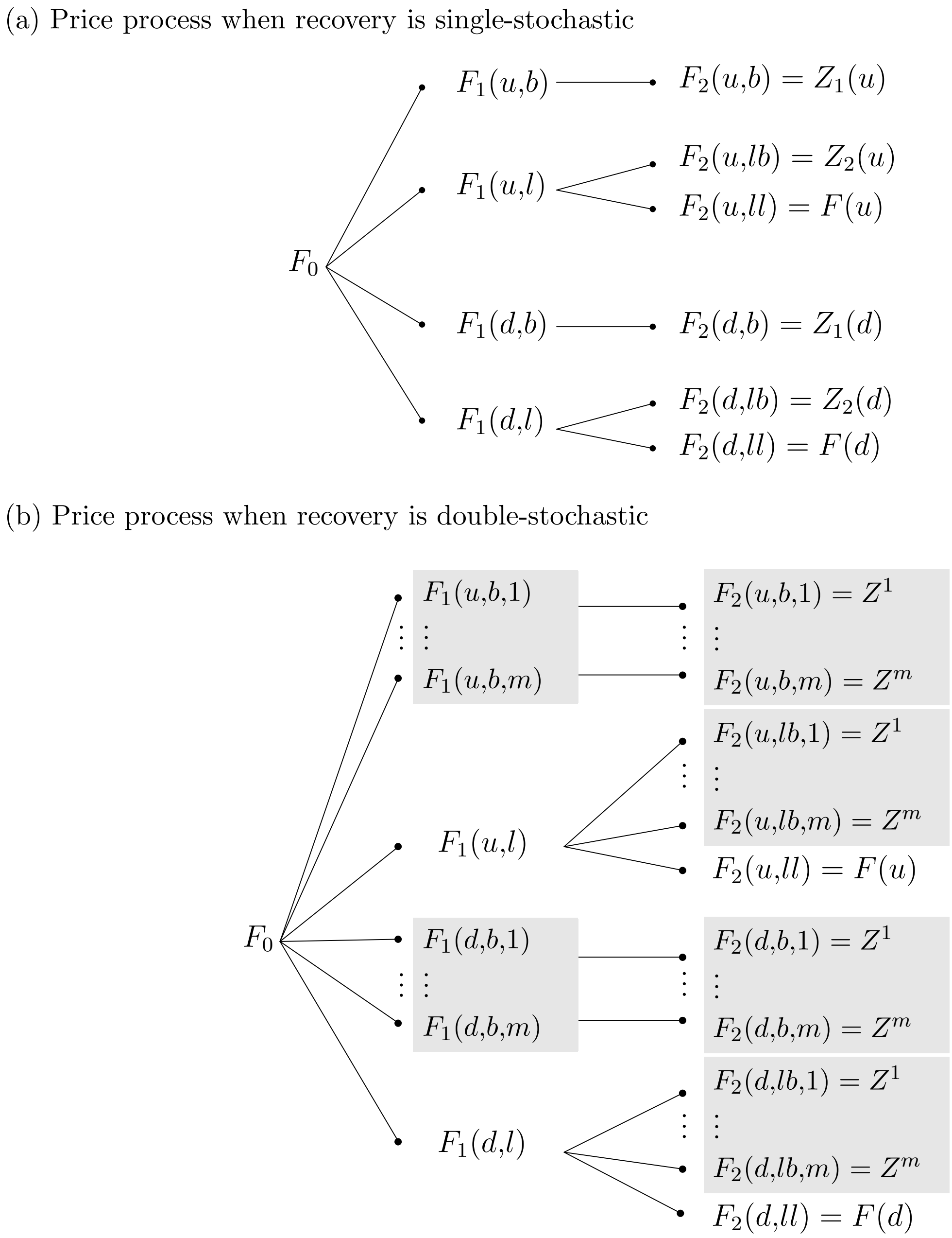

Figure 1: *Single-stochastic versus double-stochastic recovery*

Part (a) of this figure depicts the price process of a credit derivative with a recovery payment depending only on the default time ("l" denotes liquidity, "b" bankruptcy) and the term structure ("u" denotes an up-tick and "d" a down-tick of the interest rate). Conditional on default (and the given term structure) the recovery payment is known. The latter is not the case if the recovery payment is double-stochastic. Part (b) of the figure shows that conditional on default (and the given term structure) the recovery payment can take on m different values Z^1, ..., Z^m.

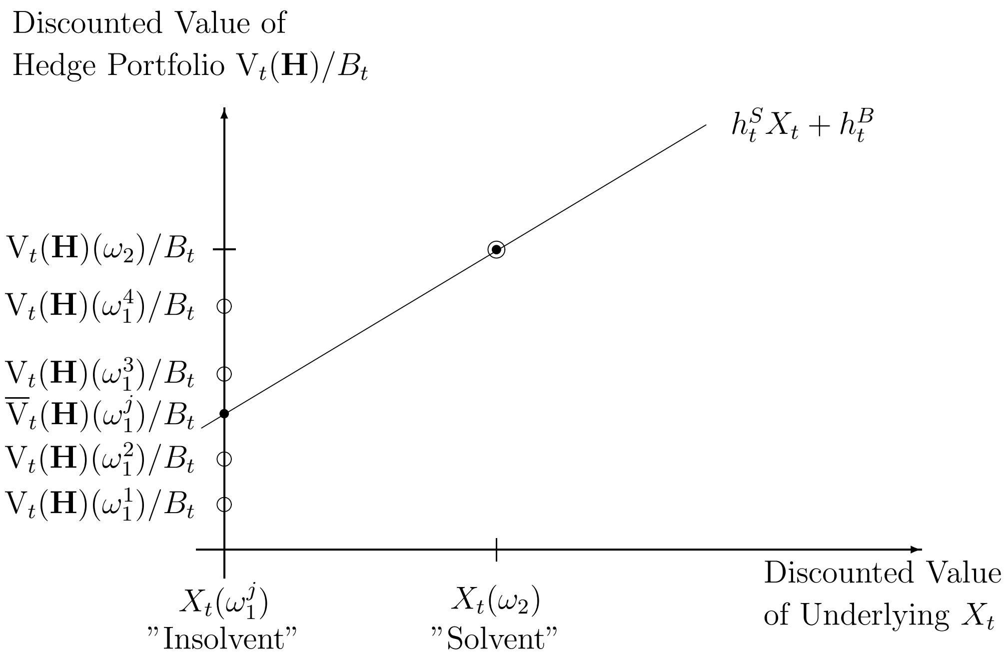

Figure 2: *LRM-strategy when recovery is double-stochastic*

If the recovery is double-stochastic, the payment at default does not only depend on the default time and the term structure but also on another risk factor. We take this risk factor into account by the superscript j in the state ω_i^j. Since the underlying (for example the stock of the firm or a corporate zero-bond with total loss at default written on the underlying firm) does not depend on the additional factor, its discounted price is always zero at default, $X_t(\omega_1^1) = X_t(\omega_1^2) = \ldots = 0$. The symbol " $\circ$ " describes a possible realization of the discounted value of the hedge portfolio. To determine the LRM-hedge we have to run a regression for the value tupels represented by the $\circ$-symbol. Alternatively, we can calculate in a first step the average value of the hedge portfolio $\overline{\mathrm{V}}_t(\mathbf{H})(\omega_1^1)/B_t = \overline{\mathrm{V}}_t(\mathbf{H})(\omega_1^2)/B_t = \ldots$, conditional on the default event occurring. The latter pairs of values are denoted with the "$\bullet$". In a second step, we identify the regression line for the points $\bullet$. The solution of this regression problem coincides with the LRM-hedge of a defaultable claim assuming no additional risk factor j for the recovery.

and in an analogous manner $\mathrm{E}_P[(X_t)^2|\mathcal{F}_{t-1}] = \mathrm{E}_P\left[(\overline{X}_t)^2|\mathcal{F}_{t-1}\right]$, $\mathrm{E}_P[X_t|\mathcal{F}_{t-1}] = \mathrm{E}_P[\overline{X}_t|\mathcal{F}_{t-1}]$, $\mathrm{E}_P[\mathrm{V}_t(\mathbf{H})X_t|\mathcal{F}_{t-1}] = \mathrm{E}_P[\overline{X}_t\overline{\mathrm{V}}_t(\mathbf{H})|\mathcal{F}_{t-1}]$. From this, it follows, that the hedge ratio (slope of the regression line) and the shares in the money market account (ordinate of the regression line) of the one-stage regression approach,

$$h_t^S = \frac{\mathrm{Cov}_P[\mathrm{V}_t(\mathbf{H}),X_t|\mathcal{F}_{t-1}]}{\mathrm{Var}[X_t|\mathcal{F}_{t-1}]B_t} \quad \text{and} \quad h_t^B = \frac{\mathrm{E}_P[V_t(\mathbf{H})|\mathcal{F}_{t-1}]}{B_t} - h_t^S\mathrm{E}_P[X_t|\mathcal{F}_{t-1}]\ ,$$

coincide with these of the two-stage procedure:

$$\overline{h_t^S} = \frac{\mathrm{Cov}_P[\overline{\mathrm{V}}_t(\mathbf{H}),\overline{X}_t|\mathcal{F}_{t-1}]}{\mathrm{Var}[\overline{X}_t|\mathcal{F}_{t-1}]B_t} \quad \text{and} \quad \overline{h_t^B} = \frac{\mathrm{E}_P[\overline{\mathrm{V}}_t(\mathbf{H})|\mathcal{F}_{t-1}]}{B_t} - \overline{h_t^S}\mathrm{E}_P[\overline{X}_t|\mathcal{F}_{t-1}]\ .$$

3 Hedging in reduced-form models

3.1 Model

This section presents a simple intensity model in continuous time which describes a possible default of a firm at time $\tau > 0$ during the time horizon $[0,T]$. Trading takes place every time $t \in [0,T]$. The credit event is specified in terms of an exogenous jump process, the so-called *default process* $H_t = \mathbb{1}_{\{\tau \leq t\}}$. In the following we assume that H is an inhomogeneous poisson process stopped at the first jump – the default time:

$$P(\tau \leq t) = P(H_t = 1) = 1 - \exp\left\{-\int_0^t \lambda(s)\,ds\right\} \quad \text{for every} \quad t \geq 0\ .$$

Here P describes the statistical probability measure and λ is a deterministic, non-negative function of the time with $\int_0^T \lambda(t)\,dt < \infty$ representing the *default intensity* under P. To simplify the following presentation we assume a deterministic term structure where the short rate $(r_t)_{t\in[0,T]}$ is only a deterministic function of time. $B_t = \exp\{\int_0^t r_s\,ds\}$ denotes the value of the money market account at time t. $X = (X_t)_{t\in[0,T]}$ denotes the discounted price process of the traded risk-free zero coupon bond with maturity date T *and total loss in case of default* given by

$$X_t = \frac{1}{B_T}\exp\left\{-\int_t^T \widehat{\lambda}(s)\,ds\right\}(1 - H_t)$$

if financial markets are frictionless and arbitrage-free. The deterministic non-negative function $\widehat{\lambda}$ with $\int_0^T \widehat{\lambda}(t)\,dt < \infty$ can be estimated via market values of defaultable financial instruments.[2] Since every probability measure Q implying a default intensity $\widehat{\lambda}$ fulfills

$$\begin{aligned}
\mathrm{E}_Q[X_t|\mathcal{F}_s] &= \mathbb{1}_{\{\tau>s\}}\left(X_t \cdot Q(\tau > t|\tau > s) + 0 \cdot Q(\tau \le t|\tau > s)\right) \\
&= (1 - H_s)\frac{1}{B_T} \exp\left\{-\int_t^T \widehat{\lambda}(s)\, ds\right\} \exp\left\{-\int_s^t \widehat{\lambda}(s)\, ds\right\} = X_s
\end{aligned}$$

for all $s \le t$, the function $\widehat{\lambda}$ specifies the default intensity under a martingale measure $Q \in \mathbb{Q}$.

Below we will determine hedging strategies for credit derivatives (Z, C, F). The defaultable claim delivers time-continuous cash flows C_t in $0 \le t \le T$ as long as no default has occurred. If the firm is still solvent at the time of maturity a payment F will also be paid. Otherwise the owner of the credit derivative receives (in addition to the cash flow stream C during the horizon $[0,\tau)$) the *uncertain* recovery payment $Z(\tau)$ in T. We assume that the recovery amount does not exceed the final value of the credit derivative's cash flow when no default occurs :

$$0 \le Z(\tau) \le B_T \int_\tau^T C_t/B_t\, dt + F \quad P\text{-a.s.} \quad \text{for all} \quad 0 < \tau \le T\,. \tag{2}$$

This assumption assures that the value of the defaultable claim (Z, C, F) is lower than the value of a default-free, but otherwise identical derivative (C, F). The value of the credit derivative at maturity amounts to

$$F(T) = \begin{cases} B_T \int_0^T C_t/B_t\, dt + F\,, & \text{if } \tau > T \\ B_T \int_0^\tau C_t/B_t\, dt + Z(\tau)\,, & \text{if } \tau \le T \end{cases}.$$

The probability distribution of Z can depend on the default time. We suppose at any time before default the recovery an expectation $\mu^Z(\tau)$ and a standard deviation $\sigma^Z(\tau)$ under P for a credit event occurring at time τ. Because of (2) we have also

$$0 \le \mu^Z(\tau) \le B_T \int_\tau^T C_t/B_t\, dt + F$$

for $0 < \tau \le T$. For technical reasons we assume $\sup_{\tau\in[0,T]} \sigma^Z(\tau) < \infty$. The information $\mathcal{F}_t$ available at the financial market at time t is given by the marked inhomogeneous poisson process $H^Z = (H, Z)$, which is stopped at the first jump:

[2] For example this procedure is introduced in Jarrow and Turnbull (1995) and Jarrow, Lando and Turnbull (1997).

$\mathcal{F}_t = \sigma(H_t^Z)$ for $t \in [0,T]$.[3] When Ω denotes the state space the economy is described by $(\Omega,\mathcal{F},P)$.

The stochastic *recovery rate* of the credit derivative (Z, C, F)

$$\delta(\tau) = \frac{B_T \int_0^\tau C_t/B_t \, dt + Z(\tau)}{B_T \int_0^T C_t/B_t \, dt + F} \in [0{,}1] \tag{3}$$

relates the final value of the defaultable claim's cash flows (Z, C, F) to the final value of the default-free, but identical derivative's cash flows (C, F).[4] Because of the assumption (2) the recovery is lower than one. If the recovery only depends on the uncertain default time, we will call it *single-stochastic*. If it is subject to another source of risk, we will denote the recovery *double-stochastic*.

3.2 Single-stochastic recovery payment

A defaultable claim with single-stochastic recovery can be duplicated by a hedging strategy $\mathbf{H} = (h^S,h^B)$ composed of h^S defaultable zeros with total loss and h^B shares in the money market account.[5]

Proposition 1 (Replication for Single-Stochastic Recovery)
The credit derivative (Z,C,F) with single-stochastic recovery is duplicated by the hedging strategy $\mathbf{H} = (h^S,h^B)$ with

$$\begin{aligned} h_t^S &= (\widetilde{C}_T B_T + F)\left[1 - \frac{1}{X_{t-} B_T}(\delta(t) - \widetilde{\mu}^\delta(t))\right] , \\ h_t^B &= V_t(\mathbf{H})/B_t - h_t^S X_{t-} = (\widetilde{C}_T + F/B_T)\delta(t) \end{aligned}$$

for $t \leq \tau$ and $h_t^S = 0$, $h_t^B = h_\tau^B$ for $t > \tau$. Here δ describes the recovery rate from (3), $\widetilde{C}_t = \int_0^t C_s/B_s \, ds$ denotes the present value of all cash flows C during $[0,t]$ assuming default has not occurred until t and the deterministic function $\widetilde{\mu}^\delta(t) = \int_t^T \delta(\tau)\widehat{\lambda}(\tau) \exp\{-\int_t^\tau \widehat{\lambda}(s)\, ds\}\, d\tau = E_Q[\delta \mathbb{1}_{\{\tau \leq T\}} | \tau > t]$ depicts at time t the under the martingale measure Q expected recovery for the credit event taking place in $(t,T]$.

[3] If the financial market is not only subject to default and recovery risk but also to further sources of risk, for example interest rate risk, the filtration $\mathcal{F}$ is generated by several filtrations – one reflecting the default and recovery risk and another describing for example the interest rate development. See Bielecki and Rutkowski (2002).

[4] Bakshi, Madan and Zhang (2006, p. 22) define the recovery rate by means of the out-standing payments. But the definition above simplifies the following formulae for the hedging strategies.

[5] Proposition 1 follows directly from Proposition 2.

The duplication strategy keeps at every time t as much in the money market account, that this position has an value relating to the maturity time corresponding to the value of the defaultable claim in maturity in the case of default at time $\tau = t$: $Z(t) + \widetilde{C}_t B_T = \delta(t)(\widetilde{C}_T B_T + F)$. The value of the position in the defaultable zeros at time $t < \tau$ equals the expected of future discounted payments minus the discounted payments in the case of default at time t:

$$\begin{aligned} h_t^S X_t &= (\widetilde{C}_T + F/B_T)\left(X_t B_T + \widetilde{\mu}^\delta(t)\right) - (\widetilde{C}_T + F/B_T)\delta(t) \\ &= (\widetilde{C}_T + F/B_T)\left(\mathrm{E}_Q[\mathbb{1}_{\{\tau > T\}}|\tau > t] + \mathrm{E}_Q[\delta \mathbb{1}_{\{\tau \le T\}}|\tau > t]\right) \\ &\quad -(\widetilde{C}_T + F/B_T)\delta(t) \\ &= \mathrm{E}_Q[F(T)/B_T|\tau > t] - (\widetilde{C}_T + F/B_T)\delta(t)\ . \end{aligned}$$

If the recovery rate is constant, this means $\delta(\tau) = \delta$ for all default times τ and therefore $\widetilde{\mu}^\delta(t) = \delta(1 - X_{t-}B_T)$, it will be possible to replicate the credit derivative (Z, C, F) with single-stochastic recovery by a *static* hedge: Buy $(1-\delta)(\widetilde{C}_T B_T + F)$ defaultable zeros with total loss and $\delta(\widetilde{C}_T + F/B_T)$ shares of the money market account.

3.3 Double-stochastic recovery payment

Every probability measure $Q \in \mathbb{Q}$ with corresponding default intensity $\widehat{\lambda}$ and arbitrary distribution of the recovery rate with values in [0,1] represents an equivalent martingale measure if the null sets of the distribution of the recovery rate under Q and P are the same. The financial market will be arbitrage-free. But it will be incomplete, if the recovery rate is not P-a.s. known, given that default occurs in τ. For this reason defaultable claims with a double-stochastic recovery can not be duplicated. The incompleteness of the financial market can also be realized as follows: There are two sources of risk – the default time and the amount of the recovery are uncertain, but there exists only one a financial instrument for hedging the occurrence of default. Therefore in this section we determine hedging strategies for defaultable claims which minimize the risk locally. More precisely, we solve Problem 2 as stated in the Appendix. This rather technical formulation is due to Schweizer (1991) and can be seen as continuous-time analogue of Problem 1.

If the recovery is single-stochastic the locally risk-minimizing hedge will concur with the replication strategy from proposition 1. To identify the LRM-hedge for credit derivatives with double-stochastic recovery we use the results of Schweizer (1991).

The discounted value process of an defaultable zero with total loss can be written as $X = X_0 + A + M$, since

$$\begin{aligned} dX_t &= \widehat{\lambda}(t)X_{t-}dt - X_{t-}dH_t \\ &= \underbrace{X_{t-}(\widehat{\lambda}(t) - \lambda(t))dt}_{dA_t:=} \underbrace{-X_{t-}d\widetilde{H}_t}_{dM_t:=} \ . \end{aligned}$$

Here $\widetilde{H}_t = H_t - \int_0^{t\wedge\tau} \lambda(s)\, ds$ denotes the *compensated default process*, A describes the continuous drift component with $A_0 = 0$, M depicts a square integrable P-martingale[6] with $M_0 = 0$, and finally the constant fulfills $X_0 = \exp\left\{-\int_0^T \widehat{\lambda}(s)\, ds\right\}/B_T$. Due to proposition A.1 and calculation rules[7] for the conditional quadratic variation it follows

$$d\langle M\rangle_t = X_{t-}^2 d\langle\widetilde{H}\rangle_t = X_{t-}^2\lambda(t)d(t\wedge\tau) = X_{t\wedge\tau-}^2\lambda(t)d(t\wedge\tau) \ .$$

Because of $dA_t = X_{t-}(\widehat{\lambda}(t) - \lambda(t))dt = X_{t\wedge\tau-}(\widehat{\lambda}(t) - \lambda(t))d(t\wedge\tau)$ we obtain $A_t = \int_0^t \widetilde{\alpha}_s\, d\langle M\rangle_s$ with

$$\widetilde{\alpha}_t = \frac{1}{X_{t\wedge\tau-}}\left(\frac{\widehat{\lambda}(t)}{\lambda(t)} - 1\right) \ ,$$

therefore $X = X_0 + \int \widetilde{\alpha}\, d\langle M\rangle + M$. Hence the conditions X(1) and X(3) from Schweizer (1991) are fulfilled. Since $P(\tau = T) = 0$ and X_T is P-a.s. continuous at T. Hence the price process X assures X(5). If the default intensities fulfill the estimation $\mathrm{E}_M[|\widetilde{\alpha}|\log^+(|\widetilde{\alpha}|)] < \infty$[8], X will also be subject to condition X(4).[9] However, due to $\langle M\rangle_t = 0$ if $t > \tau$ the condition X(2) is not fulfilled. Although we can apply the results from Schweizer (1991) to the intensity model, because after default the financial market is not subject to any risk, in addition $X_t = 0$ for $t > \tau$ and hence the locally risk-minimizing hedge must be self-financing with $h_t^S = 0$ and $h_t^B = \delta(\widetilde{C}_T + F/B_T)$ for $t > \tau$. The share in the money market account results from the requirement $\mathrm{V}_T(\mathbf{H}) = F(T) = \delta(\widetilde{C}_T B_T + F)$. Taking this into account, lemma 2.1, lemma 2.2 from Schweizer (1991) and hence considering the also

[6]Since the process $\widetilde{H}$ is a square integrable martingale and due to proposition A.1 from appendix A $[\widetilde{H},\widetilde{H}] = H$ holds, besides the process X_- is predictable and $\mathrm{E}_P[\int_0^T X_{t-}^2\, d[\widetilde{H},\widetilde{H}]_t] = \mathrm{E}_P[\int_0^T X_{t-}^2\, dH_t] < \infty$ holds, M is because of Protter (1990, p. 142) also a square integrable martingale.

[7]For example see Protter (1990).

[8]Here $\mathrm{E}_M[\cdot]$ denotes the expectation under the Doléans-Dade measure $P_M = P \times \langle M,M\rangle$.

[9]If the default intensities fulfill for example the condition $\inf_{t\in[0,T]} |\widehat{\lambda}(t) - \lambda(t)|/|\lambda(t)| > 0$, the estimation $\mathrm{E}_M[|\widetilde{\alpha}|\log^+(|\widetilde{\alpha}|)] < \infty$ holds.

applicable theorem 3.2 from Schweizer (1990) proposition 2.3 and theorem 2.4 from Schweizer (1991) maintain valid. That is the reason why the LRM-hedge can be identified via the following Föllmer-Schweizer-decomposition (FS-decomposition).[10]

Lemma 1 (FS-Decomposition of a Credit Derivative)
The credit derivative (Z, C, F) has the following strong Föllmer-Schweizer-decomposition:

$$F(T)/B_T = F(0) + \int_0^T \xi_t^F \, dX_t + L_T^F \, ,$$

whereas $\xi_t^F = (\widetilde{C}_T B_T + F)[1 - (\mu^\delta(t) - \widetilde{\mu}^\delta(t))/(B_T X_{t-})]$ holds for $t \leq \tau$ and $\xi_t^F = 0$ for $t > \tau$, the constant is $F(0) = (\widetilde{C}_T B_T + F)(X_0 + \widetilde{\mu}^\delta(0)/B_T)$ and the martingale L^F, which is orthogonal to M, is given as $L_t^F = H_t(\widetilde{C}_T + F/B_T)(\delta - \mu^\delta(\tau))$.[11] The function $\mu^\delta(\tau) = (\widetilde{C}_\tau B_T + \mu^Z(\tau))/(\widetilde{C}_T B_T + F)$ describes the expected recovery rate under the statistical probability measure assuming a default at time τ and finally we denote $\widetilde{\mu}^\delta(t) = \int_t^T \mu^\delta(\tau)\widehat{\lambda}(\tau) \exp\{-\int_t^\tau \widehat{\lambda}(s)\, ds\}\, d\tau$.

As described in Schweizer (1991) the Föllmer-Schweizer-decomposition can be evaluated by means of the minimal martingale measure P^{min}.[12] We obtain for the density of the minimal martingale measure:[13]

$$\begin{aligned} Z_t^{min} &= \mathcal{E}\left\{-\int \widetilde{\alpha}\, dM\right\}_t = \mathcal{E}\left\{\int_0^{t\wedge\tau} \lambda(s) - \widehat{\lambda}(s)\, ds + \left(\frac{\widehat{\lambda}(\tau)}{\lambda(\tau)} - 1\right) H_t\right\} \\ &= \begin{cases} \exp\{\int_0^t \lambda(s) - \widehat{\lambda}(s)\, ds\}\, , & \text{if } t < \tau \\ \frac{\widehat{\lambda}(\tau)}{\lambda(\tau)} \exp\{\int_0^\tau \lambda(s) - \widehat{\lambda}(s)\, ds\}\, , & \text{if } t \geq \tau \end{cases} . \end{aligned}$$

Thus the default intensity under P^{min} concurs with $\widehat{\lambda}$ and the recovery has the same distribution under P^{min} as under the statistical probability measure P. Therefore the function $\widetilde{\mu}^\delta(t)$ describes the expectation of the recovery for default occurring in

[10] Lemma 1 is proved in appendix A.

[11] Since δ is unknown before default and H is null for all $t < \tau$, this notation can be taken as $L_t^F = 0$ if $t < \tau$ and $L_t^F = (\widetilde{C}_T + F/B_T)(\delta - \mu^\delta(\tau))$ if $t \geq \tau$.

[12] The denomination "minimale martingal measure" has it seeds in the properties of this measure: In the context of Schweizer (1991) P^{min} is the measure that carries X over in a martingale but keeps the remaining model structure. Schweizer (1999) shows that the minimale martingale measure minimizes the reciprocal of the relative entropy $H(P/Q) = \mathrm{E}_Q[\log dQ/dP]$ for all equivalent martingale measures Q.

[13] For evaluing the stochastic exponential see, e.g., Protter (1990, p. 77).

$t < \tau \le T$ under the condition that the firm is still solvent in t:

$$\widetilde{\mu}^{\delta}(t) = \mathrm{E}_{min}[\delta \mathbb{1}_{\{\tau \le T\}}|\tau > t] = \mathrm{E}_{min}[\delta \mathbb{1}_{\{\tau \le T\}}|t < \tau \le T] P^{min}(\tau \le T|\tau > t) \ . \quad (4)$$

Due to the results of Schweizer (1991) the Föllmer-Schweizer-decomposition provides the locally risk-minimizing hedging strategy considering that the function $V_t^F = \mathrm{E}_{min}[F(T)/B_T|\mathcal{F}_t]$ fulfills

$$V_t^F = F(0) + \int_0^t \xi_s^F \, dX_s + L_t^F = \begin{cases} (\widetilde{C}_T B_T + F)(X_t + \widetilde{\mu}^{\delta}(t)/B_T) \ , & \text{if} \quad t < \tau \\ (\widetilde{C}_T + F/B_T)\delta \ , & \text{if} \quad t \ge \tau \end{cases}$$

because of the formulae (A2) and (A3) from the proof of lemma 1 in appendix A.

Proposition 2 (LRM-Hedge)
The locally risk-minimizing hedge of the credit derivative (Z, C, F) *amounts to*

$$\begin{aligned} h_t^S &= \xi_t^F = (\widetilde{C}_T B_T + F)\left[1 - \frac{1}{X_{t-} B_T}(\mu^{\delta}(t) - \widetilde{\mu}^{\delta}(t))\right] \ , \\ h_t^B &= V_{t-}^F - h_t^S X_{t-} = (\widetilde{C}_T + F/B_T)\mu^{\delta}(t) \quad \textit{for every} \quad t \le \tau \ . \end{aligned}$$

After default $t > \tau$ *we have*

$$h_t^S = 0 \ , \quad h_t^B = \delta(\tau)(\widetilde{C}_T + F/B_T) = \left(Z(\tau)/B_T + \widetilde{C}_\tau\right) \ .$$

In the case of a defaultable claim with single-stochastic recovery the locally risk-minimizing hedge collapses to the duplication strategy given in proposition 1.

Every time $t < \tau$ the locally risk-minimizing hedging strategy keeps as much in the money market account, that this position has a value in the amount of the under the statistical measure expected recovery in addition to the accumulated accrued payments $\mu^Z(t) + \widetilde{C}_t B_T = \mu^{\delta}(t)(\widetilde{C}_T B_T + F)$ until default at $\tau = t$. At default the share in the money market account makes a jump in the amount of $(Z(\tau) - \mu^Z(\tau))/B_T$ such that the value of the hedging strategy at maturity coincides with the value of the derivative $Z(\tau) + \widetilde{C}_\tau B_T = \delta(\tau)(\widetilde{C}_T B_T + F) = F(T)$. The position of defaultable zeros with total loss at time t is equal to the under the minimal martingale measure expected discounted future payments assuming no default in t minus the under the minimal martingale measure expected recovery if a credit event occur at t, because due to (4):

$$\begin{aligned} h_t^S X_t &= (\widetilde{C}_T + F/B_T)\left(X_t B_T + \widetilde{\mu}^{\delta}(t)\right) - (\widetilde{C}_T + F/B_T)\mu^{\delta}(t) \\ &= (\widetilde{C}_T + F/B_T)\left(\mathrm{E}_{min}[\mathbb{1}_{\{\tau > T\}}|\tau > t] + \mathrm{E}_{min}[\delta \mathbb{1}_{\{\tau \le T\}}|\tau > t]\right) \\ &\quad -(\widetilde{C}_T + F/B_T)\mu^{\delta}(t) \\ &= \mathrm{E}_{min}[F(T)/B_T|\tau > t] - (\widetilde{C}_T + F/B_T)\mu^{\delta}(t) \ . \end{aligned}$$

Because of the relation $C(\mathbf{H}) = V_0^F + L^F$ the LRM-hedge is self-financing at every point in time before and after default. But at default money accrues and outflows, respectively depending on the realized recovery $\delta(\tau)$ differs upwards and downwards, respectively from the expected payment at default $\mu^\delta(\tau)$. In average the locally risk-minimizing hedging strategy gets out without means. This implies that the hedge is mean-self-financing as expected according to proposition 2.3 in Schweizer (1991). If the recovery is single-stochastic the LRM-hedge will even been self-financing and therefore will depict a replication strategy. For the special case, that the expected recovery rate does not depend on the default time, i.e. $\mu^\delta(\tau) = \mu^\delta$ at $0 < \tau \leq T$, and hence $\widetilde{\mu}^\delta(t) = \mu^\delta(1 - B_T X_{t-})$ for $t \leq \tau$, the locally risk-minimizing hedge simplifies to an statical hedge:

$$\mathbf{H} = (h^S, h^B) = ((\widetilde{C}_T B_T + F)(1 - \mu^\delta), (\widetilde{C}_T + F/B_T)\mu^\delta) \ .$$

Proposition 2 shows that the locally risk-minimizing hedge depends only on the expected payment at default under the statistical probability measure, but not on other details on the probability distribution of the recovery. Hence we achieve the following result:

Proposition 3 (Impact of the Recovery Modeling)
The locally risk-minimizing hedge for a credit derivative (Z^d, C, F) *with a double-stochastic recovery concurs with the LRM-hedge for a defaultable claim* (Z^e, C, F) *with single-stochastic recovery for all points in time until default provided that the under the statistical probability measure expected recovery coincide, i.e.* $\mu^{Z^d}(\tau) = \mu^{Z^e}(\tau) = Z^e(\tau)$ *for every* $0 < \tau \leq T$.

3.4 An example

We consider a financial market where a defaultable zero of a firm with total loss at default and maturity 10 years is traded. Furthermore, we assume a flat term structure with $r = 5$ %. The default time possesses an exponential distribution with intensity $\lambda = 0{,}05$ and $\widehat{\lambda} = 0{,}20$, respectively under the statistical probability measure and the martingal measure, respectively. In the following, we determine hedging strategies of a defaultable zero with recovery payment at default. We assume a single-stochastic, one time even a *constant* recovery amount of $Z^e = \delta^e =$ 40 %, another time we consider a double-stochastic recovery which possesses at every default time a expected amount of $\mu^{Z^d} = \mu^{\delta^d} = 40$ %.

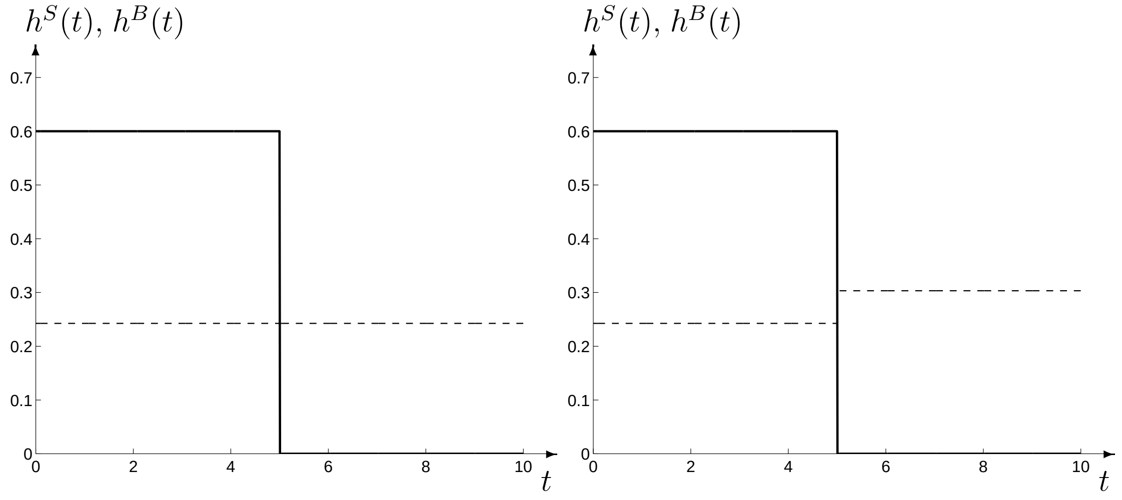

Figure 3: *LRM-hedge*

The left figure describes the locally risk-minimizing strategy of the defaultable zero with constant recovery. This hedge corresponds to the duplication. The right figure depicts the LRM-hedge of the defaultable zero with an uncertain recovery payment assuming that at default a recovery rate of 50 % is realized. The solid line describes in each time the hedge ratio and the dashed line the shares invested in the money market account.

Figure 3 shows the locally risk-minimizing hedging strategy of a zero with single- and double-stochastic recovery. We assume, that the firm defaults after 5 years and that the realized recovery rate amounts to 50 % in the case of double-stochastic recovery modeling. As due to proposition 3 expected the LRM-hedge coincides until default for the cases of a single- and a double-stochastic recovery. After the credit event the shares in the money market account of the locally risk-minimizing strategies differ since the realised payment at default are different.

If an investor prefers a self-financing hedging strategy, the so-called *super-hedging strategy*, which assures a liquidation value at maturity at least as high as the pay-off of the derivative, i.e. $\mathrm{V}_T(\mathbf{H}) \geq F(T)$ P-a.s., then the recovery modeling has got impact on the hedging strategy as the following will show. Assuming a constant recovery payment of 0,40 the super-hedge corresponds to the duplication strategy $\mathbf{H} = (h^S, h^B) = (0{,}60;\ 0{,}40/B_T)$ as well as the LRM-hedge. If the payment at default is uncertain, the super-hedge depends on the distribution of the recovery, more precisely on the domain of the recovery payment. Assuming that the recovery payment can reach values on $[0, 1]$ and $[0, 0{,}95]$, respectively, the super-hedge holds $\mathbf{H} = (h^S, h^B) = (0;\ 1/B_T)$ and $\mathbf{H} = (h^S, h^B) = (0{,}05;\ 0{,}95/B_T)$, respectively.

4 Hedging in structural models

4.1 Model

Merton (1974) models the possible default of a firm with a single liability carrying a promised terminal payoff $\bar{D}$ by comparing the total firm's value V_T at the debt's maturity with the notional value of debt $\bar{D}$: If the firm value V_T exceeds the outstanding debt the liability is repaid in full, otherwise the firm defaults and the bondholders receive the amount $V_T < \bar{D}$, i.e.

$$\tau = \begin{cases} T & \text{, if } V_T < \bar{D} \\ \infty & \text{, if } V_T \geq \bar{D} \end{cases} .$$

The default process and the price of credit derivatives depend primarily on the firm value. That is the reason why Merton's model is assigned to the class of structural models. Merton (1974) assumes the firm value V to follow a geometric Brownian motion with constant volatility σ and constant drift α,

$$\frac{dV}{V} = \alpha\, dt + \sigma\, dW \quad \text{and} \quad V_t = V_0 \exp\left\{\left(\alpha - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right\} , \tag{5}$$

where W is an standard Brownian motion under the statistical probability measure P. The financial market is characterized by the probability space $(\Omega,\mathcal{F},P)$ with filtration $\mathcal{F}_t = \sigma(W_s, 0 \leq s \leq t) = \sigma(V_s, 0 \leq s \leq t)$. Besides Merton assumes a flat term structure with interest rate r. Again, $B_t = \exp\{rt\}$ denotes the value of the money market account at t. Furthermore, trading takes place continuously in time and the financial market is frictionless. If D and E, respectively, denote the market value of debt and equity, respectively then in Merton's model the relation $V = D + E$ will always be fulfilled. At default in maturity the obligors receive V_T and the investors go away empty-handed, i.e. $D(T,V_T) = \min(\bar{D},V_T)$ and $E(T,V_T) = \max(V_T - \bar{D},0)$. Merton's model is rather simple with a single liability and default occurring at most at time T. Moreover, Merton (1974) neglects costs of financial distress.

In this section Merton's model is extended by allowing bankruptcy costs. This means a stochastic amount κV_T falls due at default as a result of the insolvency proceedings' settlement. We assume that the *percentage bankruptcy costs* $\kappa \in [0,1]$ is a random variable independent of the firm's value.[14] The market value of debt

[14] This modeling traces back to Leland (1994) and Leland and Toft (1996). However, these authors assume that the percentage bankruptcy costs are not subject to any uncertainty.

and equity at maturity are

$$D(T,V_T) = \begin{cases} \bar{D} & \text{, if } V_T \geq \bar{D} \\ (1-\kappa)V_T & \text{, if } V_T < \bar{D} \end{cases} \quad \text{and} \quad E(T,V_T) = \max(V_T - \bar{D},0) \ .$$

If $BC(T,V_T) = \kappa V_T \mathbb{1}_{\{V_T < \bar{D}\}}$ denotes the costs of financial distress, the equation

$$V_T = E(T,V_T) + D(T,V_T) + BC(T,V_T)$$

will hold. The variable V modeled according to (5) will not describe the sum of debt and equity, if we account for bankruptcy costs. But the variable V includes the value of the possibly accrued costs of financial distress as well. Therefore V depicts the *gross firm's value*. The financial market, composed of the money market account and the gross firm's value V, is arbitrage-free both with certain percentage bankruptcy costs and uncertain, since there exists an equivalent martingale measure Q. In Merton's (extended) model every equivalent martingale measure Q (defined on $\mathcal{F}_T$) arises from the statistical measure P via the following Radon-Nikodym-density:[15]

$$\frac{dQ}{dP} = \exp\left\{ -\frac{\alpha - r}{\sigma} W_T - \frac{1}{2}\left(\frac{\alpha - r}{\sigma}\right)^2 T \right\} \ . \tag{6}$$

Under a martingale measure Q the firm value process is given as

$$V_t = V_0 \exp\left\{ \left(r - \frac{1}{2}\sigma^2\right) t + \sigma \widetilde{W}_t \right\} \ , \tag{7}$$

whereas $\widetilde{W}_t = W_t + \frac{\alpha - r}{\sigma} t$ denotes a standard Brownian motion under Q.[16]

As long as the percentage bankruptcy costs κ are *constant* P-a.s., the equivalent martingale measure from (6) is unique. Otherwise every probability measure fulfilling (6) is an martingale measure with an arbitrary distribution of κ. It is equivalent, if the distribution of κ under this measure has the same null sets than under the statistical probability measure P. Since the market value of the equity at time T does not depend on the costs of financial distress (corresponding to the value of a call on the firm's value with strike $\bar{D}$), the expected discounted value of the equity holders' share under Q can be calculated via the Black-Scholes-formula. We have[17]

$$E_t = \mathrm{E}_Q[B_t/B_T E(T,V_T)|\mathcal{F}_t] = V_t \mathcal{N}(d_1) - B_t(T)\bar{D}\mathcal{N}(d_2) \tag{8}$$

[15] Girsanov's theorem proves this lemma. See theorem 7.2.3 in Elliott and Kopp (1999, p. 138) and their application to the Black-Scholes-model in Elliott and Kopp (1999, p. 154).

[16] This follows from Girsanov's theorem. See Elliott and Kopp (1999, p. 154).

[17] See, for example, Elliott and Kopp (1999, p. 165).

for all possible equivalent martingale measures Q. Where $d_1 = d_1(t) = d_1(V_t,r,\bar{D},t) = (\ln(V_t/\bar{D}) + (r + 0{,}5\sigma^2)(T-t))/(\sigma\sqrt{T-t})$ and $d_2 = d_2(V_t,r,\bar{D},t) = d_1(V_t,r,\bar{D},t) - \sigma\sqrt{T-t}$ are used. $\mathcal{N}(\cdot)$ denotes the distribution function of the standard normal distribution and $B_t(T) = \exp\{-r(T-t)\}$ describes the value of a risk-free bond with face value 1 and maturity T at point in time t. The expected discounted market value of debt and of the bankruptcy cost will depend on the martingale measure Q, if κ is not P-a.s. Otherwise we have

$$D_t = \mathrm{E}_Q[B_t/B_T D(T,V_T)|\mathcal{F}_t] = (1-\kappa)V_t\,(1-\mathcal{N}(d_1)) + B_t(T)\bar{D}\mathcal{N}(d_2)\,, \quad (9)$$
$$BC_t = \mathrm{E}_Q[B_t/B_T BC(T,V_T)|\mathcal{F}_t] = \kappa V_t\,(1-\mathcal{N}(d_1)) \quad (10)$$

for every $0 \le t \le T$ as the proof in the appendix A shows. The sum of the equity's and debt's market values $v(t,V_t) = E(t,V_t) + D(t,V_t)$ is called *net firm's value.*

Since the firm value is not traded, we are searching hedging strategies composed of money market account and stocks of the firm below. We assume that the company issued s stocks and $\bar{D}$ defaultable zeros with face value 1 and maturity T at $t = 0$. The value of a share S_t at time t is $S_t = E_t/s$. To simplify matters we assume that at any time the firm's value is known. Due to equation (8) the firm's value can be replicated by a self-financing hedging strategy, which consists of $\exp\{-rT\}\bar{D}\mathcal{N}(d_2)/\mathcal{N}(d_1)$ money market accounts and $s/\mathcal{N}(d_1)$ stocks. Therefore every trading strategy $\mathbf{H}^V = (h^V,h^B)$, composed of h^V shares of the firm's value and h^B money market accounts, can be transferred to a hedging strategy $\mathbf{H}$ consisting of $h^S = s/\mathcal{N}(d_1)h^V$ stocks and $h^B + \exp\{-rT\}\bar{D}\mathcal{N}(d_2)/\mathcal{N}(d_1)h^V$ money market accounts, so that $\mathrm{V}_t(\mathbf{H}^V) = \mathrm{V}_t(\mathbf{H})$ at any time t. Furthermore every probability measure is martingale measure of V if and only if it is one of S, assuming the value of the equity modeled by relation (8). Consequently the assertions (6) and (7) maintain valid for the financial market composed of money market account and stocks of the firm.

4.2 Single-stochastic recovery payment

Credit derivatives with a pay-off not depending on the bankruptcy costs are replicable. The following proposition specifies the duplication strategy.[18]

[18] The hedging strategies follows from the above described relation between $\mathbf{H} = (h^S,h^B)$ and $\mathbf{H}^V$ and moreover Elliott and Kopp (1999, p. 163 ff.)

Proposition 4 (Duplication in Merton's Model)
A defaultable claim, whose pay-off can be written as a function of the firm's value V_T at T, i.e. $F(T) = f(V_T)$[19] *possesses a duplication strategy $\mathbf{H} = (h^S, h^B)$ composed of h^S stocks and h^B money market accounts. With the auxiliary function*

$$G(t,v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\left(v \exp\left\{(r - 0{,}5\sigma^2)t + \sigma x \sqrt{t}\right\}\right) e^{-x^2/2}\, dx$$

it follows

$$\begin{aligned} h_t^S &= B_t(T) \cdot \frac{\partial G}{\partial v}(T-t, V_t) \cdot \frac{s}{\mathcal{N}(d_1)} , \\ h_t^B &= \exp\{-rT\} \left(G(T-t, V_t) - \frac{\partial G}{\partial v}(T-t, V_t) \left[V_t - B_t(T) \bar{D} \frac{\mathcal{N}(d_2)}{\mathcal{N}(d_1)} \right] \right) . \end{aligned}$$

The liquidation value of the hedging strategy $\mathbf{H} = (h^S, h^B)$ at time t is $V_t(\mathbf{H}) = B_t(T) G(T-t, V_t)$, particularly it holds $V_T(\mathbf{H}) = F(T)$ and the unique arbitrage-free price of the defaultable claim is $F(0) = B_0(T) \cdot G(T, V_0)$.

Due to $F(t) = \mathrm{V}_t(\mathbf{H}) = B_t(T) G(T-t, V_t)$ and $\partial S / \partial V = \mathcal{N}(d_1)/s$ the hedge ratio h^S can be rewritten to

$$h^S = \frac{\partial F}{\partial V} \Big/ \frac{\partial S}{\partial V} ,$$

such that the duplication strategy from proposition 4 corresponds to a *delta hedge*.

Corollary 1 (Duplication of the Defaultable Zero – a Special Case)
If the percentage bankruptcy costs are not subject to any risk, the defaultable zero can be replicated via the following strategy:

$$\begin{aligned} h_t^S &= -(1-\kappa) \frac{s}{\bar{D}} \left(1 - \frac{1}{\mathcal{N}(d_1)} \right) + \frac{\kappa s}{\bar{D} \sigma \sqrt{T-t}} \frac{\varphi(d_1)}{\mathcal{N}(d_1)} , \\ h_t^B &= \exp\{-rT\} \frac{\mathcal{N}(d_2)}{\mathcal{N}(d_1)} + \kappa \Bigg(\exp\{-rT\} \mathcal{N}(d_2) \left(1 - \frac{1}{\mathcal{N}(d_1)} \right) \\ &\quad - \exp\{-rt\} \frac{V_t}{\bar{D} \sigma \sqrt{T-t}} \varphi(d_1) + \exp\{-rT\} \frac{1}{\sigma \sqrt{T-t}} \varphi(d_1) \frac{\mathcal{N}(d_2)}{\mathcal{N}(d_1)} \Bigg) . \end{aligned}$$

Here $\varphi(\cdot)$ denotes the density function of the standard normal distribution.

[19] Here the function $f : (0,\infty) \to \mathbb{R}$ must fulfill the integrability condition $f(v) \leq c(1 + v^{k_1}) v^{-k_2}$ with non-negative constants c, k_1 and k_2.

4.3 Double-stochastic recovery payment

Because of proposition 4 every defaultable claim whose payoff depends only on the firm's value V_T at maturity T can be replicated. In that case the locally risk-minimizing hedging strategy coincides with die duplication strategy. Otherwise the value of the derivative is affected by the random variable κ. To determine the locally risk-minimizing strategy $\mathbf{H}$ for the defaultable zero, we first look for the LRM-hedge $\mathbf{H}^V = (h^V, h^B)$, composed of h^V shares of the firm desired value and h^B money market accounts. Then we are looking for the trading strategy $\mathbf{H}$. This two-stage approach simplifies the appropriate calculations. The discounted firm value $\widetilde{V}_t = V_t/B_t$ is a continuous semimartingale with a continuous drift component A and a square integrable martingale component M[20]:

$$\widetilde{V}_t = V_0 + \underbrace{\int_0^t \widetilde{V}_s(\alpha - r)\, ds}_{=:A_t} + \underbrace{\int_0^t \widetilde{V}_s \sigma\, dW_s}_{=:M_t} .$$

We obtain for the increment of the drift component $dA_t = \widetilde{V}_t(\alpha - r)\, dt$ and for the conditional quadratic variation of the martingale $d\langle M\rangle_t = \widetilde{V}_t^2\sigma^2\, d\langle W\rangle_t = \widetilde{V}_t^2\sigma^2\, dt$. Hence, we have

$$\widetilde{V} = V_0 + \int \widetilde{\alpha}\, d\langle M\rangle + M \quad \text{with} \quad \widetilde{\alpha}_t = \frac{\alpha - r}{\sigma^2 \widetilde{V}_t} .$$

The mean-variance-trade-off-process

$$K_t = \int_0^t \widetilde{\alpha}_s\, dA_s = \left(\frac{\alpha - r}{\sigma}\right)^2 t$$

is deterministic so that the structure condition from Schweizer (1991) is fulfilled. Obviously, the requirements from Schweizer (1991) are satisfied and the locally risk-minimizing hedge of the defaultable zero can be determined via the Föllmer-Schweizer-decomposition. The Föllmer-Schweizer-decomposition can be calculated with the minimal martingale measure. Latter is represented by

$$Z_T^{min} = \mathcal{E}\left\{-\int \widetilde{\alpha}\, dM\right\}_T = \mathcal{E}\left\{-\frac{\alpha - r}{\sigma} W\right\}_T = \exp\left\{-\frac{\alpha - r}{\sigma} W_T - \frac{1}{2}\left(\frac{\alpha - r}{\sigma}\right)^2 T\right\} .$$

[20]Since the Brownian Motion W is a square integrable martingale under P with $[W,W]_t = t$, the process $\widetilde{V}$ is continuous and hence predictable, besides $\mathrm{E}_P[\int_0^T \widetilde{V}_s \sigma\, d[W,W]_s] = \mathrm{E}_P[\int_0^T \widetilde{V}_s \sigma\, ds] < \infty$ holds, therefore due to Protter (1990, p. 142) M is also a square integrable martingale under P.

Hence under the minimal martingale measure the firm's value V is distributed as described in (7) and the percentage bankruptcy costs have the same distribution as under the statistical probability measure. The LRM-hedge is obtained via the corresponding FS-decomposition as verified in the Appendix.

Lemma 2 (FS-Decomposition of a Defaultable Zero)
Assuming random percentage bankruptcy costs κ the defaultable zero possesses the following Föllmer-Schweizer-decomposition:

$$\widetilde{B}_T(T)/B_T = \widetilde{B}_0 + \int_0^T \xi_t^{\widetilde{B}}\, d\widetilde{V}_t + L_T^{\widetilde{B}} \ ,$$

whereas $\xi_t^{\widetilde{B}} = (1-\mathcal{N}(d_1))(1-\bar{\kappa})/\bar{D} + \varphi(d_1)\bar{\kappa}/(\bar{D}\sigma\sqrt{T-t})$.[21] *The constant is $\widetilde{B}_0 = (1-\mathcal{N}(d_1(0)))\,V_0(1-\bar{\kappa})/\bar{D} + B_0(T)\mathcal{N}(d_2(0))$ and the martingale $L^{\widetilde{B}}$, which is orthogonal to M, fulfills $L_t^{\widetilde{B}} = 0$ for $t < T$ and $L_T^{\widetilde{B}} = \mathbb{1}_{\{V_T<\bar{D}\}}(\bar{\kappa}-\kappa)V_T/(B_T\bar{D})$. $\bar{\kappa} = E_P[\kappa|\mathcal{F}_t]$ denotes the percentage bankruptcy costs expected under the statistical probability measure at time $t < T$.*

Assuming the special case where percentage bankruptcy costs are certain with value $\bar{\kappa}$, the constant of the Föllmer-Schweizer-decomposition coincides with the arbitrage-free value of the defaultable zero. In this case the component $\xi^{\widetilde{B}}$ corresponds to the hedge ratio h^V of the duplication strategy $\mathbf{H}^V = (h^V, h^B)$ for a zero. Due to the results of Schweizer (1991) the locally risk-minimizing hedge composed of shares of firm's value and money market accounts

$$\begin{aligned} h_t^V &= -(1-\bar{\kappa})\frac{1}{\bar{D}}\left(\mathcal{N}(d_1)-1\right) + \frac{\bar{\kappa}}{\bar{D}\sigma\sqrt{T-t}}\varphi(d_1) \ , \\ h_t^B &= V_{t-}^{\widetilde{B}} - h_t^V \widetilde{V}_{t-} \ , \end{aligned}$$

at $t < T$, whereas $V_t^{\widetilde{B}} = (1-\bar{\kappa})\widetilde{V}_t\,(1-\mathcal{N}(d_1))\,/\bar{D} + \exp\{-rT\}\mathcal{N}(d_2)$ for $t < T$ and $V_T^{\widetilde{B}} = \mathbb{1}_{\{V_T<\bar{D}\}}\,(1-\kappa)\,\frac{V_T}{\bar{D}B_T} + \mathbb{1}_{\{V_T\geq\bar{D}\}}\frac{1}{B_T}$ as well as $\bar{\kappa} = \mathrm{E}_P[\kappa|\mathcal{F}_t]$ at $t < T$. At default it holds $h_{\tau=T}^V = 0$ and $h_{\tau=T}^B = (1-\kappa)\frac{V_T}{\bar{D}B_T}$. Since $V_t^{\widetilde{B}}$ coincides with the arbitrage-free discounted price of a defaultable zero for every point of time t before maturity assuming the percentage bankruptcy costs are surely $\bar{\kappa}$, the LRM-strategy $\mathbf{H} = (h^S, h^B)$ amounts to

Proposition 5 (LRM-Strategy of a Defaultable Zero)
The locally risk-minimizing hedging strategy $\mathbf{H} = (h^S, h^B)$ of the defaultable zero

[21] Here the last summand will converge against null P-a.s. if t approaches T.

for $t < \tau$ amounts to

$$
\begin{aligned}
h_t^S &= -(1-\bar{\kappa})\frac{s}{\bar{D}}\left(1-\frac{1}{\mathcal{N}(d_1)}\right)+\frac{\bar{\kappa}s}{\bar{D}\sigma\sqrt{T-t}}\frac{\varphi(d_1)}{\mathcal{N}(d_1)}\,,\\
h_t^B &= \exp\{-rT\}\frac{\mathcal{N}(d_2)}{\mathcal{N}(d_1)}+\bar{\kappa}\Bigg(\exp\{-rT\}\mathcal{N}(d_2)\left(1-\frac{1}{\mathcal{N}(d_1)}\right)\\
&\quad -\exp\{-rt\}\frac{V_t}{\bar{D}\sigma\sqrt{T-t}}\varphi(d_1)+\exp\{-rT\}\frac{1}{\sigma\sqrt{T-t}}\varphi(d_1)\frac{\mathcal{N}(d_2)}{\mathcal{N}(d_1)}\Bigg)
\end{aligned}
$$

as well as $h^S_{\tau=T} = 0$ and $h^B_{\tau=T} = (1-\kappa)V_T/\bar{D}B_T$.

Therefore at any time $t < \tau$ the locally risk-minimizing hedges for the defaultable zero are the same for certain $\bar{\kappa}$ and uncertain κ percentage bankruptcy costs if $\bar{\kappa} = \mathrm{E}_P[\kappa] = \mathrm{E}_P[\kappa|\mathcal{F}_t]$ $(t < T)$ holds.

Proposition 6 (Impact of the Recovery Modeling)
The locally risk-minimizing hedging strategy of a defaultable zero with double-stochastic recovery, i.e. uncertain percentage bankruptcy costs κ^d, coincides at any time before default with the locally risk-minimizing hedge of a defaultable zero with single-stochastic, i.e. certain percentage bankruptcy costs κ^e, provided $\kappa^e = E_P[\kappa^d]$.

It can be easily shown, that the Föllmer-Schweizer-decomposition and hence the LRM-hedge of a credit derivative with uncertain percentage bankruptcy costs κ can be traced back to the case of certain percentage bankruptcy costs in the amount of $\bar{\kappa} = \mathrm{E}_P[\kappa]$ – similarly as in lemma 2 described –, if the recovery of the derivative depends *linear* on the percentage bankruptcy costs. For this reason the assertion of proposition 6 holds not only for a defaultable zero but also for a whole class of credit derivatives.

4.4 An example

A corporation has issued a zero bond with maturity 10 years in $t = 0$. The firm's asset value holds $V_0 = 100$ in $t = 0$ and is modeled as a Brownian motion with drift $\mu = 8$ % and volatility $\sigma = 0{,}20$ as described in equation (5). The corporation has got a simple capital structure: it has issued $s = 100$ stocks in $t = 0$ and a single liability in terms of zeros with maturity 10 years. The face value of the debt is $\bar{D} = 75$. We assume that the percentage bankruptcy costs κ are uncertain. Under the statistical probability measure it has got an expectation in amount of $\bar{\kappa} = 20$ %

and at best it holds $\kappa^{min} = 2.5$ %. Finally, we assume a flat term structure with interest rate $r = 5$ %

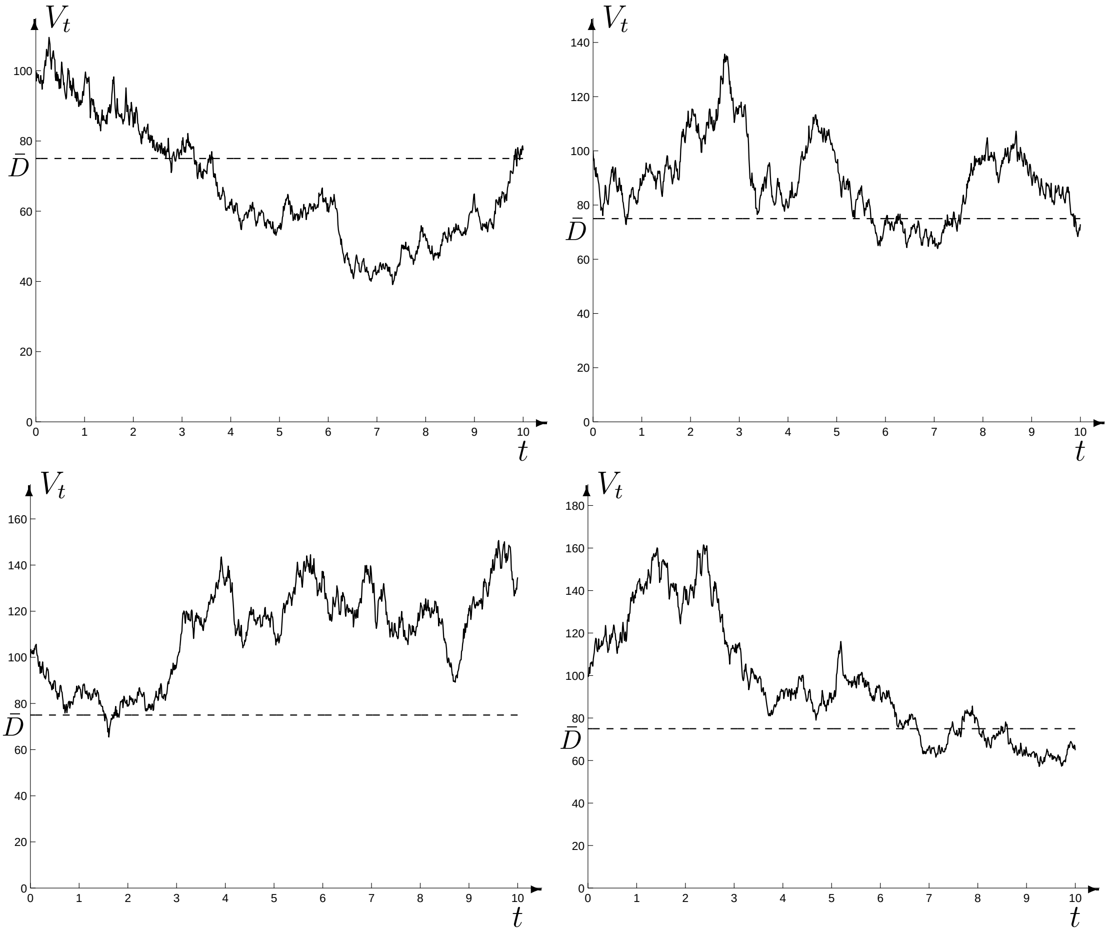

Figure 4: *Four simulated paths of the firm's gross value*

The development of the hedging strategies depends strongly on the movement of the gross firm's value V. On this account we determine hedging strategies of the defaultable zeros for four different gross developments of the firm's value. While the left lower (upper) illustration of figure 4 describes the case, that the corporation keeps (barely) solvent, the corporation defaults (barely) in the right lower (upper) illustration.

The solid lines in figure 5 illustrate the corresponding hedge ratios of the super-hedge. The dotted lines depict the hedge ratio of the LRM-hedge. Assuming the percentage costs of financial distress are certain $\bar{\kappa} = 20$ % the latter coincides with the duplication.

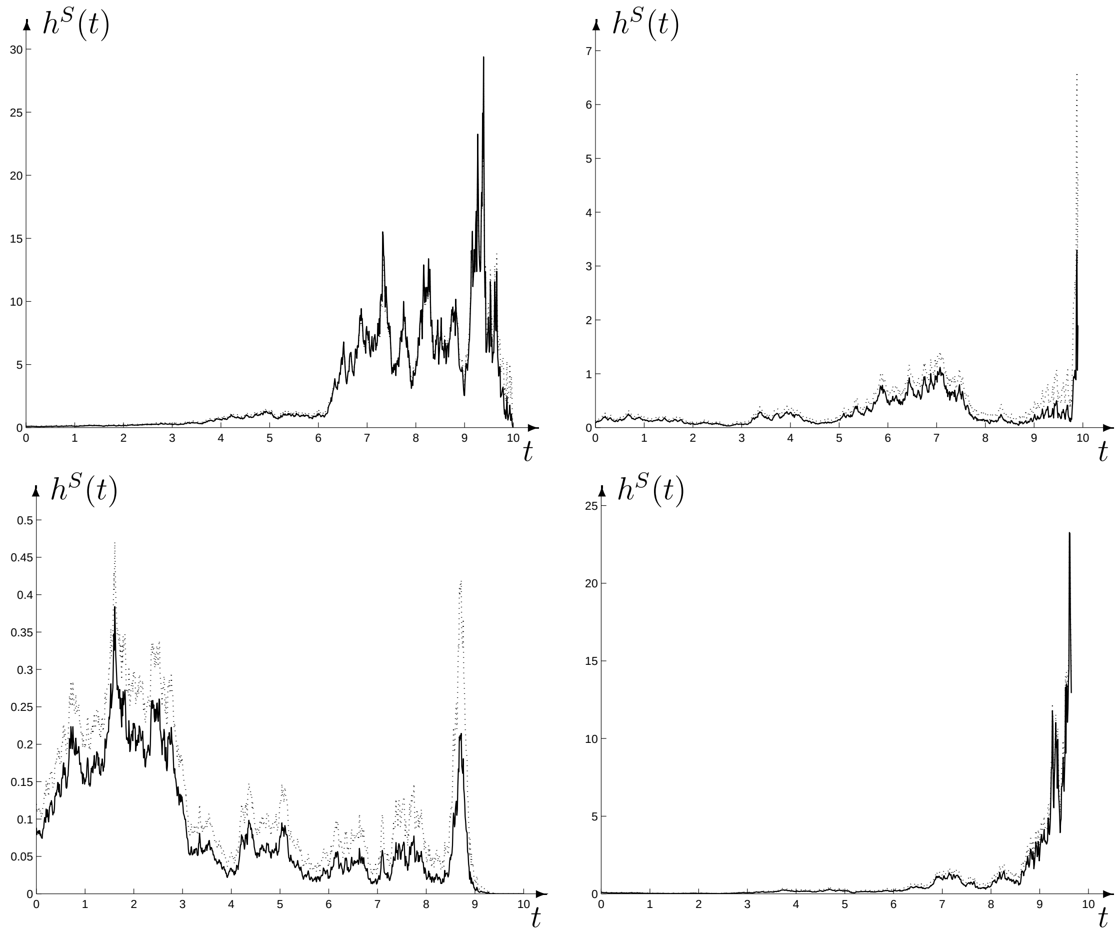

Figure 5: *Hedge ratios of the super-hedge and the LRM-hedge*

It attracts attention, that the hedge ratio fluctuates sparsely and is small, if the gross firm's value lies upon the face value of debt. If the gross firm's value is however smaller than $\bar{D}$, the hedge ratio reaches high value and is very volatile. Especially, if the remaining time to maturity of corporate bond converges against null and the firm's value is situated under the debt's face value, thus a credit event is most likely, the hedge ratio explodes. Especially, the figures on the right hand side show this. It must be pointed out, that these figures describe the hedge ratios only until approximately 9.7 and 9.9 years, because in these cases the hedge ratios converge against infinity if the remaining time until maturity tends to zero.

In figure 6 the solid lines depict the shares in the money market account of the super-hedge and the dotted lines the of the LRM-hedge. Assuming certain percentage bankruptcy costs in the amount of $\bar{\kappa} = 20\ \%$, the latter coincide with the duplication strategy.

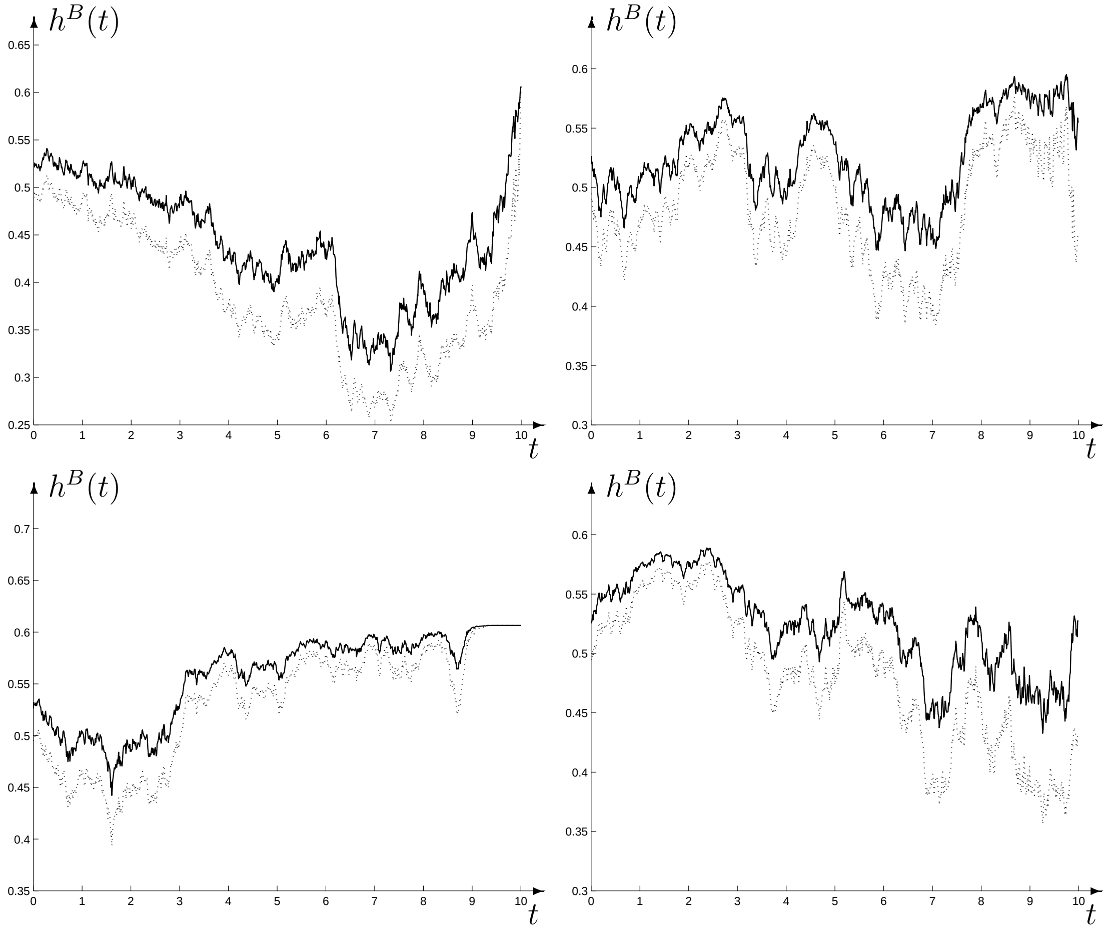

Figure 6: *Shares in the money market account of the super- and LRM-hedge*

It is obviously that the shares in the money market account fluctuate less than the hedge ratios. They will be more stable, if the default probability is small. Especially, the bottom left figure shows this for small remaining time until maturity. Comparing the hedging strategies of figure 5 and 6, it stands out, that the hedge ratios and the shares in the money market account, respectively of the super-hedge are always smaller and higher, respectively than that of the locally risk-minimizing hedge. By investing a bigger share in the risk-free money market account and reducing the investment volume of the defaultable stocks it is assured, that the super-hedge dominates the corporate zero-bond.

5 Conclusion

We derive LRM-hedging strategies for a reduced-form model as well a structural Merton-type model. The latter model differs from the original Merton-model by assuming positive bankruptcy costs, given as *percentage* of the firm value at default. As long as this percentage is a constant, we denote the corresponding recovery rate as *single-stochastic* since the recovery amount depends only on the default event. Otherwise, that is, if the percentage bankruptcy costs is random, we denote the corresponding recovery rate as *double-stochastic* since the recovery amount depends not only on the default event but also on the realization of another random variable. Corresponding model variants are examined for the reduced-form model framework.

For both model classes it turns out that the corresponding LRM-strategy is not only mean-self-financing but also self-financing if the modeled default recovery is single-stochastic. That is, as long as the recovery amount is known in the event of default, there exists a self-financing replication strategy for credit derivatives. Moreover, we find that in the more realistic case of double-stochastic default recoveries, the LRM-hedging strategy does only depend on the *expected* recovery amount, not on other characterics of its distribution. This key result of the paper helps to justify the frequent simplifying assumption that the default recovery is a constant, conditional on the default event, when valuing and hedging credit derivatives.

A Appendix

Problem 2 (Locally Risk-Minimizing Hedging in continuous time)
A trading strategy $\mathbf{H}$ *with* $V_T(\mathbf{H}) = F(T)$ *P-a.s. is called* locally risk-minimizing, *(LRM) for short, if it fulfills*

$$\liminf_{N\to\infty} r^{\mathcal{T}_N}(\mathbf{H},\Delta) \geq 0 \quad P_M\text{-a.s.}^{22}$$

for every null-convergent sequence of partitions $\mathcal{T}_N = \{t_0 = 0, t_1, \ldots, t_N = T\}$ *of* $[0,T]$, *i.e.* $\mathcal{T}_N \subset \mathcal{T}_{N+1}$ *and* $\lim_{N\to\infty} \max_{i=1,\ldots,N}(t_i^N - t_{i-1}^N) = 0$, *and every disturbance* Δ. *Here a* disturbance $\Delta = (\delta,\varepsilon)$ *is a trading strategy, such that* $\delta_T = \varepsilon_T = 0$ *and* $\int_0^T |\delta_s|\, d|A|_s$ *is bounded. Furthermore defining the* remaining risk $R_t(\mathbf{H})$, *measured as the expected quadratic increase of the discounted hedging costs,* $R_t(\mathbf{H}) = E_P\left[(C_T(\mathbf{H}) - C_t(\mathbf{H}))^2 \,|\mathcal{F}_t\right]$, *the expression*

$$r^{\mathcal{T}}(\mathbf{H},\Delta) = \sum_{i=0}^{n-1} \frac{R_{t_i}(\mathbf{H} + \Delta|_{(t_i,t_{i+1}]}) - R_{t_i}(\mathbf{H})}{E_P[\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} | \mathcal{F}_{t_i}]} \mathbb{1}_{(t_i,t_{i+1}]}$$

denotes the risk quotient *for a trading strategy* $\mathbf{H}$, *a disturbance* $\Delta = (\delta,\varepsilon)$ *and the partition* $\mathcal{T} = \{t_0 = 0, t_1, \ldots, t_n = T\}$.

Hence, a trading strategy is locally risk-minimizing if a disturbance of the strategy will raise the risk measured by the risk quotient.

Proposition A.1 ((Compensated) Default process)
The default process H, *which is an inhomogeneous Poisson process stopped at the first jump, fulfills*

$$[H,H] = H \quad \text{and} \quad \langle H,H\rangle_t = \int_0^{t\wedge\tau} \lambda(u)\, du\,.$$

The compensated default process $\widetilde{H}_t = H_t - \int_0^{t\wedge\tau} \lambda(u)\, du$ *satisfies*

$$[\widetilde{H},\widetilde{H}] = H \quad \text{and} \quad \langle \widetilde{H},\widetilde{H}\rangle_t = \langle H,H\rangle_t = \int_0^{t\wedge\tau} \lambda(u)\, du\,.$$

[22] $P_M = P \times \langle M,M\rangle$ denotes the Doléans Dade measure of $\langle M,M\rangle$ on the product space $\Omega \times [0,T]$ with the predictable σ-algebra.

Proof. The assertions for the default process H follow from Protter (1990, p. 63) and Brémaud (1981, p. 23). Since $G : t \mapsto \int_0^{t\wedge\tau} \lambda(u)\, du$ is continuous due to Jacod and Shiryaev (1987, p. 52), we have $[H,G] = 0 = [G,G]$ and hence $[\widetilde{H},\widetilde{H}] = [H-G,H-G] = [H,H] = H$. Evidently, it follows $\langle \widetilde{H},\widetilde{H}\rangle = \langle H,H\rangle$. $\square$

Proof of Lemma 1.

Considering

$$d\widetilde{\mu}^\delta(t) = \widehat{\lambda}(t)(\widetilde{\mu}^\delta(t) - \mu^\delta(t))dt \tag{A1}$$

and $dX_t = \widehat{\lambda}(t)X_{t-}d(t\wedge\tau)$ if $t<\tau$, for every $t<\tau$ we have

$$\begin{aligned} F(0) + \int_0^t \xi_s^F\, dX_s &= (\widetilde{C}_T B_T + F)(X_0 + \widetilde{\mu}^\delta(0)/B_T) \\ &\quad + (\widetilde{C}_T B_T + F)(X_t - X_0) \\ &\quad + (\widetilde{C}_T B_T + F)\frac{1}{B_T}\int_0^t \widehat{\lambda}(s)(\widetilde{\mu}^\delta(s) - \mu^\delta(s))\, ds \\ &= (\widetilde{C}_T B_T + F)(X_t + \widetilde{\mu}^\delta(t)/B_T)\ . \end{aligned} \tag{A2}$$

Hence it follows

$$\begin{aligned} F(0) + \int_0^t \xi_s^F\, dX_s &= F(0) + \int_{[0,\tau)} \xi_s^F\, dX_s + \xi_\tau^F \Delta X_\tau \\ &= (\widetilde{C}_T B_T + F)(X_{\tau-} + \widetilde{\mu}^\delta(\tau)/B_T) \\ &\quad + (\widetilde{C}_T B_T + F)\left(1 - \frac{\mu^\delta(\tau) - \widetilde{\mu}^\delta(\tau)}{B_T X_{\tau-}}\right)(-X_{\tau-}) \\ &= (\widetilde{C}_T B_T + F)\mu^\delta(\tau)/B_T \quad \text{for} \quad t \geq \tau\ . \end{aligned} \tag{A3}$$

By the definition of L^F and since $\widetilde{\mu}^\delta(T) = 0$ the equations (A2) and (A3) result in

$$F(0) + \int_0^T \xi_t^F\, dX_t + L_T^F = F(T)/B_T$$

Because of $L_0^F = 0$ it follows $\mathrm{E}_P[L_0^F] = 0$. L^F is a square-integrable martingale since

$$\begin{aligned} \mathrm{E}_P[L_s^F|\mathcal{F}_t] &= \mathbb{1}_{\{\tau\leq t\}}(\widetilde{C}_T + F/B_T)(\delta - \mu^\delta(\tau)) + \mathbb{1}_{\{\tau>t\}}\Bigg(0 \\ &\quad + (\widetilde{C}_T + F/B_T)\int_t^s (\mu^\delta(\tau) - \mu^\delta(\tau))\lambda(\tau)\exp\left\{-\int_t^\tau \lambda(s)\, ds\right\}\, d\tau\Bigg) \\ &= H_t(\widetilde{C}_T + F/B_T)(\delta - \mu^\delta(\tau)) = L_t^F\ . \end{aligned}$$

holds for $s > t$ and $\sup_{\tau\in[0,T]} \sigma^Z(\tau) < \infty$. Finally, L^F is orthogonal to M, since for $s > t$ we obtain

$$\begin{aligned}
\mathrm{E}_P[L_s^F M_s|\mathcal{F}_t] &= \mathbb{1}_{\{\tau>t\}} \left(\widetilde{C}_T + \frac{F}{B_T}\right) \int_t^s 0 \cdot M_\tau \lambda(\tau) \exp\left\{-\int_t^\tau \lambda(s)\, ds\right\} d\tau \\
&\quad +\mathbb{1}_{\{\tau\le t\}}(\widetilde{C}_T + F/B_T)(\delta - \mu^\delta(\tau))M_\tau \\
&= L_t^F M_t \ ,
\end{aligned}$$

due to $M_t = M_\tau$ for every $t \geq \tau$. $\square$

Proof of the equations (9) and (10).

Since $V_T < \bar{D}$ is equivalent to $(\widetilde{W}_T - \widetilde{W}_t)/\sqrt{T-t} < -d_2(V_t,r,\bar{D},t) = -d_2$ we have

$$\begin{aligned}
D_t\frac{B_T}{B_t} &= \mathrm{E}_Q[D(T,V_T)|\mathcal{F}_t] = \bar{D}\int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}x^2\right\} dx \\
&\quad +\frac{1-\kappa}{\sqrt{2\pi}} \int_{-\infty}^{-d_2} V_t \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\sqrt{T-t}x\right\} \exp\left\{-\frac{1}{2}x^2\right\} dx \\
&= \bar{D}\,(1-\mathcal{N}(-d_2)) \\
&\quad +\frac{1-\kappa}{\sqrt{2\pi}} \exp\{r(T-t)\} V_t \int_{-\infty}^{-d_2} \exp\left\{-\frac{1}{2}\left(x - \sigma\sqrt{T-t}\right)^2\right\} dx \\
&= \bar{D}\mathcal{N}(d_2) + (1-\kappa)V_t \exp\{r(T-t)\}\mathcal{N}(-d_2 - \sigma\sqrt{T-t}) \\
&= \bar{D}\mathcal{N}(d_2) + (1-\kappa)V_t \exp\{r(T-t)\}\,(1-\mathcal{N}(d_1)) \ .
\end{aligned}$$

The market value of the bankruptcy costs results from equation (10) due to the relation $BC = V - D - E$. $\square$

Proof of Corollary 1.

Assuming certain percentage bankruptcy costs κ the value of a defaultable zero at maturity accounts for

$$\widetilde{B}_T(T) = \begin{cases} 1 & \text{, if } V_T \geq \bar{D} \\ (1-\kappa)V_T/\bar{D} & \text{, if } V_T < \bar{D} \end{cases} \ .$$

Similar calculations as in the proof of equation (9) show that the function G amounts to

$$G(T-t,V_t) = \mathcal{N}(d_2) + (1-\kappa)\frac{V_t}{\bar{D}} \exp\{r(T-t)\}\,(1-\mathcal{N}(d_1)) \ .$$

This results in

$$\begin{aligned}
\frac{\partial G(T-t,V_t)}{\partial V_t} &= \varphi(d_2(V_t,r,\bar{D},t))\frac{\partial d_2(V_t,r,\bar{D},t)}{\partial V_t} \\
&\quad +(1-\kappa)\frac{1}{\bar{D}}\exp\{r(T-t)\}\left(1-\mathcal{N}(d_1(V_t,r,\bar{D},t))\right) \\
&\quad -(1-\kappa)\frac{V_t}{\bar{D}}\exp\{r(T-t)\}\varphi(d_1(V_t,r,\bar{D},t))\frac{\partial d_1(V_t,r,\bar{D},t)}{\partial V_t} \\
&= (1-\kappa)\frac{\exp\{r(T-t)\}}{\bar{D}}(1-\mathcal{N}(d_1))+\kappa\frac{V_t}{\bar{D}}\exp\{r(T-t)\}\varphi(d_1)\frac{\partial d_1}{\partial V_t}\ ,
\end{aligned}$$

since

$$\varphi(d_2)=\varphi(d_1)V_t\exp\{r(T-t)\}/\bar{D}$$

and $\partial d_1/\partial V_t = \partial d_2/\partial V_t = 1/\sigma V_t\sqrt{T-t}$. Now the assertion of corollary 1 follows immediately from proposition 4. □

Proof of Lemma 2.

Since $\xi^{\widetilde{B}}$ are the shares in V of the duplication strategy for a defaultable zero and $\widetilde{B}_0$ describes the arbitrage-free price of the defaultable zero-coupon bond $\widetilde{B}^{\bar{\kappa}}$ at time $t=0$, assuming the percental bankruptcy cost to be surely $\bar{\kappa}$, we have

$$\widetilde{B}_0+\int_0^T \xi_t^{\widetilde{B}}\,d\widetilde{V}_t = \frac{\widetilde{B}_T^{\bar{\kappa}}(T)}{B_T} = \mathbb{1}_{\{V_T\geq\bar{D}\}}\frac{1}{B_T}+\mathbb{1}_{\{V_T<\bar{D}\}}(1-\bar{\kappa})\frac{V_T}{\bar{D}B_T}$$

and therefore

$$\begin{aligned}
\widetilde{B}_0+\int_0^T \xi_t^{\widetilde{B}}\,d\widetilde{V}_t + L_T^{\widetilde{B}} &= \mathbb{1}_{\{V_T\geq\bar{D}\}}\frac{1}{B_T}+\mathbb{1}_{\{V_T<\bar{D}\}}(1-\bar{\kappa})\frac{V_T}{\bar{D}B_T}+\mathbb{1}_{\{V_T<\bar{D}\}}(\bar{\kappa}-\kappa)\frac{V_T}{\bar{D}B_T} \\
&= \widetilde{B}_T(T)/B_T\ .
\end{aligned}$$

Obviously, the process $L^{\widetilde{B}}$ is a martingale due to $\mathrm{E}_P[\kappa|\mathcal{F}_t]=\bar{\kappa}$ for $t<T$. Because of the continuity of the martingale component it follows for the covariation

$$\begin{aligned}
[L^{\widetilde{B}},M]_t &= L_t^{\widetilde{B}}M_t-\int_0^t M_{s-}\,dL_s^{\widetilde{B}}-\int_0^t L_{s-}^{\widetilde{B}}\,dM_s \\
&= \mathbb{1}_{\{t=T\}}\left(L_T^{\widetilde{B}}M_T-M_{T-}L_T^{\widetilde{B}}\right)=0
\end{aligned}$$

and also $\langle L^{\widetilde{B}},M\rangle=0$. Therefore $L^{\widetilde{B}}$ is orthogonal to the martingale component M.

□

References

Bakshi, Gurdip, Dilip Madan and Frank Zhang, 2006, Understanding the Role of Recovery in Default Risk Models: Empirical Comparisons and Implied Recovery Rates, *Working Paper*.

Bielecki, Tomasz R. and Marek Rutkowski, 2002, *Credit Risk: Modeling, Valuation and Hedging*, Springer, Berlin.

Brémaud, Pierre, 1981, *Point Processes and Queues: Martingale Dynamics*, Springer, New-York.

Duffie, Darrell and Kenneth J. Singleton, 2003, *Credit Risk: Pricing, Measurement and Management*, Princeton University Press, Princeton.

Elliott, Robert J. and P. Ekkehard Kopp, 1999, *Mathematics of Financial Markets*, Springer, New York.

Föllmer, Hans and Martin Schweizer, 1989, Hedging by Sequential Regression: An Introduction to the Mathematics of Option Trading, *The ASTIN Bulletin* 18, 147–160.

Föllmer, Hans and Dieter Sondermann, 1986, Hedging of Non-Redundant Contingent Claims, In: Hildenbrand, W. and A. Mas-Colell, Hrsg., Contributions to Mathematical Economics, 205–223, North-Holland, Amsterdam.

Grünewald, Barbara and Siegfried Trautmann, 1996, Option Hedging in the Presence of Jump Risk, *Working Paper*, University of Mainz, URL: http://www.finance.uni-mainz.de/192.php (28.02.2008).

Jacod, Jean and Albert N. Shiryaev, 1987, *Limit Theorems for Stochastic Processes*, Springer, New-York.

Jarrow, Robert A., David Lando and Stuart M. Turnbull, 1997, A Markov Model for the Term Structure of Credit Risk Spreads, *Review of Financial Studies* 10, 481–523.

Jarrow, Robert A. and Stuart M. Turnbull, 1995, Pricing Derivatives on Financial Securities Subject to Credit Risk, *Journal of Finance* 50, 53–85.

Lando, David, 2004, *Credit Risk Modeling: Theory and Applications*, Princeton University Press, Princeton.

Leland, Hayne E., 1994, Corporate Debt Value, Bond Covenants, and Optimal Capital Structure, *Journal of Finance* 49, 1213–1252.

Leland, Hayne E. and Klaus Bjerre Toft, 1996, Optimal Capital Structure, Endogenous Bankruptcy, and the Term Structure of Credit Spreads, *Journal of Finance* 51, 987–1019.

Merton, Robert C., 1974, On the Pricing of Corporate Debt: The Risk Structure of Interest Rates, *Journal of Finance* 29, 449–470.

Protter, Philip, 1990, *Stochastic Integration and Differential Equations: A New Approach*, Springer, New-York.

Schweizer, Martin, 1990, Risk-Minimality and Orthogonality of Martingales, *Stochastics and Stochastics Reports* 30, 123–131.

Schweizer, Martin, 1991, Option Hedging for Semimartingales, *Stochastic Processes and their Applications* 37, 339–363.

Schweizer, Martin, 1999, A Minimality Property of the Minimal Martingale Measure, *Statistics and Probability Letters* 42, 27–31.