# Pricing American Options with Mellin Transforms

Robert Frontczak and Rainer Schöbel†

This Version: May, 2008.

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**JEL Classification:** G13

**Mathematics Subject Classification (2000):** 44A15, 60H30

†Eberhard Karls University Tuebingen, Faculty of Economics and Business Administration, Sigwartstrasse 18, 72076 Tuebingen, Germany. E-mail: robert.frontczak@uni-tuebingen.de, rainer.schoebel@uni-tuebingen.de.

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Abstract

Mellin transforms in option pricing theory were introduced by Panini and Srivastav (2004). In this contribution, we generalize their results to European power options. We derive Black-Scholes-Merton-like valuation formulas for European power put options using Mellin transforms. Thereafter, we restrict our attention to plain vanilla options on dividend-paying stocks and derive the integral equations to determine the free boundary and the price of American put options using Mellin transforms. We recover a result found by Kim (1990) regarding the optimal exercise price of American put options at expiry and prove the equivalence of integral representations herein, the representation derived by Kim (1990), Jacka (1991), and by Carr et al. (1992). Finally, we extend the results obtained in Panini and Srivastav (2005) and show how the Mellin transform approach can be used to derive the valuation formula for perpetual American put options on dividend-paying stocks.



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## Introduction

Robert Hjalmar Mellin (1854-1933) gave his name to the Mellin transform that associates to a locally Lebesgue integrable function $f(x)$ defined over positive real numbers the complex function $M(f(x), \omega)$ defined by

$$M(f(x), \omega) := \widetilde{f}(\omega) = \int_0^\infty f(x)\, x^{\omega-1}\, dx.$$

The Mellin transform is defined on a vertical strip in the $\omega$-plane, whose boundaries are determined by the asymptotic behavior of $f(x)$ as $x \to 0^+$ and $x \to \infty$. The largest strip $(a, b)$ in which the integral converges is called the fundamental strip. The conditions

$$f(x) = O(x^u) \qquad \text{for} \qquad x \to 0^+$$

and

$$f(x) = O(x^v) \qquad \text{for} \qquad x \to \infty$$

when $u > v$, guarantee the existence of $M(f(x), \omega)$ in the strip $(-u, -v)$. Thus, the existence is granted for locally integrable functions, whose exponent in the order at 0 is strictly larger than the exponent of the order at infinity.

Conversely, if $f(x)$ is an integrable function with fundamental strip $(a, b)$, then if $c$ is such that $a < c < b$ and $f(c + it)$ is integrable, the equality

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \widetilde{f}(\omega)\, x^{-\omega}\, d\omega = f(x)$$

holds almost everywhere. Moreover, if $f(x)$ is continuous, then the equality holds everywhere on $(0, \infty)$.

For a proof see for example Titchmarsh (1986) or Sneddon (1972). See also Flajolet et al. (1995) for a reference.

Simple changes of variables in the definition of the Mellin transforms yield to a whole set of transformation rules and facilitate the computations. In particular, if $f(x)$ admits the Mellin transform on the strip $(a, b)$ and $\alpha, \beta$ are positive reals, then the following relations hold:

$$M(f(\alpha x), \omega) = \alpha^{-\omega} \widetilde{f}(\omega) \quad \text{on} \quad (a, b).$$

$$M(x^\alpha f(x), \omega) = \widetilde{f}(\omega + \alpha) \quad \text{on} \quad (a, b).$$

$$M(f(x^\alpha),\omega) = \frac{1}{\alpha}\widetilde{f}\Big(\frac{\omega}{\alpha}\Big) \quad ,\alpha > 0, \quad \text{on} \quad (a\alpha, b\alpha).$$

$$M(f\Big(\frac{1}{x}\Big),\omega) = -\widetilde{f}(-\omega) \quad \text{on} \quad (-b,-a).$$

$$M(x^\beta f(x^\alpha),\omega) = \frac{1}{\alpha}\widetilde{f}\Big(\frac{\omega+\beta}{\alpha}\Big) \quad ,\alpha > 0, \quad \text{on} \quad (a\alpha, b\alpha).$$

$$M(x\frac{d}{dx}f(x),\omega) = -\omega\widetilde{f}(\omega) \quad \text{on} \quad (a^*, b^*).$$

$$M(\frac{d}{dx}f(x),\omega) = -(\omega-1)\widetilde{f}(\omega-1) \quad \text{on} \quad (a^*-1, b^*-1).$$

$$M\Big(\frac{d^n}{dx^n}f(x),\omega\Big) = (-1)^n\frac{\Gamma(\omega)}{\Gamma(\omega-n)}\widetilde{f}(\omega-n) \quad \text{on} \quad (a^*-n, b^*-n).$$

For a proof of some of these relations we refer to Titchmarsh (1986) or Sneddon (1972). The change of variables $x = e^s$ shows that the Mellin transform is closely related to the Laplace transform and the Fourier transform. In particular, if $F(f(x),\omega)$ and $L(f(x),\omega)$ denote the two-sided Fourier and Laplace transform, respectively, then we have

$$M(f(x),\omega) = L(f(e^{-x}),\omega) = F(f(e^{-x}), -i\,\omega).$$

However, there are numerous applications where it proves to be more convenient to operate directly with the Mellin transform rather than the Laplace-Fourier version. This is often the case in complex function theory (asymptotics of Gamma-related functions like the Riemann zeta function), in analytic number theory (Perron's formula for the coefficients of Dirichlet series), in the analysis of algorithms (harmonic sums), and as Panini (2004) and Panini and Srivastav (2004) showed in finance. However, the applicability to problems in modern finance theory have not been studied extensively yet. Since the Mellin transform has many interesting properties, it may turn out to be very useful for specific problems.

# 1 The European Power Put Option

We consider a market where the risk neutral asset price $S_t, t \in [0,T]$, is governed by the stochastic differential equation (SDE):

$$dS_t = (r-q)\, S_t\, dt + \sigma\, S_t\, dW_t\,, \tag{1}$$

with initial value $S_0 \in (0,\infty)$, and where $r$ is the riskless interest rate, $q$ is the dividend yield, $\sigma > 0$ is the volatility, and $W_t$ is a one-dimensional Brownian motion.
A European power put option is an option with a non-linear payoff given by the difference between the strike price and the underlying asset price at maturity raised to a strictly positive power

$$P_n^E(S,T) = \max(X - S_T^n, 0) \qquad \text{for} \quad n > 0\,, \tag{2}$$

where $X$ is the strike price of the option. For $n = 1$ we have the plain vanilla put as a special case. Power options offer flexibility to investors and are of practical interest since many OTC-traded options exhibit such a payoff structure. For references to power options see for example Esser (2003) and Macovshi and Quittard-Pinon (2006). Our goal is to derive a valuation formula for European power put options using Mellin transform techniques.
Applying Ito's Lemma to $\mathcal{S}_t = S_t^n$ gives

$$d\mathcal{S}_t = \Big(n(r-q) + \frac{1}{2}n(n-1)\sigma^2\Big)\mathcal{S}_t\, dt + n\sigma\, \mathcal{S}_t\, dW_t\,, \tag{3}$$

and we observe that the new process is again a Geometric Brownian motion. Now it is straightforward to derive the partial differential equation (PDE) for any derivative $V$ written on $\mathcal{S}$:

$$\frac{\partial V}{\partial t} + n(\frac{1}{2}\sigma^2(n-1) + (r-q))\mathcal{S}\,\frac{\partial V}{\partial \mathcal{S}} + \frac{1}{2}\sigma^2 n^2\, \mathcal{S}^2\, \frac{\partial^2 V}{\partial \mathcal{S}^2} - rV = 0 \tag{4}$$

where we have abbreviated the notation slightly. Especially, for European power put options $P_n^E$ we have

$$\frac{\partial P_n^E}{\partial t} + n(\frac{1}{2}\sigma^2(n-1) + (r-q))\mathcal{S}\,\frac{\partial P_n^E}{\partial \mathcal{S}} + \frac{1}{2}\sigma^2 n^2\, \mathcal{S}^2\, \frac{\partial^2 P_n^E}{\partial \mathcal{S}^2} - rP_n^E = 0 \tag{5}$$

with boundary conditions

$$\lim_{\mathcal{S}\to\infty} P_n^E(\mathcal{S},t) = 0 \qquad \text{on} \quad [0,T)\,, \tag{6}$$

$$P_n^E(\mathcal{S},T) = \theta(\mathcal{S}) = (X - \mathcal{S})^+ \qquad \text{on} \quad [0,\infty)\,, \tag{7}$$

and

$$P_n^E(0,t) = Xe^{-r(T-t)} \qquad \text{on} \quad [0,T)\,. \tag{8}$$

Once again, for $n = 1$ PDE (5) is well known as the fundamental valuation equation or the general Black-Scholes-Merton PDE with the celebrated solution:

$$P_1^E(S,t) = X\,e^{-r(T-t)}\,N(-d_2(S,X,T)) - S\,e^{-q(T-t)}\,N(-d_1(S,X,T)) \tag{9}$$

where

$$d_1(S,X.T) = \frac{\ln\frac{S}{X} + (r - q + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\,, \tag{10}$$

$$d_2(S,X,T) = d_1(S,X,T) - \sigma\sqrt{T-t}\,, \tag{11}$$

and $N(x)$ denotes the cumulative standard normal distribution function at the point $x$.
Let $\widetilde{P}_n^E(\omega,t)$ denote the Mellin transform of $P_n^E(\mathcal{S},t)$ which is defined by the relation

$$\widetilde{P}_n^E(\omega,t) = \int_0^\infty P_n^E(\mathcal{S},t)\,\mathcal{S}^{\omega-1}\,d\mathcal{S}, \tag{12}$$

where $\omega$ is a complex variable with $0 < Re(\omega) < \infty$. Conversely, the inverse Mellin transform is defined by

$$P_n^E(\mathcal{S},t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widetilde{P}_n^E(\omega,t)\,\mathcal{S}^{-\omega}\,d\omega. \tag{13}$$

The Mellin transform of PDE (5) yields

$$\frac{\partial \widetilde{P}_n^E(\omega,t)}{\partial t} + \frac{1}{2}n^2\sigma^2\left[\omega^2 + \omega(1-\kappa_2) - \kappa_1\right]\widetilde{P}_n^E(\omega,t) = 0 \tag{14}$$

where

$$\kappa_2 = \frac{n-1}{n} + \frac{2(r-q)}{n\sigma^2}$$

and

$$\kappa_1 = \frac{2r}{n^2\sigma^2}.$$

The general solution of this ODE is given by

$$\widetilde{P}^E(\omega,t) = c(\omega)\cdot e^{-\frac{1}{2}n^2\sigma^2\cdot Q(\omega)\cdot t} \tag{15}$$

where

$$Q(\omega) = \omega^2 + \omega(1-\kappa_2) - \kappa_1 \tag{16}$$

and $c(\omega)$ a constant depending on the boundary conditions. Now, the terminal condition (7) gives

$$c(\omega) = \widetilde{\theta}(\omega, t) \cdot e^{\frac{1}{2}n^2\sigma^2 \cdot Q(\omega)\cdot T} \tag{17}$$

where

$$\widetilde{\theta}(\omega, t) = \widetilde{\theta}(\omega) = X^{\omega+1}\Big(\frac{1}{\omega} - \frac{1}{\omega+1}\Big) \tag{18}$$

is the Mellin transform of the terminal condition (7) and is independent of $n$. Using the inverse Mellin transform we see that the price of a European power put option is given by

$$\begin{aligned} P_n^E(\mathcal{S}, t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widetilde{P}_n^E(\omega, t)\, \mathcal{S}^{-\omega}\, d\omega \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widetilde{\theta}(\omega, t) \cdot e^{\frac{1}{2}n^2\sigma^2\cdot Q(\omega)\cdot(T-t)}\, \mathcal{S}^{-\omega}\, d\omega \end{aligned} \tag{19}$$

with $(\mathcal{S}, t) \in (0, \infty)\times[0, T)$, $c \in (0, \infty)$ a constant, $\{\omega \in \mathbb{C} \,|\, 0 < Re(\omega) < \infty\}$, and $\widetilde{\theta}(\omega, t)$ and $Q(\omega)$ as defined in equations (18) and (16), respectively.
To derive a "BSM-like" formula, we follow Panini and Srivastav (2004) and use the convolution property of Mellin transforms (see Sneddon (1972), p. 276)

$$P_n^E(\mathcal{S}, t) = \int_0^\infty \theta(u) \cdot \phi\Big(\frac{\mathcal{S}}{u}\Big) \cdot \frac{1}{u}\, du \tag{20}$$

where $\phi(u)$ is to be determined. First, observe that for $\beta_1 = \frac{1}{2}n^2\sigma^2(T-t)$ we have

$$\begin{aligned} \frac{1}{2}n^2\sigma^2(T-t)Q(\omega) &= \beta_1\Big[\Big(\omega + \frac{1-\kappa_2}{2}\Big)^2 - \Big(\frac{1-\kappa_2}{2}\Big)^2 - \kappa_1\Big] \\ &= \beta_1\big[(\omega+\beta_2)^2 - \beta_2^2 - \kappa_1\big] \end{aligned} \tag{21}$$

where we have set

$$\beta_2 = \frac{1-\kappa_2}{2}\,.$$

Thus, we can write for the put price

$$P_n^E(\mathcal{S}, t) = e^{-\beta_1(\beta_2^2+\kappa_1)}\, \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widetilde{\theta}(\omega, t) \cdot e^{\beta_1(\omega+\beta_2)^2}\, \mathcal{S}^{-\omega}\, d\omega. \tag{22}$$

Now, $\widetilde{\phi}(\omega)$ is the Mellin transform of

$$e^{\beta_1(\omega+\beta_2)^2} = \int_0^\infty \phi(\mathcal{S})\,\mathcal{S}^{\omega-1}\,d\mathcal{S}. \tag{23}$$

Using the transformation (see Erdelyi et al. (1954))

$$e^{\theta\omega^2} = \int_0^\infty \frac{1}{2\sqrt{\pi\theta}} e^{-\frac{(\ln\mathcal{S})^2}{4\theta}} \mathcal{S}^{\omega-1}\,d\mathcal{S} \quad , \; Re(\theta) \geq 0$$

we get

$$\phi(\mathcal{S}) = \phi(\mathcal{S},t) = \frac{\mathcal{S}^{\beta_2}}{n\sigma\sqrt{2\pi(T-t)}} e^{-\frac{1}{2}\left(\frac{\ln\mathcal{S}}{n\sigma\sqrt{T-t}}\right)^2}. \tag{24}$$

The European power put price can therefore be expressed as

$$\begin{aligned} P_n^E(\mathcal{S},t) &= \frac{e^{-\beta_1(\beta_2^2+\kappa_1)}}{n\sigma\sqrt{2\pi(T-t)}} \int_0^X (X-u)\Big(\frac{\mathcal{S}}{u}\Big)^{\beta_2} e^{-\frac{1}{2}\left(\frac{\ln\frac{\mathcal{S}}{u}}{n\sigma\sqrt{T-t}}\right)^2} \cdot \frac{1}{u}\,du \\ &= \frac{e^{-\beta_1(\beta_2^2+\kappa_1)}}{n\sigma\sqrt{2\pi(T-t)}} \cdot X \cdot \mathcal{S}^{\beta_2} \int_0^X \frac{1}{u^{\beta_2+1}} e^{-\frac{1}{2}\left(\frac{\ln\frac{\mathcal{S}}{u}}{n\sigma\sqrt{T-t}}\right)^2} du \\ &\quad - \frac{e^{-\beta_1(\beta_2^2+\kappa_1)}}{n\sigma\sqrt{2\pi(T-t)}} \cdot \mathcal{S}^{\beta_2} \int_0^X \frac{1}{u^{\beta_2}} e^{-\frac{1}{2}\left(\frac{\ln\frac{\mathcal{S}}{u}}{n\sigma\sqrt{T-t}}\right)^2} du \end{aligned} \tag{25}$$

with

$$\beta_1 = \frac{1}{2} n^2\sigma^2(T-t)\,,$$

$$\beta_2 = \frac{1-\kappa_2}{2}\,,$$

and

$$\kappa_2 = \frac{n-1}{n} + \frac{2(r-q)}{n\sigma^2}\,.$$

To evaluate the first integral use the new variable

$$\gamma = \frac{1}{n\sigma\sqrt{T-t}}\Big(\ln\Big(\frac{\mathcal{S}}{u}\Big) - \beta_2 n^2\sigma^2(T-t)\Big).$$

For the second integral use the slightly different transformation

$$\gamma = \frac{1}{n\sigma\sqrt{T-t}}\Big(\ln\Big(\frac{\mathcal{S}}{u}\Big) - (\beta_2 - 1)n^2\sigma^2(T-t)\Big).$$

Finally, the first part of (25) is determined as

$$X\, e^{-r(T-t)}\, N(-d_{2,n}(\mathcal{S}, X, T))$$

where

$$d_{2,n}(\mathcal{S}, X, T) = \frac{\ln\frac{\mathcal{S}}{X} + n\big(r - q - \frac{1}{2}\sigma^2\big)(T-t)}{n\,\sigma\,(T-t)}. \tag{26}$$

The second integral is evaluated using the transformation suggested above and the result is

$$-e^{((n-1)r - nq + \frac{1}{2}n(n-1)\sigma^2)(T-t)}\, \mathcal{S}\, N(-d_{1,n}(\mathcal{S}, X, T))$$

where

$$d_{1,n}(\mathcal{S}, X, T) = \frac{\ln\frac{\mathcal{S}}{X} + n\big(r - q + (n - \frac{1}{2})\sigma^2\big)(T-t)}{n\,\sigma\,(T-t)}. \tag{27}$$

The price of a power put option is therefore given by

$$P_n^E(\mathcal{S}, t) = X\, e^{-r(T-t)}\, N(-d_{2,n}) - e^{((n-1)r - nq + \frac{1}{2}n(n-1)\sigma^2)(T-t)}\, \mathcal{S}\, N(-d_{1,n}) \tag{28}$$

with $\mathcal{S} = S^n$, and $d_{1,n}$ and $d_{2,n}$ given by equations (27) and (26), respectively.

# 2 The American Put Option

Henceforth, we fix $n = 1$ and focus our considerations on plain vanilla options on dividend paying stocks, where we assume the dividend yield to be paid continuously at the rate $q$.

The main difference between European and American options is that an American option can be exercised by its holder at any time before and including expiry. This early exercise feature makes the pricing (and hedging) of American-styled derivatives mathematically challenging, and created a great field of research throughout the last three decades. While considerable progress has been made, no completely satisfying analytic solution has been

found, except in very few cases.[1]
The pricing of American options can be seen under several mathematical aspects, leading to different but equivalent mathematical formulations of the problem. The most prominent are

- Free boundary formulation
- Early exercise premium formulation
- Integral equation formulation
- Optimal stopping formulation
- Linear complementarity formulation
- Primal-dual formulation
- Viscosity solution formulation.

For a detailed survey of the different formulations the reader is referred to Firth (2005). As indicated above, the early exercise feature creates a free boundary problem. The free boundary is given by the critical stock price $S_t^* = S^*(t)$ which subdivides the domain $(0, \infty) \times [0, T)$ into a continuation region and an exercise region. At any time $t \in [0, T]$ it is optimal to exercise the option prematurely and receive the payoff $X - S(t)$ if $0 < S(t) \leq S^*(t)$. On the other hand, it is optimal to hold the option if $S^*(t) < S(t) < \infty$. Then the option price is the solution to the Black-Scholes-Merton PDE. Following Kwok (1998) we extend the domain of the Black-Scholes-Merton PDE by setting $P^A(S, t) = X - S(t)$ for $S(t) < S^*(t)$. Then $P^A = P^A(S, t)$ satisfies the non-homogeneous PDE:

$$\frac{\partial P^A}{\partial t} + (r - q)\, S\, \frac{\partial P^A}{\partial S} + \frac{1}{2}\sigma^2\, S^2\, \frac{\partial^2 P^A}{\partial S^2} - rP^A = f \tag{29}$$

[1] For example, the perpetual American put option problem was separately solved by McKean (1965) and Merton (1973). Samuelson (1965) derived a closed form expression for the perpetual American warrant. McKean (1965) presented a first solution to the free boundary problem inherent in American option pricing. His form is a valid mathematical representation, however, it allows no economic interpretation for the early exercise premium. Merton (1973) showed that the American call option price on a non-dividend-paying stock equals its European counterpart, since the early exercise premium is worthless.

with

$$f = f(S,t) = \begin{cases} -\,rX + qS\,, & \text{if } \; 0 < S \leq S^*(t) \\ 0\,, & \text{if } \; S > S^*(t) \end{cases} \tag{30}$$

on $(0,\infty) \times [0,T)$. Furthermore, we have the boundary conditions

$$\lim_{S\to\infty} P^A(S,t) = 0 \qquad \text{on} \quad [0,T)\,, \tag{31}$$

$$P^A(S,T) = \theta(S) = (X - S_T)^+ \qquad \text{on} \quad [0,\infty) \tag{32}$$

and

$$P^A(0,t) = X \qquad \text{on} \quad [0,T). \tag{33}$$

Arbitrage arguments show that the option's price must also satisfy the "smooth pasting conditions" at $S^*(t)$ (see Wilmott et al. (1993)):

$$P^A(S^*,t) = X - S^* \qquad \text{and} \qquad \left.\frac{\partial P^A}{\partial S}\right|_{S=S^*} = -1. \tag{34}$$

The Mellin transform of (29) is given by

$$\frac{\partial \widetilde{P}^A(\omega,t)}{\partial t} + \frac{1}{2}\sigma^2\big[\omega^2 + \omega(1-\kappa_2) - \kappa_1\big]\,\widetilde{P}^A(\omega,t) = \widetilde{f}(\omega,t) \tag{35}$$

where

$$\kappa_2 = \frac{2(r-q)}{\sigma^2}\,, \qquad \kappa_1 = \frac{2r}{\sigma^2}\,,$$

and

$$\begin{aligned} \widetilde{f}(\omega,t) &= \int_0^\infty f(S,t)\,S^{\omega-1}\,dS \\ &= -\frac{rX}{\omega}(S^*(t))^\omega + \frac{q}{\omega+1}(S^*(t))^{\omega+1}. \end{aligned} \tag{36}$$

The general solution to this non-homogeneous ODE is given by

$$\begin{aligned}
\widetilde{P}^A(\omega,t) &= c(\omega)e^{-\frac{1}{2}\sigma^2\cdot Q(\omega)\cdot t} \\
&\quad + \int_t^T \frac{rX}{\omega}(S^*(x))^{\omega} e^{\frac{1}{2}\sigma^2\cdot Q(\omega)\cdot(x-t)}\,dx \\
&\quad - \int_t^T \frac{q}{\omega+1}(S^*(x))^{\omega+1} e^{\frac{1}{2}\sigma^2\cdot Q(\omega)\cdot(x-t)}\,dx \\
&= \widetilde{\theta}(\omega)e^{\frac{1}{2}\sigma^2\cdot Q(\omega)\cdot(T-t)} \\
&\quad + \int_t^T \frac{rX}{\omega}(S^*(x))^{\omega} e^{\frac{1}{2}\sigma^2\cdot Q(\omega)\cdot(x-t)}\,dx \\
&\quad - \int_t^T \frac{q}{\omega+1}(S^*(x))^{\omega+1} e^{\frac{1}{2}\sigma^2\cdot Q(\omega)\cdot(x-t)}\,dx \qquad (37)
\end{aligned}$$

where $Q(\omega)$ is defined in equation (16) and $\widetilde{\theta}(\omega)$ is the terminal condition given in equation (18). Once again, the Mellin inversion of (37) yields

$$\begin{aligned}
&P^A(S,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widetilde{\theta}(\omega)\cdot e^{\frac{1}{2}\sigma^2\cdot Q(\omega)\cdot(T-t)}\,S^{-\omega}\,d\omega \\
&+\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T \frac{rX}{\omega}\Big(\frac{S}{S^*(x)}\Big)^{-\omega} e^{\frac{1}{2}\sigma^2\cdot Q(\omega)\cdot(x-t)}\,dxd\omega \\
&-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T \frac{qS^*(x)}{\omega+1}\Big(\frac{S}{S^*(x)}\Big)^{-\omega} e^{\frac{1}{2}\sigma^2\cdot Q(\omega)\cdot(x-t)}\,dxd\omega. \qquad (38)
\end{aligned}$$

Now, observe that the first term in equation (38) is the European put price from (19) and the last two terms capture the early exercise premium. Therefore, we finally arrive at

$$\begin{aligned}
&P^A(S,t) = P^E(S,t) \\
&+\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T \frac{rX}{\omega}\Big(\frac{S}{S^*(x)}\Big)^{-\omega} e^{\frac{1}{2}\sigma^2\cdot Q(\omega)\cdot(x-t)}\,dxd\omega \\
&-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T \frac{qS^*(x)}{\omega+1}\Big(\frac{S}{S^*(x)}\Big)^{-\omega} e^{\frac{1}{2}\sigma^2\cdot Q(\omega)\cdot(x-t)}\,dxd\omega \qquad (39)
\end{aligned}$$

where $(S,t)\in(0,\infty)\times[0,T)$, $c\in(0,\infty)$, $\{\omega\in\mathbb{C}\,|\,0<Re(\omega)<\infty\}$, and

$$Q(\omega) = \omega^2 + \omega(1-\kappa_2) - \kappa_1$$

with

$$\kappa_2 = \frac{2(r-q)}{\sigma^2}\,, \qquad \kappa_1 = \frac{2r}{\sigma^2}.$$

The free boundary is given by

$$\begin{aligned}
& X - S^*(t) = P^E(S^*(t), t) \\
& + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T rX\frac{1}{\omega}\left(\frac{S^*(t)}{S^*(x)}\right)^{-\omega} e^{\frac{1}{2}\sigma^2\cdot Q(\omega)\cdot(x-t)}\,dxd\omega \\
& - \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T \frac{qS^*(x)}{\omega+1}\left(\frac{S^*(t)}{S^*(x)}\right)^{-\omega} e^{\frac{1}{2}\sigma^2\cdot Q(\omega)\cdot(x-t)}\,dxd\omega. \qquad (40)
\end{aligned}$$

We point out that equation (40) can be used to recover an additional result derived by Kim (1990) regarding the optimal exercise price of American put options at expiry:

$$\lim_{t\to T} S^*(t) = \min\left(X, \frac{r}{q}X\right). \qquad (41)$$

Proof: If we change the time variable in equation (40), $t \mapsto \tau = T - t$, we obtain

$$\begin{aligned}
& X - S^*(\tau) = P^E(S^*(\tau), \tau) \\
& + \frac{1}{2\pi i}\int_0^\tau \int_{c-i\infty}^{c+i\infty} rX\frac{1}{\omega}\left(\frac{S^*(\tau)}{S^*(x)}\right)^{-\omega} e^{\frac{1}{2}\sigma^2\cdot Q(\omega)\cdot(\tau-x)}\,d\omega dx \\
& - \frac{1}{2\pi i}\int_0^\tau \int_{c-i\infty}^{c+i\infty} \frac{qS^*(x)}{\omega+1}\left(\frac{S^*(\tau)}{S^*(x)}\right)^{-\omega} e^{\frac{1}{2}\sigma^2\cdot Q(\omega)\cdot(\tau-x)}\,d\omega dx\,. \qquad (42)
\end{aligned}$$

A simple factorization and rearrangement produces the following implicit equation for $S^*(\tau)$:

$$\frac{S^*(\tau)}{X} = \frac{1 - e^{-r\tau} + e^{-r\tau}\,N(d_2(S^*(\tau), X, \tau)) - r\cdot I_1(\tau)}{1 - e^{-q\tau} + e^{-q\tau}\,N(d_1(S^*(\tau), X, \tau)) - q\cdot I_2(\tau)} \qquad (43)$$

where

$$I_1(\tau) = \frac{1}{2\pi i}\int_0^\tau \int_{c-i\infty}^{c+i\infty} \frac{1}{\omega}\left(\frac{S^*(\tau)}{S^*(x)}\right)^{-\omega} e^{\frac{1}{2}\sigma^2\cdot Q(\omega)\cdot(\tau-x)}\,d\omega dx \qquad (44)$$

and

$$I_2(\tau) = \frac{1}{2\pi i}\int_0^\tau \int_{c-i\infty}^{c+i\infty} \frac{1}{\omega+1}\left(\frac{S^*(\tau)}{S^*(x)}\right)^{-(\omega+1)} e^{\frac{1}{2}\sigma^2\cdot Q(\omega)\cdot(\tau-x)}\,d\omega dx\,. \qquad (45)$$

Notice first that the critical stock price is bounded from above, i.e. $S^*(\tau) \leq X, \forall \tau > 0$ (see for example Jacka (1991), Prop. 2.2.2). To find the value $S^*(0^+) = \lim_{\tau \to 0^+} S^*(\tau)$, in a first step we evaluate the limits involving $d_1$ and $d_2$. We have

$$\lim_{\tau \to 0^+} d_1(S^*(\tau), X, \tau) = \begin{cases} 0\,, & \text{for } \; S^*(0^+) = X \\ -\infty\,, & \text{for } \; S^*(0^+) < X\,. \end{cases}$$

Similarly,

$$\lim_{\tau \to 0^+} d_2(S^*(\tau), X, \tau) = \begin{cases} 0\,, & \text{for } \; S^*(0^+) = X \\ -\infty\,, & \text{for } \; S^*(0^+) < X\,. \end{cases}$$

If

$$\lim_{\tau \to 0^+} S^*(\tau) = X$$

we have

$$\lim_{\tau \to 0^+} N(d_1(S^*(\tau), X, \tau)) = \lim_{\tau \to 0^+} N(d_2(S^*(\tau), X, \tau)) = \frac{1}{2}$$

and

$$\lim_{\tau \to 0^+} \frac{S^*(\tau)}{X} = \frac{\frac{1}{2} - \lim_{\tau \to 0^+} r \cdot I_1(\tau)}{\frac{1}{2} - \lim_{\tau \to 0^+} q \cdot I_2(\tau)}\,. \tag{46}$$

It is easily verified that both expressions $I_1(\tau)$ and $I_2(\tau)$ tend to zero as $\tau \to 0^+$. As a result we have

$$\lim_{\tau \to 0^+} S^*(\tau) = X \tag{47}$$

beeing a possible solution. In the second case where

$$\lim_{\tau \to 0^+} S^*(\tau) < X\,,$$

the implicit equation for $S^*(\tau)$ reads

$$\lim_{\tau \to 0^+} \frac{S^*(\tau)}{X} = \frac{r}{q} \cdot \lim_{\tau \to 0^+} \frac{I_1(\tau)}{I_2(\tau)}\,. \tag{48}$$

But

$$I_1(\tau) = \int_0^\tau \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{\omega} \Big( \frac{S^*(\tau)}{S^*(x)} \Big)^{-\omega} e^{\frac{1}{2}\sigma^2 \cdot Q(\omega) \cdot (\tau - x)}\, d\omega dx$$

and a simple application of the residue theorem (see for example Freitag and Busam (2000)) shows that the inner integral equals

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{\omega}\left(\frac{S^*(\tau)}{S^*(x)}\right)^{-\omega}e^{\frac{1}{2}\sigma^2\cdot Q(\omega)\cdot(\tau-x)}\,d\omega = e^{-r(\tau-x)} \tag{49}$$

and thus

$$I_1(\tau) = \frac{1}{r}\left(1-e^{-r\tau}\right). \tag{50}$$

In the same manner we apply the residue theorem to the second integral to get

$$I_2(\tau) = \frac{1}{q}\left(1-e^{-q\tau}\right). \tag{51}$$

Obviously, the above calculations can be used to prove the limits in the first case, i.e. for $\lim_{\tau\to 0^+} S^*(\tau) = X$, as well.
Putting the results together we arrive at

$$\lim_{\tau\to 0^+}\frac{S^*(\tau)}{X} = \frac{r}{q}\cdot\lim_{\tau\to 0^+}\frac{\frac{1}{r}\left(1-e^{-r\tau}\right)}{\frac{1}{q}\left(1-e^{-q\tau}\right)} = \lim_{\tau\to 0^+}\frac{1-e^{-r\tau}}{1-e^{-q\tau}}. \tag{52}$$

Now, use the rule of d'Hospital to establish the second assertion. Recalling that the result holds only when $S^*(0^+) < X$, it follows that $r < q$. Combining both results confirms Kim's formula. □

# 3 The Equivalence of Integral Representations

In this section we prove explicitly the equivalence of three types of integral representations for American put options[2]. We show the equivalence of the integral representation derived herein, the representation obtained by Kim (1990), Jacka (1991), and Carr et al. (1992).
In particular, we prove that the following three representations for American put options are equivalent:

[2]Chiarella et al. (2004) use the incomplete Fourier transform to survey the integral representations of American call options.

- Representation using Mellin transforms (equation (39))

$$
\begin{aligned}
&P^A(S,t) = P^E(S,t) \\
&+\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T \frac{rX}{\omega}\Big(\frac{S}{S^*(x)}\Big)^{-\omega} e^{\frac{1}{2}\sigma^2\cdot Q(\omega)\cdot(x-t)}\,dxd\omega \\
&-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T \frac{qS^*(x)}{\omega+1}\Big(\frac{S}{S^*(x)}\Big)^{-\omega} e^{\frac{1}{2}\sigma^2\cdot Q(\omega)\cdot(x-t)}\,dxd\omega \qquad (53)
\end{aligned}
$$

with $Q(\omega)$ defined in equation (16).

- Representation obtained by Kim (1990) and Jacka (1991)

$$
\begin{aligned}
P^A(S,\tau) \;=\;& P^E(S,\tau) \\
&+\int_0^\tau rX\,e^{-r(\tau-\xi)}\cdot N(-d_2(S,S^*(\xi),\tau-\xi))\,d\xi \\
&-\int_0^\tau qS\,e^{-q(\tau-\xi)}\cdot N(-d_1(S,S^*(\xi),\tau-\xi))\,d\xi \qquad (54)
\end{aligned}
$$

where $\tau = T-t, S = S(\tau), S \geq S^*(\tau)$, and

$$
d_1(x,y,t) = \frac{\ln\frac{x}{y} + (r-q-\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\,,
$$

$$
d_2(x,y,t) = d_1(x,y,t) - \sigma\sqrt{t}.
$$

- Decomposition derived by Carr et al. (1992))

$$
\begin{aligned}
&P^A(S,\tau) = max(X-S,0) + \frac{1}{2}\sigma^2 S\int_0^\tau \frac{1}{\sigma\sqrt{\tau-\xi}}e^{-q(\tau-\xi)}\cdot N'(-d_1(S,X,\tau-\xi))\,d\xi \\
&+\int_0^\tau rX\,e^{-r(\tau-\xi)}\Big[N(-d_2(S,S^*(\xi),\tau-\xi)) - N(-d_2(S,X,\tau-\xi))\Big]\,d\xi \\
&-\int_0^\tau qS\,e^{-q(\tau-\xi)}\Big[N(-d_1(S,S^*(\xi),\tau-\xi)) - N(-d_1(S,X,\tau-\xi))\Big]\,d\xi \qquad (55)
\end{aligned}
$$

where $\tau = T-t, S = S(\tau), S \geq S^*(\tau)$, and $d_1$ and $d_2$ as above.

Proof: A change of the time variable in the "Mellin representation" $t \mapsto \tau = T - t$ leads to

$$\begin{aligned} & P^A(S,\tau) = P^E(S,\tau) \\ & + \frac{1}{2\pi i}\int_0^\tau \int_{c-i\infty}^{c+i\infty} rX\frac{1}{\omega}\Big(\frac{S}{S^*(x)}\Big)^{-\omega} e^{\frac{1}{2}\sigma^2\cdot Q(\omega)\cdot(\tau-x)}\, d\omega dx \\ & - \frac{1}{2\pi i}\int_0^\tau \int_{c-i\infty}^{c+i\infty} \frac{qS^*(x)}{\omega+1}\Big(\frac{S}{S^*(x)}\Big)^{-\omega} e^{\frac{1}{2}\sigma^2\cdot Q(\omega)\cdot(\tau-x)}\, d\omega dx \end{aligned}$$

or using a more compact form

$$P^A(S,\tau) = P^E(S,\tau) - \int_0^\tau \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widetilde{f}(\omega,x)\cdot\widetilde{\phi}(\omega,x)\cdot S^{-\omega}\, d\omega dx$$

with

$$\widetilde{f}(\omega,x) = -\frac{rX}{\omega}(S^*(x))^{\omega} + \frac{q}{\omega+1}(S^*(x))^{\omega+1}$$

and

$$\widetilde{\phi}(\omega,x) = e^{\frac{1}{2}\sigma^2\cdot Q(\omega)\cdot(\tau-x)}$$

the Mellin transforms of $f(S,x)$ and $\phi(S,x)$, respectively. Using the convolution theorem (see Sneddon (1972), p. 276) we can write

$$P^A(S,\tau) = P^E(S,\tau) - \int_0^\tau \int_0^\infty f(u,x)\cdot\phi\Big(\frac{S}{u},x\Big)\cdot\frac{1}{u}\,du\,dx.$$

Now, from (25) we have

$$P^A(S,\tau) = P^E(S,\tau) - \int_0^\tau h(S,x)\,dx \tag{56}$$

where

$$\begin{aligned} h(S,x) \;=\; & -rX\,e^{-\beta_1(\beta_2^2+\kappa_1)}\frac{S^{\beta_2}}{\sigma\sqrt{2\pi(\tau-x)}}\int_0^{S^*(x)} \frac{1}{u^{\beta_2+1}}\,e^{-\frac{1}{2}\left(\frac{\ln\frac{S}{u}}{\sigma\sqrt{\tau-x}}\right)^2} du \\ & + q\,e^{-\beta_1(\beta_2^2+\kappa_1)}\frac{S^{\beta_2}}{\sigma\sqrt{2\pi(\tau-x)}}\int_0^{S^*(x)} \frac{1}{u^{\beta_2}}\,e^{-\frac{1}{2}\left(\frac{\ln\frac{S}{u}}{\sigma\sqrt{\tau-x}}\right)^2} du\,, \end{aligned} \tag{57}$$

and

$$\beta_1 = \frac{1}{2}\sigma^2\,(\tau-x)\,,$$

$$\beta_2 = \frac{1-\kappa_2}{2}$$
$$\kappa_1 = \frac{2r}{\sigma^2}$$
$$\kappa_2 = \frac{2(r-q)}{\sigma^2}.$$

Transforming variables

$$\gamma = \frac{1}{\sigma\sqrt{\tau - x}}\Big(\ln\left(\frac{S}{u}\right) - \beta\sigma^2(\tau - x)\Big) \tag{58}$$

for the first integral in (57), and

$$\gamma = \frac{1}{\sigma\sqrt{\tau - x}}\Big(\ln\left(\frac{S}{u}\right) - (\beta - 1)\sigma^2(\tau - x)\Big) \tag{59}$$

for the second yields to

$$\begin{aligned} h(S,x) \;&=\; -rX\,e^{-r(\tau-x)} \cdot N(-d_2(S, S^*(x), \tau - x)) \\ &\quad +qS\,e^{-q(\tau-x)} \cdot N(-d_1(S, S^*(x), \tau - x)). \end{aligned} \tag{60}$$

Finally, change the variables from $x$ to $\xi$ and the equivalence of (53) and (54) follows.
For the second equivalence, observe that we can write the European put as

$$\begin{aligned} P^E(S,\tau) \;&=\; X \cdot H(X - S) - X \cdot H(X - S) \\ &\quad +X\,e^{-r\tau}\,N(-d_2(S,X,\tau)) - S\,e^{-q\tau}\,N(-d_1(S,X,\tau)) \end{aligned} \tag{61}$$

where $H(x)$ is the Heaviside step function given by

$$H(x) = \begin{cases} 1\,, & \text{for } \; x > 0 \\ \frac{1}{2}\,, & \text{for } \; x = 0 \\ 0\,, & \text{for } \; x < 0\,. \end{cases} \tag{62}$$

The reason for the appearance of the factor 1/2 at the point of discontinuity will become obvious below. Given the limit result that

$$\lim_{\tau\to 0} d_1(S,X,\tau) = \lim_{\tau\to 0} d_2(S,X,\tau) = \begin{cases} \infty & \text{for } \; S > X \\ 0 & \text{for } \; S = X \\ -\infty & \text{for } \; S < X \end{cases} \tag{63}$$

we can express $P^E(S,\tau)$ as

$$
\begin{aligned}
P^E(S,\tau) &= X \cdot H(X-S) - S\,e^{-q\tau} N(-d_1(S,X,\tau)) + \left[X\,e^{-r\xi} N(-d_2(S,X,\xi))\right]\Big|_0^{\tau} \\
&= X \cdot H(X-S) - S\,e^{-q\tau} N(-d_1(S,X,\tau)) \\
&\quad + X \int_0^{\tau} \Big[e^{-r\xi} N^{'}(-d_2(S,X,\xi)) \cdot \frac{\partial}{\partial \xi}(-d_2(S,X,\xi)) \\
&\quad - r\,e^{-r\xi} N(-d_2(S,X,\xi))\Big]\, d\xi \\
&= X \cdot H(X-S) - S\,e^{-q\tau} N(-d_1(S,X,\tau)) \\
&\quad - r\,X \int_0^{\tau} e^{-r\xi} N(-d_2(S,X,\xi))\, d\xi \\
&\quad + X \int_0^{\tau} e^{-r\xi} N^{'}(-d_2(S,X,\xi)) \frac{\partial}{\partial \xi}\big[-(d_1(S,X,\xi) - \sigma\sqrt{\xi})\big]\, d\xi \\
&= X \cdot H(X-S) - S\,e^{-q\tau} N(-d_1(S,X,\tau)) \\
&\quad - r\,X \int_0^{\tau} e^{-r\xi} N(-d_2(S,X,\xi))\, d\xi \\
&\quad + X \int_0^{\tau} e^{-r\xi} N^{'}(-d_2(S,X,\xi)) \frac{\partial}{\partial \xi}(-d_1(S,X,\xi))\, d\xi \\
&\quad + X \int_0^{\tau} e^{-r\xi} N^{'}(-d_2(S,X,\xi)) \frac{\sigma}{2\sqrt{\xi}}\, d\xi\,, \qquad (64)
\end{aligned}
$$

where $N^{'}(x) = n(x)$ is the density function of a standard normal distributed random variable $x$. Now, we have

$$N^{'}(-d_2(S,X,\xi)) = N^{'}(d_2(S,X,\xi)) \qquad (65)$$

$$N^{'}(-d_1(S,X,\xi)) = N^{'}(d_1(S,X,\xi)) \qquad (66)$$

and

$$S\,e^{-q\xi}\,N^{'}(d_1(S,X,\xi)) = X\,e^{-r\xi}\,N^{'}(d_2(S,X,\xi))\,. \qquad (67)$$

Thus,

$$
\begin{aligned}
P^E(S,\tau) &= (X-S)\cdot H(X-S) + S\cdot H(X-S) - S\,e^{-q\tau}N(-d_1(S,X,\tau)) \\
&\quad -r\,X\int_0^\tau e^{-r\xi}N(-d_2(S,X,\xi))\,d\xi \\
&\quad +S\int_0^\tau e^{-q\xi}N'(-d_1(S,X,\xi))\frac{\partial}{\partial\xi}(-d_1(S,X,\xi))\,d\xi \\
&\quad +S\int_0^\tau e^{-q\xi}N'(-d_1(S,X,\xi))\frac{\sigma}{2\sqrt{\xi}}\,d\xi \\
&= max(X-S,0) \\
&\quad +\frac{1}{2}\sigma^2\,S\int_0^\tau e^{-q\xi}\,N'(-d_1(S,X,\xi))\frac{1}{\sigma\sqrt{\xi}}\,d\xi \\
&\quad -r\,X\int_0^\tau e^{-r\xi}N(-d_2(S,X,\xi))\,d\xi \\
&\quad -S\Big[e^{-q\tau}N(-d_1(S,X,\tau)) - H(X-S) \\
&\quad -\int_0^\tau e^{-q\xi}N'(-d_1(S,X,\xi))\frac{\partial}{\partial\xi}(-d_1(S,X,\xi))\Big]\,d\xi. \qquad (68)
\end{aligned}
$$

Finally,

$$
\begin{aligned}
P^E(S,\tau) &= max(X-S,0) + \frac{1}{2}\sigma^2\,S\int_0^\tau e^{-q\xi}\cdot N'(-d_1(S,X,\xi))\frac{1}{\sigma\sqrt{\xi}}\,d\xi \\
&\quad -r\,X\int_0^\tau e^{-r\xi}N(-d_2(S,X,\xi))\,d\xi \\
&\quad -S\Big[\Big[e^{-q\xi}N(-d_1(S,X,\xi))\Big]\Big|_0^\tau \\
&\quad -\int_0^\tau e^{-q\xi}N'(-d_1(S,X,\xi))\frac{\partial}{\partial\xi}(-d_1(S,X,\xi))\,d\xi\Big]. \qquad (69)
\end{aligned}
$$

Changing the integration variable from $\xi$ to $\tau-\xi$ gives

$$
\begin{aligned}
P^E(S,\tau) &= max(X-S,0) + \frac{1}{2}\sigma^2\,S\int_0^\tau \frac{1}{\sigma\sqrt{\tau-\xi}}e^{-q(\tau-\xi)}\cdot N'(-d_1(S,X,\tau-\xi))\,d\xi \\
&\quad -r\,X\int_0^\tau e^{-r(\tau-\xi)}N(-d_2(S,X,\tau-\xi))\,d\xi \\
&\quad +qS\int_0^\tau e^{-q(\tau-\xi)}N(-d_1(S,X,\tau-\xi))\,d\xi\,. \qquad (70)
\end{aligned}
$$

Now, substitute this expression into Kim's representation and rearrange terms. $\square$

Remark: We found a second proof for the first equivalence that makes no explicit use of the convolution theorem. Our starting point is equation (5.1.16) in Panini and Srivastav (2004). Including dividends it is straightforward to extend the result and show that equation (53) is equivalent to

$$P^A(S,\tau) = P^E(S,\tau) + \int_0^\tau I_1(\xi)\,d\xi - \int_0^\tau I_2(\xi)\,d\xi\,, \tag{71}$$

where

$$I_1(\xi) = \frac{rX}{2\sqrt{\pi\bar{\zeta}}}\,e^{-r\xi}\,e^{-\bar{\zeta}c^2+\beta c}\int_0^\infty e^{-cx}\,e^{-\frac{(\beta-x)^2}{4\bar{\zeta}}}\,dx \tag{72}$$

and

$$I_2(\xi) = \frac{qS^*(\tau-\xi)}{2\sqrt{\pi\bar{\zeta}}}\,e^{-r\xi}\,e^{-\bar{\zeta}c^2+\beta c}\int_0^\infty e^{-(c+1)x}\,e^{-\frac{(\beta-x)^2}{4\bar{\zeta}}}\,dx \tag{73}$$

with $\xi = \tau - x$, $\bar{\zeta} = \frac{1}{2}\sigma^2\xi$ and

$$\beta = \bar{\zeta}(2c+1-\kappa_2) - \ln\left(\frac{S(\tau)}{S^*(\tau-\xi)}\right). \tag{74}$$

Now, the integrals can be expressed as

$$I_1(\xi) = \frac{rX}{2\sqrt{\pi\bar{\zeta}}}\,e^{-r\xi}\,e^{-\bar{\zeta}c^2+\beta c}\,e^{-\frac{b}{4\bar{\zeta}}}\int_0^\infty e^{-\frac{x^2}{4\bar{\zeta}}}\,e^{-\frac{a_1 x}{4\bar{\zeta}}}\,dx \tag{75}$$

and

$$I_2(\xi) = \frac{qS^*(\tau-\xi)}{2\sqrt{\pi\bar{\zeta}}}\,e^{-r\xi}\,e^{-\bar{\zeta}c^2+\beta c}\,e^{-\frac{b}{4\bar{\zeta}}}\int_0^\infty e^{-\frac{x^2}{4\bar{\zeta}}}\,e^{-\frac{a_2 x}{4\bar{\zeta}}}\,dx \tag{76}$$

with

$$a_1 = 2\bar{\zeta}(\kappa_2-1) + 2\ln\left(\frac{S(\tau)}{S^*(\tau-\xi)}\right),$$

$$a_2 = 2\bar{\zeta}(\kappa_2+1) + 2\ln\left(\frac{S(\tau)}{S^*(\tau-\xi)}\right),$$

and $b = \beta^2$. From Gradshteyn and Ryzhik (2007), p.336, we have

$$\int_0^\infty \exp\Big(-\frac{x^2}{4\beta}-\gamma x\Big)dx = \sqrt{\pi\beta}\,\exp\left(\beta\gamma^2\right)\Big[1-\Phi(\gamma\sqrt{\beta})\Big] \tag{77}$$

for $Re(\beta) > 0$ and where $\Phi(x)$ denotes the error function at $x$. After simplifying the expressions for $I_1(\xi)$ and $I_2(\xi)$ become, respectively:

$$\begin{aligned} I_1(\xi) &= \frac{rX}{2}\Big(\frac{S^*(\tau-\xi)}{S(\tau)}\Big)^c e^{-r\xi}\, e^{\bar{\zeta}c(c+1-\kappa_2)}\, e^{\frac{1}{4\bar{\zeta}}\left(\frac{a_1^2}{4}-b\right)}\Big[1-\Phi\Big(\frac{a_1}{4\bar{\bar{\zeta}}}\Big)\Big] \\ &= \frac{rX}{2}\Big(\frac{S^*(\tau-\xi)}{S(\tau)}\Big)^c\Big(\frac{S^*(\tau-\xi)}{S(\tau)}\Big)^{-c} e^{-r\xi}\Big[1-\Phi\Big(\frac{a_1}{4\bar{\bar{\zeta}}}\Big)\Big] \\ &= \frac{rX}{2}\, e^{-r\xi}\Big[1-\Phi\Big(\frac{a_1}{4\bar{\bar{\zeta}}}\Big)\Big] \end{aligned} \tag{78}$$

and

$$\begin{aligned} I_2(\xi) &= \frac{qS^*(\tau-\xi)}{2}\Big(\frac{S^*(\tau-\xi)}{S(\tau)}\Big)^c e^{-r\xi}\, e^{\bar{\zeta}c(c+1-\kappa_2)}\, e^{\frac{1}{4\bar{\zeta}}\left(\frac{a_2^2}{4}-b\right)}\Big[1-\Phi\Big(\frac{a_2}{4\bar{\bar{\zeta}}}\Big)\Big] \\ &= \frac{qS^*(\tau-\xi)}{2}\Big(\frac{S^*(\tau-\xi)}{S(\tau)}\Big)^c\Big(\frac{S(\tau)}{S^*(\tau-\xi)}\Big)^{c+1} e^{-r\xi}\, e^{\frac{1}{4}\bar{\zeta}[(\kappa_2+1)^2-(\kappa_2-1)^2]}\Big[1-\Phi\Big(\frac{a_2}{4\bar{\bar{\zeta}}}\Big)\Big] \\ &= \frac{qS(\tau)}{2}\, e^{-q\xi}\Big[1-\Phi\Big(\frac{a_2}{4\bar{\bar{\zeta}}}\Big)\Big]\,. \end{aligned} \tag{79}$$

Using the connection between the error function and the cumulative standard normal distribution function

$$\Phi(x) = 2\,N(\sqrt{2}\,x)-1 \tag{80}$$

we have, respectively:

$$I_1(\xi) = rX\,e^{-r\xi}\,N\Big(-\frac{a_1}{2}\frac{1}{\sqrt{2\bar{\bar{\zeta}}}}\Big) \tag{81}$$

and

$$I_2(\xi) = qS(\tau)\,e^{-q\xi}\,N\Big(-\frac{a_2}{2}\frac{1}{\sqrt{2\bar{\bar{\zeta}}}}\Big)\,. \tag{82}$$

Finally, observe that

$$-\frac{a_1}{2}\frac{1}{\sqrt{2\bar{\bar{\zeta}}}} = \frac{1}{2}\sigma\,(1-\kappa_2)\sqrt{\xi}-\frac{1}{\sigma\sqrt{\xi}}\ln\Big(\frac{S(\tau)}{S^*(\tau-\xi)}\Big) \tag{83}$$

and

$$-\frac{a_2}{2}\frac{1}{\sqrt{2\zeta}} = -\frac{1}{2}\sigma\,(1+\kappa_2)\sqrt{\xi} - \frac{1}{\sigma\sqrt{\xi}}\ln\Big(\frac{S(\tau)}{S^*(\tau-\xi)}\Big) \tag{84}$$

and Kim's integral representation follows immediately by inserting the corresponding expressions into equations (81), (82) and thereafter (71). This completes the second proof.

# 4 Perpetual American Puts and Mellin Transforms

In this section, we show how to use the Mellin transform approach to derive closed-form solutions for perpetual American put options. We extend the results obtained by Panini and Srivastav (2005) to dividend-paying stocks. First, observe that the roots of $Q(\omega)$ defined in (16) are given by

$$\omega_1 = \frac{\kappa_2-1}{2} - \frac{\sqrt{(\kappa_2-1)^2+4\kappa_1}}{2}$$

and

$$\omega_2 = \frac{\kappa_2-1}{2} + \frac{\sqrt{(\kappa_2-1)^2+4\kappa_1}}{2}.$$

Thus, we have

$$Q(\omega) = (\omega-\omega_1)(\omega-\omega_2)$$

with $\omega_1 \leq -1 < 0 < \omega_2 \leq \kappa_1$. The limiting cases $\omega_1 = -1$ and $\omega_2 = \kappa_1$ are special roots for $q = 0$. We will determine the unknown critical stock price $S^*(t)$ using the second smooth pasting condition from equation (34).
Notice, that for the valuation formula (39) to hold as $T \to \infty$, it is necessary that $Re(Q(\omega)) < 0$, i.e. $0 < Re(\omega) < \omega_2$.
Using the second smooth pasting condition in (34) we obtain as $T \to \infty$

$$-1 = \frac{\partial P^A}{\partial S}\Big|_{S=S^*} = \frac{\partial P^E}{\partial S}\Big|_{S=S^*} + \frac{\partial P_1}{\partial S}\Big|_{S=S^*} + \frac{\partial P_2}{\partial S}\Big|_{S=S^*} \tag{85}$$

where the free boundary $S^* = S^*_\infty$ is now independent of time, and $P_1$ and $P_2$ denote the second and third term in the valuation formula (39), respectively. Now, the delta of a European put option on a dividend-paying stock is determined as

$$\frac{\partial P^E}{\partial S} = -e^{-q(T-t)}\cdot N(-d_1(S,X,T)) \tag{86}$$

with

$$d_1(S,X,T) = \frac{\ln\frac{S}{X} + (r - q + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

It follows that as $T \to \infty$

$$\frac{\partial P^E}{\partial S}\Big|_{S=S^*_\infty} \to 0. \tag{87}$$

Now consider the $P_1$ term. The limit $T \to \infty$ gives

$$\begin{aligned}\frac{\partial P_1}{\partial S} &= -\frac{rX}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{S}\Big(\int_t^\infty\Big(\frac{S}{S^*_\infty}\Big)^{-\omega}e^{\frac{1}{2}\sigma^2\cdot Q(\omega)\cdot(x-t)}\,dx\Big)d\omega\\ &= -\frac{rX}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{S}\Big(\frac{S}{S^*_\infty}\Big)^{-\omega}\Big[\frac{1}{\frac{1}{2}\sigma^2\cdot Q(\omega)}e^{\frac{1}{2}\sigma^2\cdot Q(\omega)\cdot(x-t)}\Big|_t^\infty\Big]d\omega\end{aligned} \tag{88}$$

Therefore,

$$\frac{\partial P_1}{\partial S}\Big|_{S=S^*_\infty} = \frac{\kappa_1 X}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{S^*_\infty}\cdot\frac{1}{(\omega-\omega_1)(\omega-\omega_2)}\,d\omega. \tag{89}$$

Similarly, the $P_2$ term is determined as

$$\begin{aligned}\frac{\partial P_2}{\partial S} &= \frac{q}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Big(\int_t^\infty\frac{\omega}{\omega+1}\Big(\frac{S}{S^*_\infty}\Big)^{-(\omega+1)}e^{\frac{1}{2}\sigma^2\cdot Q(\omega)\cdot(x-t)}\,dx\Big)d\omega\\ &= -\frac{2q}{\sigma^2}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\omega}{\omega+1}\Big(\frac{S}{S^*_\infty}\Big)^{-(\omega+1)}\cdot\frac{1}{Q(\omega)}\,d\omega.\end{aligned} \tag{90}$$

Therefore,

$$\frac{\partial P_2}{\partial S}\Big|_{S=S^*_\infty} = (\kappa_2-\kappa_1)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\omega}{(\omega+1)(\omega-\omega_1)(\omega-\omega_2)}\,d\omega. \tag{91}$$

To evaluate both integrals we consider the following integration path (or contour path) in the complex plane:

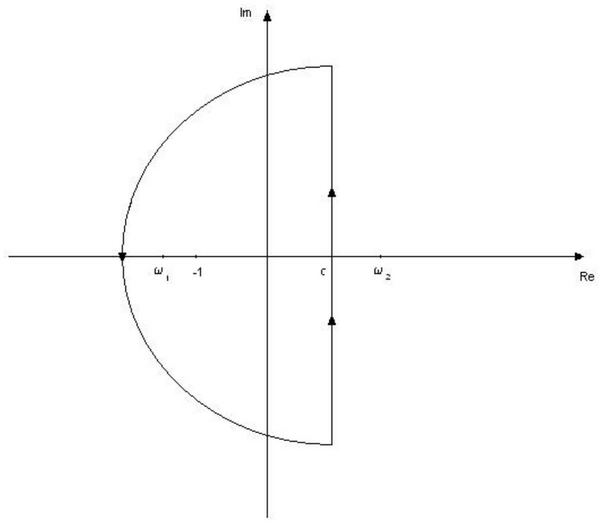


Figure 1: Integration path for the critical stock price.

An application of the residue theorem (see Freitag and Busam (2000)) gives

$$\frac{\partial P_1}{\partial S}\Big|_{S=S^*_\infty} = \kappa_1 \, X \frac{1}{S^*_\infty(\omega_1 - \omega_2)} \tag{92}$$

and

$$\frac{\partial P_2}{\partial S}\Big|_{S=S^*_\infty} = (\kappa_2 - \kappa_1)\Big[\frac{\omega_1}{(\omega_1+1)(\omega_1-\omega_2)} - \frac{1}{(\omega_1+1)(\omega_2+1)}\Big]. \tag{93}$$

Finally, we get for the critical stock price[3]

$$\begin{aligned} S^*_\infty &= \frac{\kappa_1(\omega_1+1)}{\omega_1(\kappa_1-\kappa_2)} \, X \\ &= \frac{\omega_2}{\omega_2+1} \, X. \end{aligned} \tag{94}$$

[3]Merton's result (1973)

$$S^*_\infty = \frac{\kappa_1}{\kappa_1+1} \, X$$

is obtained as a special case for $q = 0$.

Observe that since $S^*(t)$ is non-decreasing in $t$ (see Kim (1990), p. 560, Jacka (1991), Proposition 2.2.2 for a reference) we have the lower and upper bounds for $S^*(t)$ given by

$$S^*_\infty \leq S^*(t) \leq S^*(T) = \min\left(X, \frac{r}{q}X\right) \qquad \forall t \in [0,T]. \tag{95}$$

The price for the perpetual American put is given by

$$\begin{aligned} P^A_\infty(S,t) &= -\frac{\kappa_1 X}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{S}{S^*_\infty}\right)^{-\omega}\frac{1}{\omega(\omega-\omega_1)(\omega-\omega_2)}\,d\omega \\ &+\frac{2q}{\sigma^2}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}S^*_\infty\left(\frac{S}{S^*_\infty}\right)^{-\omega}\frac{1}{(\omega+1)(\omega-\omega_1)(\omega-\omega_2)}\,d\omega. \end{aligned} \tag{96}$$

Once again, we apply the residue theorem to determine the first integral as

$$\left(\frac{S}{S^*_\infty}\right)^{-\omega_2}\frac{\kappa_1 X}{\omega_2(\omega_2-\omega_1)}. \tag{97}$$

The second integral is evaluated as

$$-\frac{2q}{\sigma^2}\left(\frac{S}{S^*_\infty}\right)^{-\omega_2}\frac{S^*_\infty}{(\omega_2+1)(\omega_2-\omega_1)}. \tag{98}$$

Thus, we finally get for the perpetual American put price

$$\begin{aligned} P^A_\infty(S,t) &= \left(\frac{S}{S^*_\infty}\right)^{-\omega_2}\frac{\kappa_1 X}{\omega_2(\omega_2-\omega_1)} - \frac{2q}{\sigma^2}\left(\frac{S}{S^*_\infty}\right)^{-\omega_2}\frac{S^*_\infty}{(\omega_2+1)(\omega_2-\omega_1)} \\ &= \left(\frac{S}{S^*_\infty}\right)^{-\omega_2}\frac{X}{\omega_2+1} \\ &= \left(\frac{S}{S^*_\infty}\right)^{-\omega_2}(X - S^*_\infty)\,, \qquad \text{for } S > S^*_\infty\,. \end{aligned} \tag{99}$$

# 5 Conclusion

We have extended a technique proposed by Panini (2004) and Panini and Srivastav (2004) and derived a Black-Scholes-Merton-like valuation formula for European power put options on dividend-paying stocks. Focusing on American plain vanilla put options, we used the Mellin transform approach to derive the valuation formulas for the option's price and its free boundary. To place emphasis on the generality of our results, we have proved the equivalence of the valuation formula derived herein to the meanwhile classical results presented by Kim (1990), Jacka (1991), and Carr et al. (1992). Additionally, we have recovered interesting properties of American options using the new method.

The analysis presented in this contribution is based on the Black-Scholes-Merton framework. The valuation formulas for the American put option and its free boundary may be used to derive new approximations for the American put option. Also, the method can be extended to pricing more complex European- and American-styled derivatives, like European and American basket options (see Panini and Srivastav (2004)) or path-dependent options. Extensions are also possible to other stochastic price processes, stochastic volatility models, and jump diffusions.

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