Modelling credit dynamics – a tractable first-passage time model with jumps

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The payoff of many credit derivatives is subject to spread risk, i.e., it depends on the evolution of credit spreads through time. To motivate our analysis we consider the leveraged credit-linked note, which is particularly sensitive to jumps in credit spreads. In the framework of first-passage time models we define a credit quality process with stochastic volatility. Using a representation of the credit quality process as a time-changed Brownian motion, we derive a formula for the dynamics of default probabilities, which in turn provides the dynamics of credit spreads. As an example for a volatility process we consider the square root of a Lévy-driven Ornstein-Uhlenbeck process and we show that jumps in the volatility translate into jumps in credit spreads. We show that the dynamics can be computed efficiently, and we demonstrate that the model calibrates well to a given term structure of credit spreads.

Keywords: CDS, credit spreads, credit dynamics, first-passage time models, stochastic volatility, Lévy processes, general Ornstein-Uhlenbeck processes

JEL classification: G12, G13, G24, C69

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1. Introduction

The market for credit derivatives has grown considerably over the past few years. Statistics published by the Bank of International Settlements (BIS) reveal that the outstanding notional of credit derivatives has grown from USD 14 trillion in December 2005 to over USD 42 trillion in June 2007.

The plain vanilla credit derivative is the credit default swap (CDS), an instrument that provides insurance against the default of a borrower, e.g. a company. According to the British Bankers' Association, CDS make up for about a third of the credit derivatives market. Other single-name credit

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derivatives include credit spread options, credit-linked notes and credit index trades. According to BIS statistics, the outstanding notional in single-name instruments was USD 24 trillion in June 2007. Multi-name credit derivatives provide insurance against defaults in a portfolio of companies. Popular multi-name credit derivatives are basket products and collateralized debt obligations (CDOs) (outstanding notional according to BIS statistics: USD 18 trillion in June 2007).

Other than being subject exclusively to default risk, the payoff of some credit derivatives is determined explicitly by the level of CDS spreads. In this case the dynamics of CDS spreads play a significant role in product valuation. Examples of such so-called spread products are credit spread options and particular types of credit-linked notes. We focus on single-name credit derivatives, and to motivate our work, we present the leveraged credit-linked note as a credit derivative that is sensitive to both default risk and spread risk, and we discuss the impact of jumps in CDS spreads on product valuation.

There are generally two approaches to modelling credit risk: the structural and the reduced-form approach. In the former, it is assumed that observable economic variables, such as the value of the firm under consideration, trigger default. Typically the dynamics of a firm value are modelled with default occurring when the firm value falls below a certain threshold. This approach was introduced by [Merton, 1974], and further developed by [Black and Cox, 1976], [Longstaff and Schwartz, 1995], and many others. Often the firm value and the threshold are assumed to be continuous processes, in which case credit spreads vanish as maturity tends to 0, contradicting empirical observation. This can be overcome by introducing jumps in the model as by [Merton, 1976].

In a more general setting, we consider a model to be of structural type when default is the first hitting time of a certain threshold by an abstract observable *credit quality process*, where we understand “observable” as the property of being adapted to the underlying flow of information.

In the reduced-form approach, the default event is not directly linked to economic observables, but it is an unpredictable Poisson event. This approach is followed by e.g. [Lando, 1998], [Duffie and Singleton, 1999], [Duffie and Lando, 2001] and many others. Its main advantages lie in the capability of reproducing a given credit spread term structure well and in its close analogy to interest-rate term structure modelling. The link between the economic environment of a firm and its default is often not clear in this type of model. For a detailed exposition on structural and reduced-form models, we refer to [Bielecki and Rutkowski, 2002].

[Zhou, 2001] proposes a structural model that incorporates desirable features of the reduced-form approach. Here, a firm value process is modelled as a jump-diffusion, thereby including economic variables in the model and at the same time allowing for unpredictable default events. Some recent papers consider the dynamics of the term structure of CDS spreads in a structural approach, amongst them [Baxter, 2007] and [Cariboni and Schoutens, 2007] who propose models based on Lévy processes in order to yield satisfactory calibration results, [Hull and White, 2007] who model the default probability of an entity as a stochastic process in a binomial tree representation, and [Kiesel and Scherer, 2007] who present a generalization of the model by [Zhou, 2001].

We extend the class of existing models for CDS spread dynamics by a model that is mathematically tractable and at the same time allows for meaningful dynamics of CDS spreads. In particular, the model includes jumps in the evolution of CDS spreads, which allows for calibration to a wide range of term structures and for valuation of spread products whose payoff is particularly sensitive to jumps. Although the spread dynamics exhibit jumps, we are able to formulate our model in a way that allows to draw on results from the theory of continuous stochastic processes. The model can be implemented very efficiently.

Our basic idea follows [Overbeck and Schmidt, 2005] who propose a structural model for valuing credit derivatives whose payoff is sensitive to default risk. Here, a credit quality process is a continuous stochastic process with deterministic time-varying volatility. The model calibrates analytically to any given term structure of credit spreads. In our model, the volatility of the credit quality process is a stochastic process, and it turns out that this is the key to providing meaningful dynamics of CDS spreads.

The paper is structured as follows: In Section 2 we introduce credit default swaps (CDS) and we present the leveraged credit-linked note as a credit derivative that is sensitive to CDS spread dynamics, in particular to jumps in CDS spreads. We summarise the Overbeck-Schmidt model (OS-model) and we examine its dynamics in Section 4. In Section 5, we define the credit quality process with stochastic volatility, we derive a formula for default probabilities conditional on the information flow, we give a concrete example with the volatility process the square root of an Ornstein-Uhlenbeck process driven by a compound Poisson process, and we examine the relationship between jumps in the Ornstein-Uhlenbeck process and jumps in default probabilities and CDS spreads, respectively. We present an algorithm for efficient computation conditional default probabilities of term structures of default probabilities and CDS spreads in Section 6. As an example we consider valuation of the leveraged credit-linked note in Section 7. We discuss calibration to a given term structure of default probabilities in Section 8 and we discuss the impact of the parameters involved on the dynamics.

2. Credit dynamics and spread risk

The fundamental product of the credit derivatives market is the *credit default swap (CDS)*. Given an underlying entity, such as a company, it is a contract between two counterparties, the protection buyer and the protection seller, that insures the protection buyer against the default event (i.e., failure to fulfill a financial obligation) of the underlying entity in a fixed time interval. The protection buyer regularly pays a constant premium, the *credit spread (or CDS spread)*, that is fixed at inception, up until maturity of the CDS or the default event, whichever occurs first. This stream of payments is termed the *premium leg* of the CDS. In return, the protection seller agrees to compensate the protection buyer for the loss incurred by default of the underlying entity at the time of default in case this occurs before maturity. This constitutes the *protection leg* of the CDS.

Following the principle of no-arbitrage, we consider the value of a financial claim to be the discounted expectation of its payoff under a risk-neutral measure. The *fair CDS spread* is the CDS spread that makes the value of both legs equal. We only consider fair CDS spreads in this paper, hence we just speak of CDS spreads. The mapping of CDS spreads with respect to their maturity is called the *term structure of CDS spreads*. In general, we shall assume market-given CDS spreads to be fair spreads and use the no-arbitrage principle to derive risk-neutral default probabilities; this relationship is made precise in Section 3. CDS for large firms and sovereigns are liquidly traded, and typically CDS spreads for maturities 1, 3, 5, 7, 10 years are quoted in the market. The mark-to-market value of an existing CDS position is the cost of unwinding the transaction by entering into an offsetting CDS position. At default, the mark-to-market value is just the loss incurred by the default event.

The term structure of credit spreads has been thoroughly studied, both theoretically (cf. e.g. [Zhou, 2001]) and empirically (cf. e.g. [Collin-Dufresne et al., 2001]). The fact that CDS spreads do not vanish when time-to-maturity tends to zero indicates that the market assumes that an entity may default unexpectedly and instantaneously at any time. In general, two components of credit risk contribute to the behaviour of CDS spreads: jump-to-default risk and the risk of credit quality changes over time, with the short end of the term structure dominated by jump-to-default risk. [Schneider et al., 2007] observe that CDS spreads exhibit frequent positive jumps in their movement through time. These jumps are attributed to the arrival of bad news, and typically they affect CDS spreads of all maturities. On the other hand, good news tend to propagate gradually.

Credit derivatives whose payoff depends not only on default risk but also on the spread levels are called *spread products*. Examples are options on CDS and particular types of *credit-linked notes (CLN)*. For a general description of CLNs we refer to [Bielecki and Rutkowski, 2002, Section 1.3.3].

As an example of a spread product, consider the *leveraged credit-linked note*. This note is particularly sensitive to jumps in CDS spreads, even if a jump does not lead to default. The principal idea is that an investor sells protection on an amount of default risk that is a multiple k, the *leverage factor*, of his investment amount. The motivation for taking leveraged exposure is to earn a certain

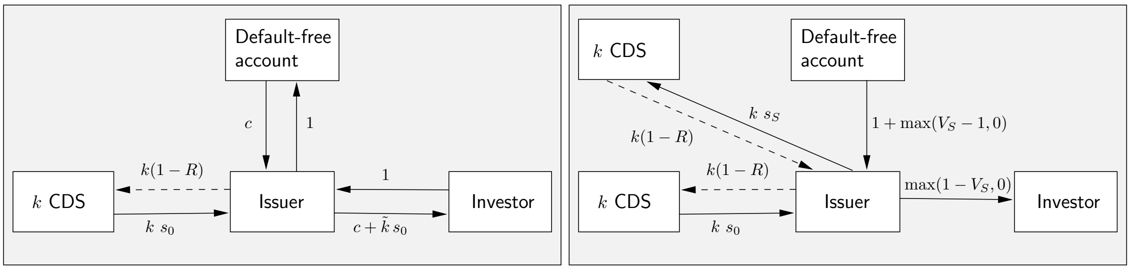

Fig. 1. Leveraged credit-linked note with leverage factor k and notional 1. Left: Cash-flows at inception and while the note is alive. Right: Cash-flows at trigger time S.

multiple $\tilde{k}$ of the credit spread. Most likely, his investment will not suffice to compensate the loss incurred by default. Therefore, a trigger is agreed to terminate the structure while the cost of closing the position is still likely to be sufficiently covered by the investment amount. The cost of closing the position depends on the level of credit spreads, hence the investor is exposed mainly to spread risk and to default risk only to a lesser extent.

In more detail, the issuer structures the note as follows (see Figure 1): For simplicity, assume constant default-free interest rates c and an investment amount of 1, which is deposited in a default-free account earning a coupon c. In addition, protection is sold by entering a fair CDS with notional k earning a spread of $k\,s_0$. The investor receives a fixed coupon until either maturity of the note or until a trigger event takes place. The size of the coupon is $c + \tilde{k}\,s_0$ with $\tilde{k}\,s_0$, $\tilde{k} \leq k$, the premium associated with the note. This premium is financed by the CDS position. The trigger event is determined as follows: denote by V_t^k the mark-to-market value at time t of the underlying CDS with notional k from the point of view of the CDS protection buyer. The trigger event takes place at time $S = \inf\{t \in (0,T] : V_t^k > K\}$, with $K \leq 1$ a pre-defined trigger level. At S, the note is unwound by withdrawing the investment amount 1 from the deposit account and by closing the CDS position at a cost of V_S^k. Observe that possibly $V_S^k > 1$, in which case the issuer must cover the missing amount required to unwind the CDS position. For this type of risk, called *gap risk*, the issuer is compensated with a premium of $(k - \tilde{k})\,s_0$. In the case where $V_S^k \leq 1$, the investor receives the remainder of the structure, $1 - V_S^k$. Given K, valuation of the note essentially means determining the fair factor $\tilde{k}$.

Clearly, the trigger time S depends on the evolution of the underlying CDS spread. Furthermore, the amount of the redemption payment $\max(1 - V_S^k, 0)$ is undetermined until S. Assuming a model in which CDS spreads evolve continuously, the mark-to-market value V evolves continuously as well, and the trigger time is $S = \inf\{t \in (0,T] : V_t^k = K\}$, unless a default takes place. Hence $V_S^k \leq 1$, and gap risk is limited to the default case. On the contrary, upward jumps in CDS spreads translate into upward jumps in the mark-to-market value of the CDS, and possibly $V_S^k > 1$ so the issuer faces gap risk, even when no default takes place.

We will consider valuation of the leveraged CLN in Section 7.

3. Notation and preliminaries

Throughout, assume given a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ with right-continuous filtration, and with $\mathbf{P}$ a risk-neutral martingale measure. We assume that the probability space is rich enough to support any random elements that we define. If not otherwise stated, all processes are assumed to be $(\mathcal{F}_t)_{t\geq 0}$-adapted. The filtration generated by a process X, and augmented if necessary, is denoted by $(\mathcal{F}_t^X)_{t\geq 0}$. An $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion W is a Brownian motion W that is $(\mathcal{F}_t)_{t\geq 0}$-adapted and such that for any $s \leq t$, $W_t - W_s$ is independent of $\mathcal{F}_s$. We omit explicit reference to the underlying entity when we speak of a default event, default time, etc.. Denote by τ the random time of the default event. The distribution function of τ conditional on the information flow $(\mathcal{F}_t)_{t\geq 0}$ is denoted by $P(t,T) := \mathbf{P}(\tau \leq T | \mathcal{F}_t)$. For fixed maturity T, $(P(t,T))_{t\leq T}$ is a process whose dynamics are determined by the information flow $(\mathcal{F}_t)_{t\geq 0}$. Observe that for each T the process

$(P(t,T))_{t\le T}$ is a martingale with a càdlàg modification, cf. [Karatzas and Shreve, 1998, Theorem 1.3.13], and we shall choose this modification whenever it is necessary. Furthermore, $\mathbf{P}(\cdot|\mathcal{F}_t)$ has a regular version, so we may treat $\mathbf{P}(\cdot|\mathcal{F}_t)(\omega)$ like a probability measure for $\mathbf{P}$-almost all $\omega\in\Omega$.

Let $s(t,T)$ be the fair spread at time t of a CDS with maturity T. Then $(s(t,T))_{t\le T}$ is the spread process with maturity T and $(s(t,T+t))_{t\ge 0}$ is the spread process with time to maturity t.

From conditional default probabilities, $\mathbf{P}(\tau\le u|\mathcal{F}_t)$, $t<u\le T$, we derive the fair CDS spread $s(t,T)$ as follows: Let $r=(r_t)_{t\ge 0}$ be the short rate process, and for any $0\le t\le T$, let $B(t,T)=\mathbb{E}\left(\mathbf{e}^{-\int_t^T r_s\,\mathrm{d}s}|\mathcal{F}_t\right)$ the time-t price of a default-free zero coupon bond maturing at T. We shall assume that for any $s\ge 0$ and $t\ge s$, $(r_u)_{u\ge s}$ and $\mathbf{1}_{\{\tau>t\}}$ are conditionally independent given $\mathcal{F}_s$. On a CDS with maturity T started at t, the protection buyer continuously pays the premium $s(t,T)$ until default or maturity, whichever occurs first. Taking risk-neutral expectation with respect to $\mathcal{F}_t$, the value of the premium leg at time t is given by

$$\begin{aligned}\mathbb{E}\left(s(t,T)\int_t^T \mathbf{e}^{-\int_t^u r_v\,\mathrm{d}v}\mathbf{1}_{\{\tau>u\}}\,\mathrm{d}u\middle|\mathcal{F}_t\right) &= s(t,T)\int_t^T \mathbb{E}\left(\mathbf{e}^{-\int_t^u r_v\,\mathrm{d}v}\,\mathbf{1}_{\{\tau>u\}}|\mathcal{F}_t\right)\mathrm{d}u\\ &= s(t,T)\int_t^T B(t,u)\,\mathbf{P}(\tau>u|\mathcal{F}_t)\,\mathrm{d}u.\end{aligned}$$

The protection seller pays the fractional amount $(1-R)$ of the notional at default, with R the recovery rate, assumed to be known and constant. The corresponding value of the protection leg is

$$\mathbb{E}\left((1-R)\,\mathbf{e}^{-\int_t^\tau r_v\,\mathrm{d}v}\mathbf{1}_{\{t<\tau\le T\}}|\mathcal{F}_t\right)=(1-R)\int_t^T B(t,u)\,\mathbf{P}(\tau\in\mathrm{d}u|\mathcal{F}_t).$$

Solving for the fair CDS spread so that the value of both legs is equal yields

$$s(t,T):=\frac{(1-R)\int_t^T B(t,u)\,\mathbf{P}(\tau\in\mathrm{d}u|\mathcal{F}_t)}{\int_t^T B(t,u)\,\mathbf{P}(\tau>u|\mathcal{F}_t)\,\mathrm{d}u},\tag{1}$$

on $\{\tau>t\}$. Otherwise the spread $s(t,T)$ does not exist at t.

Clearly, the information generated by $N_t:=\mathbf{1}_{\{\tau\le t\}}$ does not suffice to model the dynamics of default probabilities and credit spreads. Consequently, the filtration $(\mathcal{F}_t)_{t\ge 0}$ will, in addition to $(\mathcal{F}_t^N)_{t\ge 0}$, contain the information flow generated by some driving state variables. The specification of such state variables will be our occupation in the following Sections.

We state the following well-known result without proof.

Lemma 1. *Let $\mathcal{G}\subset\mathcal{F}$ be a σ-algebra and let X, Y be random elements with values in the Borel space $(E,\mathscr{E})$ and such that X is $\mathcal{G}$-measurable and Y is independent of $\mathcal{G}$. Let $h:E\times E\to\mathbb{R}$ be a bounded $\mathscr{E}\times\mathscr{E}$-measurable function. Define $g(x):=\mathbb{E}h(x,Y)$. Then $g(X)$ is a version of the conditional expectation $\mathbb{E}(h(X,Y)|\mathcal{G})$.*

Finally, recall some standard notation: For a càdlàg process Y, we write $Y_{t-}:=\lim_{s\uparrow t}Y_s$ and $\Delta Y_t:=Y_t-Y_{t-}$. We say that Y has a *jump at t* if $\Delta Y_t\neq 0$. A stochastic process Y is *stochastically continuous* or *continuous in probability* if, for every $t\ge 0$ and $\varepsilon>0$, $\lim_{s\to t}\mathbf{P}(|X_s-X_t|>\varepsilon)=0$.

4. Overbeck-Schmidt model

[Overbeck and Schmidt, 2005] present a model that allows for straightforward analytic calibration to a given term structure of default probabilities, $F(t):=\mathbf{P}(\tau\le t)$, $t\ge 0$, where τ denotes the time of default of an underlying entity. In the Overbeck-Schmidt model, τ is determined as the first time that a *credit quality process* $X=(X_t)_{t\ge 0}$ hits a barrier $b<X_0$, i.e., $\tau=\inf\{t\ge 0: X_t\le b\}$. The principal idea is to model X as a time-changed Brownian motion. Given a Brownian motion W and a deterministic, strictly increasing and continuous time transformation $T=(T_t)_{t\ge 0}$, $T:[0,\infty)\to[0,\infty)$, with $T_0=0$, set

$$X_t:=W_{T_t},\quad t\ge 0.$$

[Overbeck and Schmidt, 2005] show that if the time change T is given by

$$T_t = \left(\frac{b}{\mathrm{N}^{(-1)}\left(\frac{F(t)}{2}\right)} \right)^2, \quad t \geq 0, \tag{2}$$

where $\mathrm{N}^{(-1)}$ denotes the inverse of the Normal distribution function, then τ admits the distribution $F(t)$, $t \geq 0$. Furthermore, if the distribution of τ admits a density, then the time change T is absolutely continuous, so that

$$T_t = \int_0^t \sigma_s^2 \, \mathrm{d}s, \tag{3}$$

with $\sigma : [0, \infty) \to [0, \infty)$ a square-integrable function. The volatility σ can be interpreted as the *default speed*, in the sense that the higher default speed the higher the likelihood of crossing the default barrier. The quadratic variation $[X, X]$ of X is just $[X, X] = T$, so that there exists a representation of X as a stochastic integral

$$X_t = \int_0^t \sigma_s \, \mathrm{d}\tilde{W}_s, \tag{4}$$

for some Brownian motion $\tilde{W}$.

Although the OS-model is not intended to value spread products, it exhibits dynamics by specification of the process X. These dynamics are fully determined by calibration to market-given default probabilities, and it is not possible to assign different dynamics to the same term structure of default probabilities. The following Proposition allows us to analyse the dynamics of the OS-model in more detail.

Proposition 2. *Let $X = (X_t)_{t \geq 0}$, with $X_t = W_{T_t}$ and $(T_t)_{t \geq 0}$ given by Equation (2), and let $\tau := \inf\{t \geq 0 : X_t \leq b\}$, $b < X_0$.*

(i) Then, on $\{\tau > s\}$, the probability of default until t conditional on $\mathcal{F}_s$, $P(s, t)$, is given by

$$P(s, t) = 2\mathbb{E}\left(\mathrm{N}\left(\frac{b - X_s}{\sqrt{T_t - T_s}} \right) \Big| X_s \right). \tag{5}$$

(ii) Assume further that the time-change T admits a representation as in Equation (3). Then, For any $t \geq 0$, the conditional default probability process $(P(s, t))_{s \leq t}$ is continuous in s.

Proof.

(i) Observe that the events $\{\tau \leq t\}$ and $\{\min_{u \leq t} X_u \leq b\}$ are equivalent. It is well-known that the hitting-time distribution of a Brownian motion starting at 0 is, cf. [Karatzas and Shreve, 1998, Section 2.6.A],

$$\mathbf{P}\left(\min_{s \leq t} W_s < b \right) = 2\mathrm{N}(b/\sqrt{t}), \quad b < 0. \tag{6}$$

We obtain, on $\{\tau > s\}$,

$$\begin{aligned} P(s, t) &= \mathbf{P}(s < \tau \leq t | \mathcal{F}_s) = \mathbf{P}\left(\min_{s < u \leq t} X_u \leq b \big| \mathcal{F}_s \right) \\ &= \mathbf{P}\left(\min_{s < u \leq t} W_{T_u} - W_{T_s} \leq b - W_{T_s} \big| \mathcal{F}_s \right) = 2\mathbb{E}\left(\mathrm{N}\left(\frac{b - W_{T_s}}{\sqrt{T_t - T_s}} \Big| \mathcal{F}_s \right) \right), \end{aligned}$$

where the last step is an application of Lemma 1, since $(W_{T_u} - W_{T_s})_{u \geq s}$ is a Brownian motion independent of $\mathcal{F}_s$ and $b - W_{T_s}$ is $\mathcal{F}_s$-measurable. Finally, by the Markov property of Brownian motion me may condition under $\sigma(X_s)$ instead of $\mathcal{F}_s$.

(ii) Taking into account that $\mathrm{N}(\cdot)$, X, $\sqrt{\cdot}$ and T are continuous, for any sequence $s_n \to s$, as $n \to \infty$,

$$\lim_{s_n \to s} P(s_n, t) = \lim_{s_n \to s} \mathbb{E}\left(2\mathrm{N}\left(\frac{b - X_{s_n}}{\sqrt{T_t - T_{s_n}}}\right) \Big| \mathcal{F}_{s_n}\right) = \lim_{s_n \to s} 2\mathrm{N}\left(\frac{b - X_{s_n}}{\sqrt{T_t - T_{s_n}}}\right)$$
$$= \mathrm{N}\left(\frac{b - X_s}{\sqrt{T_t - T_s}}\right) = P(s,t). \qquad \square$$

Inspection of Equation (5) reveals that $P(s,t)$ depends merely on the distance-to-default of X_s. The time-change and consequently the default speed, being deterministic, have no impact on the dynamics of $(P(s,t))_{s \le t}$. The second part of the Proposition tells us that it is impossible to generate jumps in a default probability process when the time-change is continuous.[a] We will establish in Section 5.4 that this implies continuity of credit spread processes. Recall that the short end of the credit term structure is governed by jump-to-default risk. The absence of jumps forces jump-to-default risk to be compensated by the credit quality risk, and consequently calibration to a given spread term structure leads to poor dynamics.

An important consequence of Proposition 2 is that in the setting of the OS-model a necessary condition for random jumps in default probability processes is that the time-change be stochastic.

5. A hitting-time model with stochastic volatility

Can we extend the Overbeck-Schmidt model to allow for better and meaningful dynamics of default probabilities and CDS spreads? Naturally, we would like to retain the tractability of the OS-model as much as possible. If X is a credit quality process as in Equation (4), then by Proposition 2 $P(t,T) = F(t,T,X_t)$ for some function F by the Markov property of X. This shows that $(P(t,T))_{t \le T}$ is driven by the credit quality process X. Can we enrich the dynamics in the sense that $P(t,T) = F(t,T,X_t,Y_t)$, i.e., $(P(t,T))_{t \le T}$ is driven by Markov process (X,Y) with X the credit quality process and Y some other process? Furthermore, we have seen that jumps play an important role in aligning market data with meaningful dynamics. Can we incorporate jumps in default probabilities and credit spreads?

We proceed as follows: The credit quality process X will be defined as a stochastic integral with respect to a Brownian motion with a stochastic volatility σ (Section 5.1). We then derive a formula for conditional default probabilities (Section 5.2). In this framework, we specify σ^2 as a Generalised Ornstein-Uhlenbeck process, i.e., an Ornstein-Uhlenbeck process driven by a Lévy process. In our setting, the driving Lévy process will be a compound Poisson process (Section 5.3). We show that, although X is a continuous process, default probabilities and credit spreads exhibit jumps triggered by jumps in the volatility σ (Section 5.4).

5.1. *Model setup*

Definition 3. *The* credit quality process $X = (X_t)_{t \ge 0}$ *of a risky entity is defined to be*

$$X_t = \int_0^t \sigma_s \, \mathrm{d}W_s, \quad t \ge 0,$$

where W is an $(\mathcal{F}_t)_{t \ge 0}$-Brownian motion and σ is a strictly positive $(\mathcal{F}_t)_{t \ge 0}$-adapted càdlàg process independent of W with $\mathbf{P}(\int_0^t \sigma_s^2 \, \mathrm{d}s < \infty) = 1$ and $\lim_{t \to \infty} \int_0^t \sigma_s^2 \, \mathrm{d}s = \infty$ $\boldsymbol{P}$–a.s.[b].

To emphasise the association with σ, we may speak of X as a *credit quality process with volatility* σ. Denote the quadratic variation process of X by $\Lambda = (\Lambda_t)_{t \ge 0}$, with $\Lambda_t = \int_0^t \sigma_s^2 \, \mathrm{d}s$. Observe that Λ is continuous, strictly increasing and $(\mathcal{F}_t)_{t \ge 0}$-adapted.

[a]It is straight-forward to see that for time-changes with discontinuities any jumps in a default probability process are deterministic.

[b]The requirement $\lim_{t \to \infty} \int_0^t \sigma_s^2 \, \mathrm{d}s = \infty$ $\mathbf{P}$–a.s. will ensure that $\tau < \infty$ $\mathbf{P}$–a.s..

Define the family of $(\mathcal{F}_t)_{t\geq 0}$-stopping times $\tau_t = \inf\{s \geq 0 : \Lambda_s > t\}$, $t \geq 0$. By application of the Theorem of Dambis, Dubins-Schwarz, cf. [Karatzas and Shreve, 1998, Theorem 3.4.6], the process B, with $B_t = X_{\tau_t}$, $t \geq 0$, is an $(\mathcal{F}_{\tau_t})$-Brownian motion. Conversely, given B, X can be expressed as a time-changed Brownian motion, i.e., $X_t = B_{\Lambda_t}$, $t \geq 0$. We shall refer to B as the DDS-Brownian motion of X.

The default time τ of the risky entity associated with the credit quality process X is the first time that X hits a barrier $b < 0$:

$$\tau = \inf\{t \geq 0 : X_t \leq b\}.$$

In this model setup, we would like to compute the probability distribution of τ conditional on the evolution of X and σ. We state some preliminary results first.

Proposition 4. *Let X be a credit quality process with volatility process σ, and let B be the DDS-Brownian motion of X. Then, B is an $(\mathcal{F}_{\tau_t})_{t\geq 0}$-Brownian motion independent of σ.*

Proof. Denote by $(C, \mathcal{B}(C))$ the measurable space of continuous functions $f : \mathbb{R}_+ \to \mathbb{R}$ with $\mathcal{B}(C)$ the Borel sets of C and by $(D, \mathcal{B}(D))$ the measurable space of càdlàg functions $f : \mathbb{R}_+ \to \mathbb{R}$, with $\mathcal{B}(D)$ the Borel sets of D, respectively. For every $\Gamma \in \mathcal{B}(C)$, $\Delta \in \mathcal{B}(D)$, we check that

$$\mathbf{P}(B \in \Gamma, \sigma \in \Delta) = \mathbf{P}(B \in \Gamma)\,\mathbf{P}(\sigma \in \Delta).$$

It is straightforward that this holds for sets Δ with $\mathbf{P}(\sigma \in \Delta) \in \{0, 1\}$. Choose Δ such that $\mathbf{P}(\sigma \in \Delta) \in (0, 1)$, and denote by $\mathscr{D}$ the σ-algebra generated by $\{\sigma \in \Delta\}$. Using properties of conditional expectation, we obtain

$$\mathbf{P}(B \in \Gamma, \sigma \in \Delta) = \mathbb{E}\left(\mathbf{1}_{\{\sigma\in\Delta\}}\,\mathbf{P}(B \in \Gamma|\mathscr{D})\right). \tag{7}$$

Writing $D_1 = \{\sigma \in \Delta\}$ and $D_2 = \{\sigma \notin \Delta\}$, and since $0 < \mathbf{P}(D_1), \mathbf{P}(D_2) < 1$, it is easy to check that a version of the conditional probability of $A \in \mathcal{F}$ with respect to $\mathscr{D}$ is

$$\mathbf{P}(A|\mathscr{D})(\omega) = \sum_{i=1,2} \mathbf{P}(A \cap D_i)/\mathbf{P}(D_i)\,\mathbf{1}_{\{D_i\}}(\omega), \quad \omega \in \Omega.$$

Fix this version of the conditional probability. For every $\omega \in \Omega$, $\mathbf{P}(\cdot|\mathscr{D})(\omega)$ is a probability measure (and thus it is a variant of the regular conditional probability with respect to $\mathscr{D}$). Furthermore, $\mathbf{P}(\cdot|\mathscr{D})(\omega)$ is absolutely continuous with respect to $\mathbf{P}$, i.e., $\mathbf{P}(\cdot|\mathscr{D})(\omega) \ll \mathbf{P}$.

In order to exploit properties of $\int_0^\cdot \sigma_s \,\mathrm{d}W_s$ under $\mathbf{P}(\cdot|\mathscr{D})(\omega)$, we establish that X and $\int_0^\cdot \sigma_s \,\mathrm{d}W_s$ (under $\mathbf{P}(\cdot|\mathscr{D})(\omega)$) are $\mathbf{P}(\cdot|\mathscr{D})(\omega)$-indistinguishable. By independence of W and $\mathscr{D}$ it follows that W is still a Brownian motion under $\mathbf{P}(\cdot|\mathscr{D})(\omega)$. Let $(\sigma^n)_{n\geq 0}$ be a sequence of simple adapted processes such that

$$\mathbf{P}\left(\lim_{n\to\infty} \int_0^t \sigma_s^{(n)} \,\mathrm{d}W_s = \int_0^t \sigma_s \,\mathrm{d}W_s, 0 \leq t < \infty\right) = 1.$$

The existence of such a sequence follows from e.g. [Karatzas and Shreve, 1998, Section 3.2]. From $\mathbf{P}(\cdot|\mathscr{D})(\omega) \ll \mathbf{P}$, it follows that this holds $\mathbf{P}(\cdot|\mathscr{D})(\omega)$-a.s. for the same sequence $(\sigma^{(n)})_{n\geq 0}$, and hence the stochastic integral $\int_0^\cdot \sigma_r \,\mathrm{d}W_r$ under $\mathbf{P}$ is $\mathbf{P}(\cdot|\mathscr{D})(\omega)$-indistinguishable of the stochastic integral $\int_0^\cdot \sigma_r \,\mathrm{d}W_r$ under $\mathbf{P}(\cdot|\mathscr{D})(\omega)$.

Since W is a Brownian motion under $\mathbf{P}(\cdot|\mathscr{D})(\omega)$ it follows that X is a continuous local martingale under $\mathbf{P}(\cdot|\mathscr{D})(\omega)$. The quadratic variation, as a limit in probability, is invariant to absolutely continuous changes of measure. It follows that B is an $(\mathcal{F}_{\tau_t})$-Brownian motion under $\mathbf{P}(\cdot|\mathscr{D})(\omega)$, or in other words, $\mathbf{P}(B \in \cdot|\mathscr{D})(\omega) = \mathbf{P}_W(\cdot)$, where $\mathbf{P}_W$ is the Wiener measure on $(C, \mathcal{B}(C))$. Inserting into Equation (7) yields

$$\mathbb{E}(\mathbf{1}_{\{\sigma\in\Delta\}}\mathbf{P}(B \in \Gamma|\mathscr{D})) = \mathbb{E}(\mathbf{1}_{\{\sigma\in\Delta\}}\mathbf{P}_W(\Gamma)) = \mathbb{E}(\mathbf{1}_{\{\sigma\in\Delta\}}\mathbf{P}(B \in \Gamma)) = \mathbf{P}(B \in \Gamma)\,\mathbf{P}(\sigma \in \Delta). \quad \square$$

Corollary 5. *Let X be a credit quality process with volatility process σ, and let B be the DDS-Brownian motion of X. Furthermore, let S be a $\boldsymbol{P}$–a.s. finite $(\mathcal{F}_{\tau_t})$-stopping time, and define $\tilde{B} = (\tilde{B}_u)_{u\geq 0}$, with $\tilde{B}_u := B_{S+u} - B_S$. Then $\tilde{B}$ is an $(\mathcal{F}_t^{\tilde{B}})_{t\geq 0}$-Brownian motion independent of σ.*

Proof. By the properties of Brownian motion, $\tilde{B}$ is a Brownian motion independent of $\mathcal{F}_{\tau_S}$. In the notation of the previous proof, since B is a Brownian motion under $\mathbf{P}(\cdot|\mathscr{D})(\omega) = \mathbf{P}_W(\cdot)$, for $\mathbf{P}$-almost all $\omega \in \Omega$, so is $\tilde{B}$, and the claim follows. $\square$

Corollary 6. *Let X be a credit quality process with volatility process σ, and let B be the DDS-Brownian motion of X. Let S be a $\boldsymbol{P}$–a.s. finite $(\mathcal{F}_{\tau_t})_{t\geq 0}$-stopping time. Denote by $\mathcal{F}_\infty^\sigma$ the σ-algebra generated by σ. Then, $\tilde{B} = (\tilde{B}_u)_{u\geq 0}$, with $\tilde{B}_u := B_{S+u} - B_S$ is an $(\mathcal{F}_t^{\tilde{B}})_{t\geq 0}$-Brownian motion independent of $\mathcal{F}_{\tau_S} \vee \mathcal{F}_\infty^\sigma$, the smallest σ-algebra containing $\mathcal{F}_{\tau_S}$ and $\mathcal{F}_\infty^\sigma$.*

Proof. Let $\xi : \Omega \to \mathbb{R}$ be any $\mathcal{F}_{\tau_S}$-measurable random variable. With the notation of the proof of Proposition 4, for $\Gamma \in \mathcal{C}$, $\Delta \in \mathcal{D}$, $\mathscr{D} = \sigma(\{\sigma \in \Delta\})$, $\Xi \in \mathcal{B}(\mathbb{R})$ for $\mathbf{P}$-almost all $\omega \in \Omega$,

$$\mathbf{P}(\tilde{B} \in \Gamma, \xi \in \Xi|\mathscr{D})(\omega) = \mathbf{P}(\tilde{B} \in \Gamma|\mathscr{D})(\omega)\, \mathbf{P}(\xi \in \Xi|\mathscr{D})(\omega),$$

since $\tilde{B}$ is a Brownian motion independent of $\mathcal{F}_{\tau_S}$ under $\mathbf{P}(\cdot|\mathscr{D})$. Thus,

$$\begin{aligned}\mathbf{P}(\tilde{B} \in \Gamma, \xi \in \Xi, \sigma \in \Delta) &= \mathbb{E}\left(\mathbf{1}_{\{\sigma\in\Delta\}}\, \mathbf{P}(\tilde{B} \in \Gamma, \xi \in \Xi|\mathscr{D})\right) = \mathbb{E}\left(\mathbf{1}_{\{\sigma\in\Delta\}}\, \mathbf{P}(\tilde{B} \in \Gamma)\, \mathbf{P}(\xi \in \Xi|\mathscr{D})\right) \\ &= \mathbf{P}(\tilde{B} \in \Gamma)\, \mathbb{E}\left(\mathbf{P}(\sigma \in \Delta, \xi \in \Xi|\mathscr{D})\right) = \mathbf{P}(\tilde{B} \in \Gamma)\, \mathbf{P}(\sigma \in \Delta, \xi \in \Xi). \qquad \square\end{aligned}$$

5.2. *Conditional default probabilities*

In this Section, we derive a formula for default probabilities conditional on the information flow $\mathcal{F}_s$ when the default time τ is determined by a credit quality process as in Definition 3.

Proposition 7. *Let X be a credit quality process with volatility process σ and quadratic variation Λ. Let $\tau = \inf\{t \geq 0 : X_t \leq b\}$ be the associated default time. For $s \leq t$, on $\{\tau > s\}$, the probability of default until time t, conditional on $\mathcal{F}_s$, is given by*

$$\mathbf{P}(\tau \leq t|\mathcal{F}_s) = \mathbb{E}\left(2\mathrm{N}\left(\frac{b - X_s}{\sqrt{\Lambda_t - \Lambda_s}}\right)\Big|\mathcal{F}_s\right) \qquad \boldsymbol{P}\text{–a.s..}$$

Proof. Let B be the DDS-Brownian motion of X, and recall that $B_{\Lambda_t} = X_t$, $t \geq 0$. By continuity of Λ and by properties of conditional expectation, $\mathbf{P}$–a.s.,

$$\begin{aligned}\mathbf{P}(\tau \leq t|\mathcal{F}_s) &= \mathbf{P}\left(\min_{s<u\leq t} X_u \leq b\Big|\mathcal{F}_s\right) = \mathbf{P}\left(\min_{s<u\leq t} B_{\Lambda_u} \leq b\Big|\mathcal{F}_s\right) \\ &= \mathbf{P}\left(\min_{\Lambda_s<u\leq\Lambda_t} B_u \leq b\Big|\mathcal{F}_s\right) = \mathbf{P}\left(\min_{0<u\leq\Lambda_t-\Lambda_s} B_{\Lambda_s+u} \leq b\Big|\mathcal{F}_s\right) \\ &= \mathbf{P}\left(\min_{0<u\leq\Lambda_t-\Lambda_s} B_{\Lambda_s+u} - B_{\Lambda_s} \leq b - B_{\Lambda_s}\Big|\mathcal{F}_s\right) \\ &= \mathbb{E}\left(\mathbf{P}\left(\min_{0<u\leq\Lambda_t-\Lambda_s} B_{\Lambda_s+u} - B_{\Lambda_s} \leq b - B_{\Lambda_s}\Big|\mathcal{F}_s \vee \sigma(\Lambda_t)\right)\Big|\mathcal{F}_s\right). \end{aligned} \tag{8}$$

The random time Λ_s is an $(\mathcal{F}_{\tau_t})_{t\geq 0}$-stopping time, and with $\mathcal{F}_{\tau_{\Lambda_s}} = \mathcal{F}_s$ it follows from Proposition 6 that $(B_{\Lambda_s+u} - B_{\Lambda_s})_{u\geq 0}$, is a Brownian motion independent of $\mathcal{F}_s \vee \sigma(\Lambda_t) \subseteq \mathcal{F}_s \vee \mathcal{F}_\infty^\sigma$. On the other hand, the random variables $\Lambda_t - \Lambda_s$ and $b - B_{\Lambda_s}$ are $\mathcal{F}_s \vee \sigma(\Lambda_t)$-measurable. By Lemma 1 and the first-passage time distribution of Brownian motion, cf. Equation (6),$\mathbf{P}$–a.s.,

$$\mathbf{P}\left(\min_{0<u\leq\Lambda_t-\Lambda_s} B_{\Lambda_s+u} - B_{\Lambda_s} \leq b - B_{\Lambda_s}\Big|\mathcal{F}_s \vee \sigma(\Lambda_t)\right) = 2\mathrm{N}\left(\frac{b - B_{\Lambda_s}}{\sqrt{\Lambda_t - \Lambda_s}}\right).$$

Inserting into Equation (8) yields, $\mathbf{P}$–a.s.,

$$\mathbb{E}\left(\mathbf{P}\left(\min_{0<u\leq\Lambda_t-\Lambda_s} B_{\Lambda_s+u} - B_{\Lambda_s} \leq b - B_{\Lambda_s}|\mathcal{F}_s \vee \sigma(\Lambda_t)\right)\Big|\mathcal{F}_s\right) = \mathbb{E}\left(2\mathrm{N}\left(\frac{b - B_{\Lambda_s}}{\sqrt{\Lambda_t - \Lambda_s}}\right)\Big|\mathcal{F}_s\right).$$

□

Corollary 8. *Let X be a credit quality process with volatility process σ, and assume further that (X, σ) has the Markov property. Let τ be the associated default time. Then, for $t > s$, on $\{\tau > s\}$, the conditional default distribution is*

$$\mathbf{P}(\tau \leq t|\mathcal{F}_s) = \mathbf{P}(\tau \leq t|X_s, \sigma_s) = \mathbb{E}\left(2\mathrm{N}\left(\frac{b - X_s}{\sqrt{\Lambda_t - \Lambda_s}}\right)\Big|X_s, \sigma_s\right) \quad \boldsymbol{P}\text{–a.s.}. \tag{9}$$

Proof. For the first step, observe that on $\{\tau > s\}$ the events $\{s < \tau \leq t\}$ and $\{\min_{s<r\leq t} X_r \leq b\}$ are equal, and the latter is conditionally independent of $\mathcal{F}_s$ given (X_s, σ_s) by the Markov property. For the second step, taking into account that $\Lambda_t - \Lambda_s = \int_s^t \sigma_u^2 \,\mathrm{d}u$, it follows that $2\mathrm{N}((b - X_s)/\sqrt{\Lambda_t - \Lambda_s})$ is a bounded random variable that is measurable with respect to $\sigma(X_r, r \geq s) \vee \sigma(\sigma_r, r \geq s)$. The assertion then follows from Proposition 7 and the Markov property. □

By setting $s = 0$ we obtain a formula for unconditional default probabilities:

Corollary 9. *Let X be a credit quality process with volatility process σ, and let τ be the associated default time. Assume further that σ_0 is non-random. Then the default distribution is given by*

$$\mathbf{P}(\tau \leq t) = 2\mathbb{E}\left(\mathrm{N}\left(\frac{b}{\sqrt{\Lambda_t}}\right)\right). \tag{10}$$

Clearly, for a deterministic time-change Λ we recover the Overbeck-Schmidt model, cf. Equation (2).

5.3. *Variance as Lévy-driven Ornstein-Uhlenbeck process*

We now put our model to work. We specify the variance as a mean-reverting process with jumps. This leads to the notion of a Lévy-driven Ornstein-Uhlenbeck. For details on Lévy-driven OU processes, we refer to [Norberg, 2004] and [Cont and Tankov, 2004, Chapter 15.3.3].

Let Z be a compound Poisson process, defined by $Z_t = \sum_{i=1}^{N_t} Y_i$, where N is a Poisson process with intensity λ and $Y_1, Y_2, \ldots$ are i.i.d. random variables independent of N. Then, for any $t \geq 0$, Z_t has a compound Poisson distribution with intensity λt and compounding variate $Y \sim Y_1$ and we write $Z_t \sim \mathrm{CPO}(\lambda t, Y)$.

Proposition 10. *Let W be an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion, and let Z be a compound Poisson process (with respect to $(\mathcal{F}_t)_{t\geq 0}$)) independent of W such that $Z_t \sim CPO(\lambda t, Y)$, $t \geq 0$, with $Y > 0$ $\boldsymbol{P}$–a.s.. Let $a \in \mathbb{R}_+$ and let θ be a strictly positive, bounded and càdlàg function, Then the stochastic process $X = (X_t)_{t\geq 0}$,*

$$X_t = \int_0^t \sigma_s \,\mathrm{d}W_s, \quad t \geq 0,$$

with σ such that σ^2 is the solution to the SDE

$$\mathrm{d}\sigma_t^2 = a(\theta(t) - \sigma_{t-}^2)\,\mathrm{d}t + \mathrm{d}Z_t, \quad t \geq 0, \tag{11}$$

with $\sigma_0^2 > 0$, is a credit quality process in the sense of Definition 3. Moreover, (X, σ) is Markov process with respect to $(\mathcal{F}_t)_{t\geq 0}$.

Before checking that X satisfies the conditions of a credit quality process, we state explicit formulas for the variance σ^2 and time-change Λ. The solution of the Lévy-driven OU process from Equation (11) is obtained by applying the Itô formula to $\mathbf{e}^{at}\,\sigma_t^2$, and is given by

$$\sigma_t^2 = \mathbf{e}^{-at}\,\sigma_0^2 + \int_0^t \mathbf{e}^{-a(t-s)}\,a\,\theta(s)\,\mathrm{d}s + \sum_{0<s\leq t} \mathbf{e}^{-a(t-s)}\,\Delta Z_s. \tag{12}$$

Increments of the time-change process $\Lambda = \int_0^{\cdot} \sigma_r^2 \, \mathrm{d}r$ are given by

$$\Lambda_t - \Lambda_s = \left(1 - \mathbf{e}^{-a(t-s)}\right) \frac{\sigma_s^2}{a} + \int_s^t \theta(r) \left(1 - \mathbf{e}^{-a(t-r)}\right) \mathrm{d}r + \frac{L_{s,t}}{a}, \tag{13}$$

with

$$L_{s,t} := \sum_{s<r\leq t} \left(1 - \mathbf{e}^{-a(t-r)}\right) \Delta Z_r, \quad s \leq t. \tag{14}$$

This is obtained by integrating each term of Equation (12).

Proof of Proposition 10. That σ is positive is straight-forward from the conditions on θ and Z and Equation (12). That $\mathbf{P}(\int_0^t \sigma_s^2 \, \mathrm{d}s < \infty) = 1$, $t \geq 0$, follows from Equation (13) and the fact that Z is a Lévy process with paths of finite variation $\mathbf{P}$–a.s.. Similarly, taking into account that $\lambda > 0$, it follows that $\lim_{t\to\infty} \int_0^t \sigma_s^2 \, \mathrm{d}s = \infty$.

That (X, σ) is a Markov process follows e.g. from Theorem 32 of [Protter, 2005], which states conditions for the solution of a Lévy-driven SDE to be a Markov process). □

5.4. *Jumps in conditional default probabilities and CDS spreads*

We investigate the pathwise propagation of jumps of the volatility process σ to conditional default probabilities and credit spreads for the Lévy-driven OU variance process. Recall that in Proposition 2 we have already established the absence of jumps in a model with deterministic volatility.

Proposition 11. *Let X be the credit quality process with Lévy-driven OU variance process as in Proposition 10. Let $\tau = \inf\{t > 0 : X_t \leq b\}$ be the associated default time. Fix $t > 0$ and let $(P(s,t))_{s\leq t}$ be the associated conditional default probability process. Then, for $\mathbf{P}$-almost all $\omega \in \{\tau > s\}$, $(P(s,t))_{s\leq t}$ is a process whose jumps are positive and*

$$\Delta\sigma_s(\omega) = 0 \iff \Delta P(s,t)(\omega) = 0, \quad s < t.$$

Proof. By the Markov property of (X, σ) and by Equation (9),

$$P(s,t) = \mathbb{E}\left(2\mathrm{N}\left(\frac{b - X_s}{\sqrt{\Lambda_t - \Lambda_s}}\right) \Big| X_s, \sigma_s\right), \tag{15}$$

with $\Lambda_t - \Lambda_s$ given by (cf. Equation (13)),

$$\Lambda_t - \Lambda_s = \left(1 - \mathbf{e}^{-a(t-s)}\right) \frac{\sigma_s^2}{a} + h(s,t) + \frac{L_{s,t}}{a},$$

with

$$\begin{aligned} h(s,t) &= \int_s^t \theta(r) \left(1 - \mathbf{e}^{-a(t-r)}\right) \mathrm{d}r \\ L_{s,t} &= \sum_{s<r\leq t} \left(1 - \mathbf{e}^{-a(t-r)}\right) \Delta Z_r. \end{aligned}$$

By Lemma 1, a version of the conditional probability of Equation (15) is given by $g_{s,t}(X_s, \sigma_s)$ with

$$g_{s,t}(x,y) := \mathbb{E}\left(2\mathrm{N}\left(\frac{b - x}{\sqrt{\left(1 - \mathbf{e}^{-a(t-s)}\right) y^2/a + h(s,t) + L_{s,t}/a}}\right)\right). \tag{16}$$

To derive the claim of the Proposition we require the following:

(i) For fixed $s \leq t$, $L_{s-,t} = L_{s,t}$ $\mathbf{P}$–a.s.,
(ii) for any sequence $(s_n, x_n, y_n) \to (s, x, y)$,

$$g_{s_n,t}(x_n, y_n) \to g_{s,t}(x, y), \tag{17}$$

(iii) for $(b-x)<0$, $g_{s,t}(x,y)$ is strictly increasing in y.

Property (i) follows from the càdlàg property and stochastic continuity of $L_{\cdot,t}$, cf. [Sato, 1999, p. 6]. For (ii) observe that

$$\frac{b-x_n}{\sqrt{\left(1-\mathbf{e}^{-a(t-s_n)}\right)y_n^2/a+h(s_n,t)+L_{s_n,t}/a}} \to \frac{b-x}{\sqrt{\left(1-\mathbf{e}^{-a(t-s)}\right)y^2/a+h(s,t)+L_{s,t}/a}}, \quad \mathbf{P}\text{–a.s.,} \quad \text{as } n\to\infty,$$

since all the terms in the sum of the denominator converge and the limit of the denominator is greater 0. Equation (17) is obtained by continuity of the Normal distribution and Dominated Convergence. For (iii) observe that the denominator in Equation (16) is strictly increasing in y and that for $(b-x)<0$, $t\mapsto \mathrm{N}((b-x)/\sqrt{t})$ is strictly increasing.

Fix $g_{s,t}(X_s,\sigma_s)$ as the version of the conditional default probability from Equation (15). Then, taking into account that X is continuous $\mathbf{P}$–a.s., and that on $\{\tau>s\}$ we have $(b-X_s)<0$, we obtain $\mathbf{P}$–a.s. for every sequence $s_n\uparrow s$,

$$P(s-,t)=\lim_{s_n\uparrow s} g_{s_n,t}(X_{s_n},\sigma_{s_n})=g_{s,t}(X_s,\sigma_{s-})\begin{Bmatrix} = g_{s,t}(X_s,\sigma_s), & \text{if } \Delta\sigma_s=0\\ < g_{s,t}(X_s,\sigma_s), & \text{if } \Delta\sigma_s>0\end{Bmatrix}=P(s,t). \qquad \square$$

As a consequence of this Proposition, a jump at time s in the volatility process induces a jump in the conditional default probability process $(P(s,t))_{s\le t}$ for each $t>s$.

To compute CDS spreads from default probabilities, assume for simplicity that the short rate is constant, $r_t=r$, $t\ge 0$, so that the formula for CDS spreads, Equation (1), becomes[c]

$$s(t,T):=\frac{(1-R)\int_t^T \mathbf{e}^{-r(t-u)}\,\mathbf{P}(\tau\in \mathrm{d}u|\mathcal{F}_t)}{\int_t^T \mathbf{e}^{-r(t-u)}\,\mathbf{P}(\tau>u|\mathcal{F}_t)\,\mathrm{d}u}, \quad \text{on } \{\tau>t\}. \tag{1*}$$

Proposition 12. *Let X be a credit quality process with Lévy-driven OU variance process as in Proposition 10. Let $\tau=\inf\{t>0: X_t\le b\}$ be the associated default time, let $(P(t,u))_{0\le t\le u}$, $u>0$, be conditional default probability processes, and let $(s(t,T))_{0\le t\le T}$ be the CDS spread process for maturity T. Then $(s(t,T))_{0\le t\le T}$ is càdlàg, and for $t\le T$ and $\omega\in\{\tau>t\}$,*

(i) $\Delta s(t,T)(\omega)=0$ $\boldsymbol{P}$–a.s., whenever $\Delta P(t,u)(\omega)=0$, $u>t$ and $u\le T$,
(ii) $\Delta s(t,T)(\omega)>0$ $\boldsymbol{P}$–a.s., whenever $\Delta P(t,u)(\omega)>0$, $u>t$ and $u\le T$.

Proof. Consider first the integral in the numerator of Equation (1*). For any sequence $t_n\uparrow t$, as $n\to\infty$,

$$\lim_{t_n\uparrow t}\int_{t_n}^T \mathbf{e}^{-r(u-t_n)}\mathbf{P}(\tau\in \mathrm{d}u|\mathcal{F}_{t_n}) = \underbrace{\lim_{t_n\uparrow t}\int_{t_n}^t \mathbf{e}^{-r(u-t_n)}\mathbf{P}(\tau\in \mathrm{d}u|\mathcal{F}_{t_n})}_{=0}+\lim_{t_n\uparrow t}\int_{t}^T \mathbf{e}^{-r(u-t_n)}\mathbf{P}(\tau\in \mathrm{d}u|\mathcal{F}_{t_n}) \tag{18}$$

where for the first integral, taking into account that $\{\tau>t\}$,

$$\lim_{t_n\uparrow t}\int_{t_n}^t \mathbf{e}^{-r(u-t_n)}\mathbf{P}(\tau\in \mathrm{d}u|\mathcal{F}_{t_n})\le \lim_{t_n\uparrow t}\int_{t_n}^t \mathbf{P}(\tau\in \mathrm{d}u|\mathcal{F}_{t_n})=P^t_{t-}=0.$$

[c]In the general case we have to assume that the short rate is continuous.

(i) By assumption, $\lim_{t_n \to t} \mathbf{P}(\tau \le u|\mathcal{F}_{t_n})(\omega) = \mathbf{P}(\tau \le u|\mathcal{F}_t)(\omega)$, as $n \to \infty$, $t < u \le T$. Choosing the regular version of $\mathbf{P}(\tau \in \cdot|\mathcal{F}_{t_n})$ for each t_n, $\mathbf{P}(\tau \in \cdot|\mathcal{F}_{t_n})(\omega)$ converges weakly, i.e., $\mathbf{P}(\tau \in \cdot|\mathcal{F}_{t_n})(\omega) \xrightarrow{w} \mathbf{P}(\tau \in \cdot|\mathcal{F}_t)(\omega)$. It follows by continuity of the integrand and weak convergence that

$$\lim_{t_n \to t} \int_{t_n}^{T} \mathbf{e}^{-r(u-t_n)} \mathbf{P}(\tau \in \mathrm{d}u|\mathcal{F}_{t_n}) = \int_{t}^{T} \mathbf{e}^{-r(u-t)} \mathbf{P}(\tau \in \mathrm{d}u|\mathcal{F}_t), \quad n \to \infty,$$

and consequently by Equation (18) the numerator converges from below. For the denominator of the CDS spread formula continuity at t follows from Dominated convergence.

(ii) Now assume $\lim_{t_n \uparrow t} \mathbf{P}(\tau \le u|\mathcal{F}_{t_n})(\omega) < \mathbf{P}(\tau \le u|\mathcal{F}_t)(\omega)$, as $n \to \infty$, $t < u \le T$. Observe that there is more probability mass beyond u when conditioning on $\mathcal{F}_{t_n}$ than when conditioning on $\mathcal{F}_t$, and since $\mathbf{e}^{-ru}$ is decreasing in u,

$$\lim_{t_n \uparrow t} \int_{t_n}^{T} \mathbf{e}^{-r(u-t_n)} \mathbf{P}(\tau \in \mathrm{d}u|\mathcal{F}_{t_n}) < \int_{t}^{T} \mathbf{e}^{-r(u-t)} \mathbf{P}(\tau \in \mathrm{d}u|\mathcal{F}_t), \quad n \to \infty.$$

Consequently, by Equation (18), the numerator has a positive jump at t. For the denominator of the CDS spread formula, as $n \to \infty$,

$$\begin{aligned}
&\lim_{t_n \uparrow t} \int_{t_n}^{T} \mathbf{e}^{-r(u-t_n)} \mathbf{P}(\tau > u|\mathcal{F}_{t_n})\,\mathrm{d}u \\
&\qquad = \underbrace{\lim_{t_n \uparrow t} \int_{t_n}^{t} \mathbf{e}^{-r(u-t_n)} \mathbf{P}(\tau > u|\mathcal{F}_{t_n})\,\mathrm{d}u}_{\le \lim_{t_n \uparrow t} \int_{t_n}^{t} 1\,\mathrm{d}u = 0} + \lim_{t_n \uparrow t} \int_{t}^{T} \mathbf{e}^{-r(u-t_n)} \mathbf{P}(\tau > u|\mathcal{F}_{t_n})\mathrm{d}u \\
&\qquad > \int_{t}^{T} \mathbf{e}^{-r(u-t)} \mathbf{P}(\tau > u|\mathcal{F}_t)\,\mathrm{d}u \qquad \text{(Dominated Convergence)}.
\end{aligned}$$

Hence, $\Delta s_t^T > 0$. For sequences $t_n \downarrow t$, convergence follows as in (i). □

In a similar way, noting that Λ is continuous, we obtain a corresponding result for CDS spread processes with a fixed time-to-maturity.

Corollary 13. *Let $(s(t, t+T))_{t \ge 0}$ be the CDS spread process for time-to-maturity T. Then, under the assumptions of Proposition 12, $(s(t, t+T))_{t \ge 0}$ is càdlàg and for $t \le T$ and $\omega \in \{\tau > t\}$,*

(i) $\Delta s(t, t+T)(\omega) = 0$ $\boldsymbol{P}$–a.s., whenever $\Delta P(t, u)(\omega) = 0$, $u > t$ and $u \le T + t$,
(ii) $\Delta s(t, t+T)(\omega) > 0$ $\boldsymbol{P}$–a.s., whenever $\Delta P(t, u)(\omega) > 0$, $u > t$ and $u \le T + t$.

In the previous results we established that a positive jump in σ_s induces a positive jump at time s in every default probability process $P(s, t)$, $t > s$. In turn, this induces a positive jump at time s in each spread process $s(s, t)$, $t > s$. In turn, whenever σ is continuous at s, the conditional default processes and the CDS spread processes are continuous at s. Hence, the credit quality process model does not include events where CDS spreads jump processes for different maturities jump at different times. However, recall the observation stated in Section 2 that CDS spreads of different maturities tend to jump together, cf. [Schneider et al., 2007], the reasoning being that (positive) jumps in CDS spreads are triggered by the arrival of bad news about the underlying entity and generally this affects the CDS of all maturities written on that entity.

6. Implementation

The valuation of financial claims in the credit quality process model is done by Monte Carlo simulation. If we have a means for efficiently computing conditional default probabilities $P(s, t) = \mathbf{P}(\tau \le t|X_s, \sigma_s)$, $0 \le s \le t$, then Monte Carlo simulation reduces to simulating X and σ. An important feature of our algorithm is that for valuing a product that involves the quantities $P(s, t)$ or the spread $s(s,t)$, we need to simulate only until time s not t. For example, valuation of an option on a CDS requires simulation until maturity of the option instead of simulation until maturity of the underlying CDS.

We first examine numerical computation of conditional default probabilities $\mathbf{P}(\tau \leq t | X_s, \sigma_s)$, $0 \leq s \leq t$, cf. Equation (9). This requires computing the distribution of $L_{s,t}$ from Equation (14). We make use of the following result (see e.g. [Norberg, 2004] for the proof):

Proposition 14. *Let $Z = (Z_t)_{t \geq 0}$ be a compound Poisson process such that $Z_t \sim CPO(\lambda t, Y)$, $t \geq 0$, and let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function. Then*

$$\int_s^t f(s)\,\mathrm{d}Z_s \sim CPO(\lambda(t-s), f(S)Y), \quad s \leq t,$$

where S is uniformly distributed over $(s,t]$ and independent of Y.

In our case, for $0 \leq s \leq t$, $L_{s,t}$ follows a compound Poisson distribution,

$$L_{s,t} \sim \mathrm{CPO}\left(\lambda(t-s), \left(1 - \mathbf{e}^{-a(t-S)}\right) Y\right), \tag{19}$$

with S uniformly distributed on $(s,t]$. Inspection of Equation (14) reveals that $L_{s,t}$ and L_{t-s} follow the same distribution, $L_{s,t} \stackrel{\mathcal{L}}{=} L_{t-s}$, hence it suffices to compute the distributions of $L_t := L_{0,t}$, $t \geq 0$. The following result is useful for efficient numerical computation of the distribution of L_t.

Proposition 15. *Let $L_t := L_{0,t}$ be a random variable with compound Poisson distribution as in Equation (19), with $\mathbf{P}(Y > 0)$ $\boldsymbol{P}$–a.s.. Then, for $x \geq 0$, the distribution function of the compounding variate $\left(1 - \mathbf{e}^{a(t-S)}\right) Y$ is given by*

$$F(x) = \mathbb{E}\left(-\frac{\ln(1 - x/Y)}{at} \boldsymbol{1}_{\{[0,1-\mathbf{e}^{-at})\}}(x/Y)\right) + \mathbf{P}\left(Y \leq \frac{x}{1 - \mathbf{e}^{-at}}\right), \quad x \geq 0. \tag{20}$$

Proof. Conditioning under a larger filtration yields

$$F(x) = \mathbf{P}\left(\left(1 - \mathbf{e}^{-a(t-S)}\right) Y \leq x\right) = \mathbb{E}\left(\mathbf{P}\left(\left(1 - \mathbf{e}^{-a(t-S)}\right) Y \leq x \middle| Y\right)\right). \tag{21}$$

By Lemma 1 and the independence of S and Y, a version of the conditional probability is given by $g_x(Y)$ with $g_x(y) := \mathbf{P}\left(\left(1 - \mathbf{e}^{-a(t-S)}\right) y \leq x\right)$. Since $S \in (0,t]$,

$$g_x(y) = \begin{cases} 0, & x < (1 - \mathbf{e}^{-a(t-t)})y = 0, \\ 1, & x >= (1 - \mathbf{e}^{-at})y, \\ -\frac{\ln(1-x/y)}{at}, & x \in [0, (1 - \mathbf{e}^{-at})y), \end{cases}$$

where the result for $x \in [0, (1 - \mathbf{e}^{-at}y)$ is obtained by making use of the fact that S/t is uniformly distributed on $(0,1]$. Inserting into Equation (21) yields

$$\begin{aligned} F(x) &= \mathbb{E}\left(-\frac{\ln(1 - x/Y)}{at} \mathbf{1}_{\{[0,1-\mathbf{e}^{-at})\}}(x/Y) + \mathbf{1}_{\{[1-\mathbf{e}^{-at},\infty)\}}(x/Y)\right) \\ &= \mathbb{E}\left(-\frac{\ln(1 - x/Y)}{at} \mathbf{1}_{\{[0,1-\mathbf{e}^{-at})\}}(x/Y)\right) + \mathbf{P}\left(Y \leq \frac{x}{1 - \mathbf{e}^{-at}}\right). \end{aligned} \qquad \square$$

The distribution of L_t can be computed efficiently using the method of [Panjer, 1981], who states a recursive evaluation formula for a family of compound distributions (see also [McNeil et al., 2005, Chapter 10]). In our implementation it has proven to be numerically more stable to assume a discrete distribution of the compounding variate, even though the distribution function of the compounding variate is continuous. For the compound Poisson case, assume N to be a Poisson distribution random variable with intensity λ and assume the compounding variate distribution to be discrete and defined on the positive integers by f_i, $i = 1, 2, \ldots$. The result by Panjer states that the compound Poisson distribution is given by

$$g_i = \frac{\lambda}{i} \sum_{j=1}^{i} j\, f_j\, g_{i-j}, \quad i = 1, 2, \ldots,$$

whereas the usual form is

$$g_i = \sum_{n=0}^{i} f_i^{(n)} \mathbf{P}(N = n), \quad i = 1, 2, \ldots,$$

where $f_i^{(n)}$ denotes the n-fold convolution of f at i. By proper scaling the method can also be used for discrete positive compounding variates not restricted to integers.

To illustrate the pickup in computational speed using Panjer recursion, we compare the computation of the distribution of Λ_{t_i}, with $t_i = i/10$, $i = 0, \ldots, 200$, for points $x = (x_i)_{i=0,\ldots,8000}$, using Monte Carlo simulation and Panjer recursion. CPU time of various computations are given in Table 1. The parameters used for the computations are given in Table 3.

Table 1. Monte Carlo simulation vs. Panjer recursion. Computation of distribution of Λ_t at 200 time points with 8000 grid points each. Mean square error is computed with respect to the result obtained by Panjer recursion. The CPU time consumed by the Panjer recursion implementation was 331.33 CPU seconds.

number of simulations	CPU time (seconds)	MSE (at $t = 5$)
1000	164.28	0.9369
2000	333.70	0.2725
5000	1966.54	0.0589
10000	8554.59	0.0403

For Monte Carlo simulation, we simulate paths of σ and X on a discrete time grid. The simulation algorithm is given in Algorithm 1. For each time step we compute the default probability term structure for desired maturities through Equation (9). Note that since $L_{s,t} \stackrel{\mathcal{L}}{=} L_{t-s}$ the distributions of L_t, $t \geq 0$, need only be computed once. Additionally, we assume that the deterministic function θ is time-homogeneous, i.e., at time s the conditional default probability is based on $\theta(r - s)$, $s \leq r \leq t$. This ensures that properties of the original term structure that depend on time-to-maturity are preserved. Simulating on a discrete time grid underestimates the occurrence of the default event. This is overcome by sampling an indicator variable that determines whether default has occurred or not between two time points. Taking into account that X is a Brownian motion with a continuous time-change, the indicator takes value 1 with probability $\mathbf{e}^{-2(b-X_{t_{i-1}})(b-X_{t_i})/(\Lambda_{t_i}-\Lambda_{t_{i-1}})}$ (cf. [Karatzas and Shreve, 1998, p. 265] or [Glasserman, 2004, p. 368]).

To compute CDS spreads from default probabilities we use a well-known approximation called the *credit triangle*. Under the assumptions that the interest rate curve and the CDS spread curves are flat and that CDS spreads are paid continuously, the formula for the CDS spread, Equation (1), becomes

$$s(t, T) = \lambda_{t,T}(1 - R), \quad t \leq T,$$

with $\lambda_{t,T}$ the hazard rate derived from $P(t, T) = 1 - \mathbf{e}^{-\lambda_{t,T}(T-t)}$.

7. Valuation of the leveraged credit-linked note

As an example we illustrate valuation of the leveraged credit-linked note (leveraged CLN) of Section 2. Recall that the note is unwound at the trigger time $S := \inf\{t \geq 0 : V_t^k > K\}$, where V_t^k is

Input: $t_1 = 0 < \ldots < t_n$ // time discretisation
$x_1 = 0 < \ldots < x_m$ // space discretisation
$u_1, \ldots, u_r$ // desired maturities
N // number of simulations
b // default barrier
a, θ, λ, σ_0^2, F // volatility process parameters, F jump size distribution

// Panjer recursion
for $i = 1$ to n **do**
    for $j = 1$ to m **do**
        compute $\mathbf{P}(L_{t_i} \in [x_{j-1}, x_j))$
    end for
end for
// simulation step
for $k = 1$ to N **do**
    $\tau^k \leftarrow \infty$ // default time of k-th simulation
    for $i = 1$ to n **do**
        simulate $\sigma_{t_i}^k$ and $X_{t_i}^k$
        sample $d \leftarrow \mathbf{1}_{\{\min_{t_{i-1} < s \leq t_i} X_s \leq b\}}$ conditional on $X_{t_{i-1}}$ and X_{t_i} // (see text)
        if $d = 1$ or $X_{t_i}^k \leq b$ **then**
            $\tau^k \leftarrow t_i$
            next k // exit k-th simulation
        end if
        for $l = 1$ to r **do**
            if $u_l > t_i$ **then**
                $h \leftarrow \left(1 - \mathbf{e}^{-a(u_l - t_i)}\right) \frac{\sigma_{t_i}^2}{a} + \int_0^{u_l - t_i} \theta(r) \left(1 - \mathbf{e}^{-a(u_l - t_i - r)}\right) \mathrm{d}r$
                $P(t_i, u_l) \leftarrow 2 \sum_{k=1}^m \mathrm{N}\left(\frac{b - X_{t_i}}{\sqrt{h + x_{k-1}/a}}\right) \mathbf{P}\left(L_{u_l - t_i} \in [x_{k-1}, x_k)\right)$
                $s(t_i, u_l) \leftarrow (1 - R) \frac{-\ln(1 - P(t_i, u_l))}{u_l - t_i}$ // credit triangle
            end if
        end for
    end for
end for

Algorithm 1: Computation of conditional default probabilities

the mark-to-market value (from the point of view of a CDS protection buyer) of k CDS positions entered at time 0 with maturity T, and K is the trigger level. The notional of the note, 1, is then used to finance the closing of the position. However, in the case where $V_S^k > 1$, the issuer must cover the resulting gap. To compensate for this risk, the issuer receives the gap risk fee $(k - \tilde{k})s(0,T)$.

Again, for simplicity, assume that interest rates are constant. The mark-to-market of a CDS entered at time 0 with maturity T at time t on $\{\tau > t\}$ is

$$V_t = (s(t,T) - s(0,T)) \int_t^T \mathbf{e}^{-r(u-t)} \, \mathbf{P}(\tau > u | \mathcal{F}_t) \, \mathrm{d}u.$$

Set $V_\tau := (1 - R)$. Assuming continuous CDS spread payments, the value of the leveraged CLN at time 0 is obtained by taking expectation under a risk-neutral measure of the discounted cash-flows of the note issuer,

$$\begin{aligned} V_0^{\mathrm{LCLN}} &= \mathbb{E}\left((k - \tilde{k})s(0,T) \int_0^T \mathbf{e}^{-ru} \mathbf{1}_{\{S > u\}} \, \mathrm{d}u - \mathbf{e}^{-rS} \, \max(V_S^k - 1, 0)\right) \\ &= (k - \tilde{k})s(0,T) \int_0^T \mathbf{e}^{-ru} \mathbf{P}(S > u) \, \mathrm{d}u - \int_0^T \mathbf{e}^{-ru} \, \max(V_u^k - 1, 0) \, \mathbf{P}(S \in \mathrm{d}u). \end{aligned}$$

The fair gap risk fee is obtained by setting $V_0^{\mathrm{LCLN}} = 0$, i.e.,

$$(k - \tilde{k})s(0,T) = \frac{\int_0^T \mathbf{e}^{-ru} \max(V_u^k - 1, 0)\, \mathbf{P}(S \in \mathrm{d}u)}{\int_0^T \mathbf{e}^{-ru}\, \mathbf{P}(S > u)\, \mathrm{d}u}.$$

From the output of the Monte Carlo algorithm, for each simulation path, we compute the required mark-to-market values, determine the trigger time S, and compute the gap risk fee. Averaging over all scenarios is then an estimator for the fair gap risk fee.

As an example we compute the fair factor $\tilde{k}$ for a leveraged credit-linked note with a maturity of 5 years, a leverage factor of $k = 5$ and a trigger level of $K = 60\%$. The initial CDS spread is 180 bp; the interest rate is $r = 5\%$. The variance process σ^2 of the credit quality process has a jump intensity of $\lambda = 1$, where the jump size is 0.1 with probability 0.95 and 50 with probability 0.05, respectively. The full parameter set of the credit quality process is given in Table 3 and will be explained in more detail in the following Section. The factor $\tilde{k}$ computed as the mean of 1000 simulations is 3.611 (standard deviation 0.778). Compare this to a model without jumps ($\lambda = 0$), where the fair factor $\tilde{k} = k = 5$ as there is no gap risk.

8. Calibration

Calibration of the model consists of assigning the following parameters:

(i) Default barrier b: The choice of b is arbitrary, provided $b < 0$. Fixing a time-horizon T we set

$$b := \sqrt{T} \cdot \mathrm{N}^{(-1)}\left(\frac{\mathbf{P}(\tau \leq T)}{2}\right).$$

This implies $\mathbf{P}(\tau \leq T) = 2\mathrm{N}(b/\sqrt{T})$, which is the probability of hitting the barrier b until T if the credit quality process were a Brownian motion.

(ii) Deterministic function θ and initial variance σ_0: Given the other parameters, θ and σ_0 are chosen as to reproduce a given term structure of default probabilities using Equation (10).

(iii) Mean reversion a, jump intensity λ, jump size distribution F: These parameters determine the dynamics of default probabilities and CDS spreads. As it turns out, they also influence the quality of the calibration to the given term structure (this is explained below).

In order to assign these parameters recall the stylised facts of the term structure of credit spreads that were mentioned in Section 2: The two risk types that govern credit spreads are jump-to-default risk and the risk of credit quality changes. CDS spreads of short time-to-maturity are dominated by jump-to-default risk. CDS spreads exhibit frequent positive jumps, which are attributed to the arrival of bad news.

Calibrating to the given term structure means assigning the deterministic function θ, given all other parameters, such that the given term structure is recovered. We choose θ as a piecewise constant function based on the time-grid of given default probabilities. Typically, due to the constraints on θ, perfect calibration to the given term structure is not possible. Calibration errors at the short end of the credit spread term structure indicate that the parameters for the dynamics are not chosen well.[d] In particular, large values of θ at the short end indicate a poor calibration, since essentially jump-to-default risk is compensated by the deterministic function θ. A reasonable calibration should therefore lead to values of θ that are near zero at the short end. Likewise, a low initial volatility σ_0 leads to better calibration results. Figure 2 shows a calibration example to a given term structure of survival probabilities, where the survival probability is given $\mathbf{P}(\tau > t) = \mathbf{e}^{-ht}$, with $h = 3\%$ the

[d]Theoretically, relaxing the constraint that θ be bounded and non-negative (merely requiring suitable integrability), and assuming a deterministic model, i.e., $\lambda = 0$, perfect calibration to the market-given spot curve is achieved by determining θ using the time transformation of the Overbeck-Schmidt model, Equation (2). In practice, this turns out to be numerically unstable. In particular, interpolation may lead to negative values for the variance.

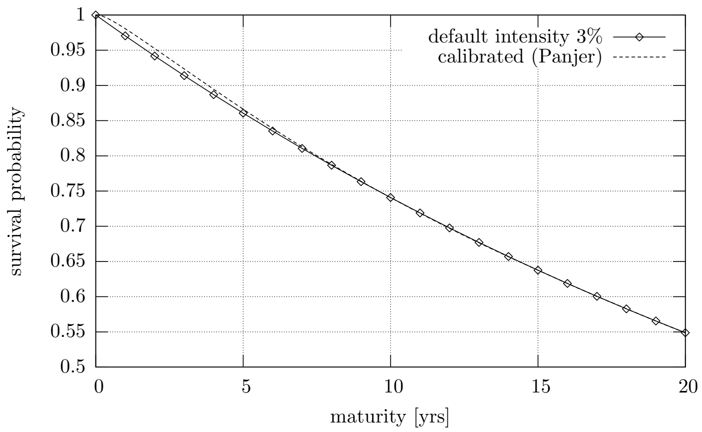

Fig. 2. Survival curve, original data and fitted data. See Tables 3 and 4 for parameters.

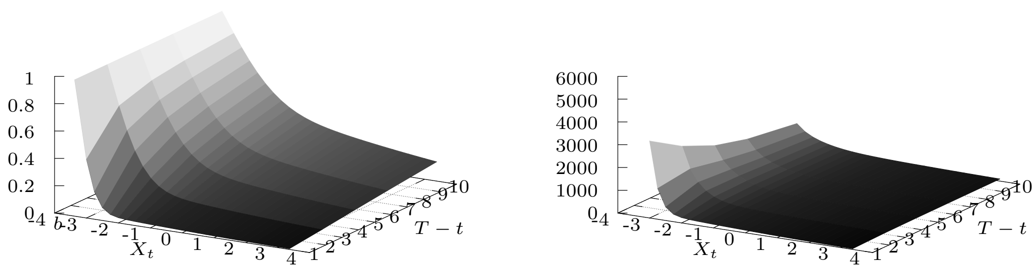

Fig. 3. Default probabilities (left) and CDS spreads (right) as a function of the level of the credit quality process X_t and $\sigma_t^2 = 0.2$ for time-to-maturities $(T-t)$ of 3, 5, 7 and 10 years. Parameters as Tables 3 and 4.

hazard rate. The parameters of the example are given in Table 3, and the values for θ resulting from calibration are listed in Table 4.

The remaining parameters determine the dynamics of the model. It is useful to distinguish those parameters that control large jumps in volatility and those that control small jumps and the diffusion component of the volatility process. Even though our model does not include jump-to-default events, a large jump in volatility may lead to a default in a very short time interval. Small jumps and the continuous component of the volatility process in turn determine the risk of credit quality changes. When calibrating to a given term structure, we find that "near-jump-to-default" risk and credit quality risk offset each other, i.e., assigning increasing risk of large jumps (either in terms of jump size or probability) leads to a reduction in credit quality risk and vice versa.

Consider the Lévy-driven OU process with discrete jump size distribution $c = (0.1, 50)$ with corresponding probabilities $p = (0.95, 0.05)$ and jump intensity $\lambda = 1$ (by Propositions 11 and 12 this is also the jump intensity of default probabilities and CDS spreads). The full set of parameters is given in Table 3. Figure 3 shows the default probabilities and CDS spreads for various time-to-maturities as a function of the level of the credit quality process X_t and a fixed level of $\sigma_t^2 = 0.2$. The probabilities of X hitting a particular level within a time horizon is given in Table 2. Figures 4 and 5 show the jump size in default probabilities and CDS spreads with respect to the level of X and σ^2 when a jump in the volatility process of size 0.1 and 50, respectively, occurs. Finally, two example scenarios are given in Figure 6.

Table 2. From left to right: (a) probability of hitting a level k when $X_t = 0$: $\mathbf{P}(\min_{t<s\le T-t} X_s \le k | X_t = 0)$; (b) probability of hitting the default barrier b from a given level $X_t = k$: $\mathbf{P}(\min_{t<s\le T-t} X_s \le b | X_t = k)$; (c) CDS spreads that correspond to default probabilities in (b). Parameters as in Tables 3 and 4.

	(a)			(b)			(c)		
k	$T-t=3$	5	10	$T-t=3$	5	10	$T-t=3$	5	10
3	0.08	0.15	0.28	0.04	0.07	0.15	79.44	90.93	98.24
2	0.12	0.22	0.39	0.05	0.09	0.18	101.38	113.06	119.80
1	0.34	0.48	0.62	0.06	0.11	0.22	127.69	139.14	145.45
-1	0.34	0.48	0.62	0.10	0.18	0.34	211.54	241.79	248.18
-2	0.12	0.22	0.39	0.22	0.35	0.52	487.84	519.04	439.20
-3	0.08	0.15	0.28	0.71	0.79	0.85	2484.59	1845.13	1143.56

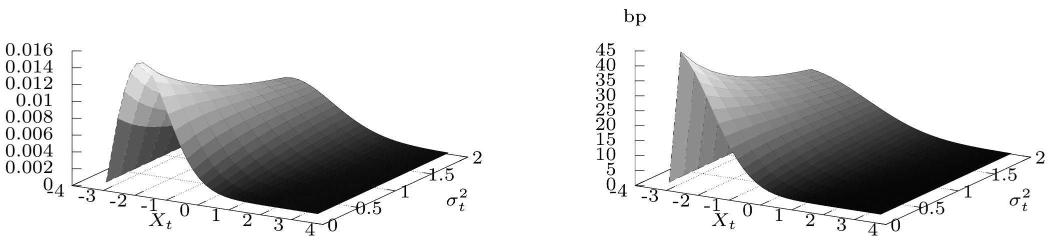

Fig. 4. Jump size of 5-year default probability (left) and 5-year CDS spread (right) with respect to level of X_t and σ_t when a jump of size 0.1 occurs in σ_t^2. Parameters as in Tables 3 and 4.

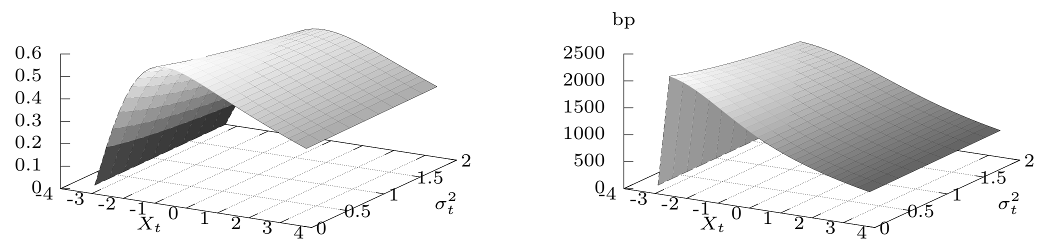

Fig. 5. Jump size of of 5-year default probability (left) and 5-year CDS spread (right) with respect to level of X and σ when a jump of size 50 occurs in σ_t^2. Parameters as in Tables 3 and 4.

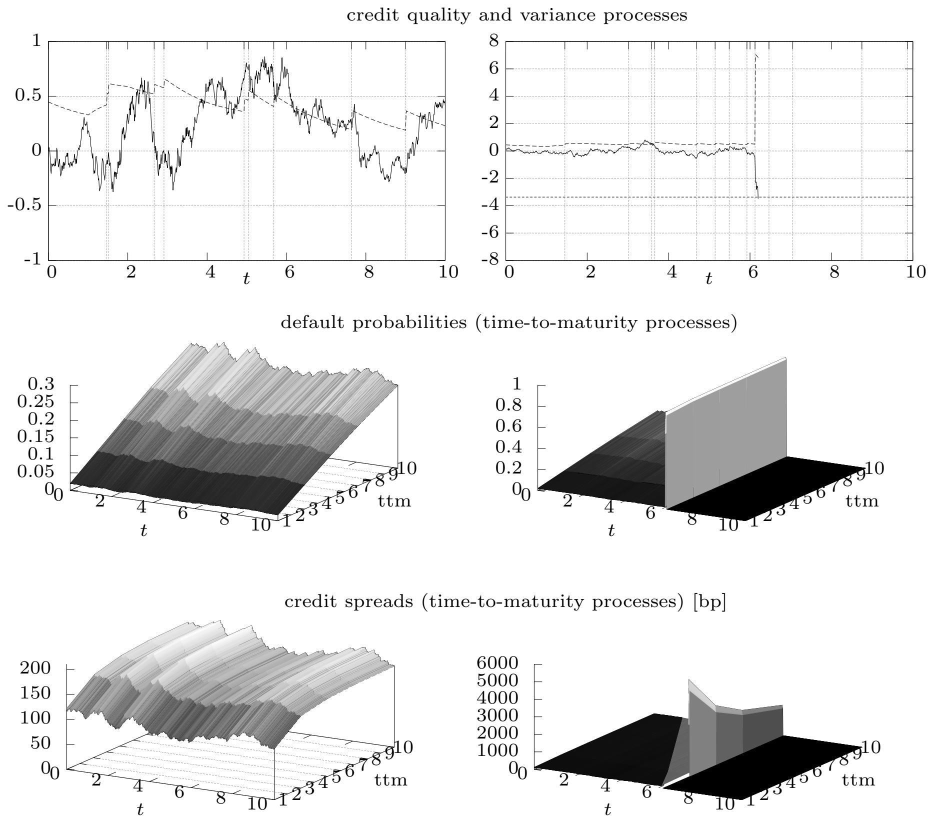

Fig. 6. Two scenarios (left and right) over a time horizon of 10 years. Top: credit quality process and volatility (dashed), jump times are marked in the grid; middle: default probabilities for 3, 5, 10 years time-to-maturity (ttm). bottom: CDS spreads (bottom) for 3, 5, 10 years time-to-maturity. Parameters as in Tables 3 and 4. The scenario on the right has a large jump, which leads to default.

Table 3. Parameters used in example

Parameter	Value
hazard rate h	0.03
recovery rate R	0.4
mean reversion a	1
jump intensity λ	1
jump size c	$\{0.1, 50\}$ with prob. $\{0.95, 0.05\}$
barrier b	-3.3695
initial variance σ_0^2	0.2
discretization for CPO distribution	$x = (x_i)_{i=0,\ldots,8000}$, $x_i = i/20$

Table 4. θ as determined by calibration

t	$\theta(t)$	t	$\theta(t)$
0	1.24630	10	0.00482
1	0.05480	11	0.00485
2	0.29490	12	0.01452
3	0.19158	13	0.00122
4	0.07052	14	0.00696
5	0.08591	15	0.03297
6	0.00007	16	0.08675
7	0.00002	17	0.02841
8	0.00150	18	0.10331
9	0.00052	19	0.11849

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