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On the risk-neutral valuation of life insurance contracts with numerical methods in view

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Abstract In recent years, market-consistent valuation approaches have gained an increasing importance for insurance companies. This has triggered an increasing interest among practitioners and academics, and a number of studies on such valuation approaches have been published. However, despite the fact that many models are structurally similar, no generic modeling setup has been proposed so far.

In this paper, we present such a generic model for the valuation of life insurance contracts and embedded options. Furthermore, we describe various numerical valuation approaches within our generic setup. We particularly focus on contracts containing early exercise features since these present (numerically) challenging valuation problems.

Based on an example of participating life insurance contracts, we illustrate the different approaches and compare their efficiency in a simple and a generalized Black-Scholes setup, respectively. Moreover, we study the impact of the considered early exercise feature on our example contract and analyze the influence of model risk by additionally introducing an exponential Lévy model.

Keywords Life insurance · Risk-neutral valuation · Embedded options · Bermudan options · Nested simulations · PDE methods · Least-squares Monte Carlo

1 Introduction

In recent years, market-consistent valuation approaches for life insurance contracts have gained an increasing practical importance.

In 2001, the European Union initiated the "Solvency II" project to revise and extend current solvency requirements, the central intention being the incorporation of a risk-based framework for adequate risk management and option pricing techniques for insurance valuation. Furthermore, in 2004 the International Accounting Standards Board issued the new International Financial Reporting Standard (IFRS) 4 (Phase I) which is also concerned with the valuation of life insurance liabilities. Although Phase I just constitutes a temporary standard, experts agree

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that fair valuation will play a major role in the future permanent standard (Phase II), which is expected to be in place by 2010 (see [29]).

However, so far, most insurance companies only have little knowledge about risk-neutral valuation techniques and, hence, mostly rely on simple models and brute force Monte Carlo simulations. This is mainly due to the fact that predominant software solutions (e.g. Moses, Prophet, or ALM.IT) were initially designed for deterministic forecasts of an insurer's trade accounts and only subsequently extended to perform Monte Carlo simulations. In academic literature, on the other hand, there exists a variety of different articles on the valuation of life insurance contracts. There is, however, no generic modeling setup and there are hardly any detailed comparisons of different numerical valuation approaches. Moreover, some studies do not apply methods from financial mathematics appropriately to the valuation of life insurance products (e.g. questionable worst-case scenarios in [23] and [33]; see Sec. 3.1 below for details).

The objective of this article is to formalize the valuation problem for insurance contracts in a general way and to provide a survey on concrete valuation methodologies. We particularly focus on the valuation of insurance contracts containing early-exercise features, such as surrender options, withdrawal guarantees, or options to change the premium payment method. While almost all insurance contracts contain such features, insurers usually do not include these in their price and risk management computations even though they may add considerably to the value of the contract.

The remainder of the text is organized as follows: In Sect. 2, we present a generic model for life insurance contracts. Subsequently, in Sect. 3, we describe different numerical valuation approaches. Based on an example of participating life insurance contracts, we carry out numerical experiments in Sect. 4. Similarly to most prior literature on the valuation of life contingencies from a mathematical finance perspective, we initially assume a general Black-Scholes framework. We compare the obtained results as well as the efficiency of the different approaches and analyze the influence of a surrender option on our example contract. However, as is well-known from various empirical studies, several statistical properties of financial market data are not described adequately by Brownian motion and, in general, guarantees and options will increase in value under more suitable models. Therefore, we analyze the model risk for our valuation problem by introducing an exponential Lévy model and comparing the obtained results for our example to those from the Black-Scholes setup. We find that the qualitative impact of the model choice depends on the particular model parameters, i.e. that there exist (realistic) parameter choices for which either model yields higher values, which is in contrast to the general notion. Finally, the last section summarizes our main results.

2 Generic contracts

We assume that financial agents can trade continuously in a frictionless and arbitrage-free financial market with finite time horizon T.[1] Let $\left(\Omega^F, \mathscr{F}^F, \mathscr{Q}^F, \mathbb{F}^F = (\mathscr{F}_t^F)_{t\in[0,T]}\right)$ be a complete, filtered probability space, where $\mathscr{Q}^F$ is a pricing measure and $\mathbb{F}^F$ is assumed to satisfy the usual conditions. In this probability space, we introduce the q_1-dimensional vector $(Y_t^F)_{t\in[0,T]} = (Y_t^{F,(1)}, \ldots, Y_t^{F,(q_1)})_{t\in[0,T]}$ of locally bounded, adapted Lévy processes, and call it the state process of the financial market.

Within this market, we assume the existence of a locally risk-free asset $(B_t)_{t\in[0,T]}$ with $B_t = \exp\left\{\int_0^t r_u\,du\right\}$, where $r_t = r\left(t, Y_t^F\right)$ is the short rate. Moreover, we allow for $n \in \mathbb{N}$ other risky assets $(A_t^{(i)})_{t\in[0,T]}$, $1 \le i \le n$, traded in the market with[2]

$$A_t^{(i)} = A^{(i)}(t, Y_t^F), \qquad 1 \le i \le n.$$

[1] In actuarial modeling, it is common to assume a so-called limiting age meaning that a finite time horizon naturally suffices in view of our objective.

[2] We denote by $A_t^{(i)}$ only assets which are not solely subject to interest rate risk, e.g. stocks or immovable property. The price processes of non-defaultable bonds traded in the market are implicitly given by the short rate process.

In order to include the mortality component, we fix another probability space $\left(\Omega^M, \mathscr{G}^M, \mathscr{P}^M\right)$ and a homogenous population of x-year old individuals at inception. Similar to [12,20], we assume that a q_2-dimensional vector of locally bounded, adapted Lévy processes $(Y_t^M)_{t\in[0,T]} = (Y_t^{M,(q_1+1)}, \ldots, Y_t^{M,(q)})_{t\in[0,T]}$, $q = q_1 + q_2$, on $\left(\Omega^M, \mathscr{G}^M, \mathscr{P}^M\right)$ is given. Now let $\mu(\cdot,\cdot) : \mathbb{R}_+ \times \mathbb{R}^{q_2} \to \mathbb{R}_+$ be a positive continuous function and define the time of death T_x of an individual as the first jump time of a Cox process with intensity $\left(\mu(x+t, Y_t^M)\right)_{t\in[0,T]}$, i.e.

$$T_x = \inf\left\{ t \,\middle|\, \int_0^t \mu(x+s, Y_s^M)\, ds \geq E \right\},$$

where E is a unit-exponentially distributed random variable independent of $(Y_t^M)_{t\in[0,T]}$ and mutually independent for different individuals. Also, define subfiltrations $\mathbb{F}^M = \left(\mathscr{F}_t^M\right)_{t\in[0,T]}$ and $\mathscr{H} = (\mathscr{H}_t)_{t\in[0,T]}$ as the augmented subfiltrations generated by $(Y_t^M)_{t\in[0,T]}$ and $(\mathbb{1}_{\{T_x\leq t\}})_{t\in[0,T]}$, respectively. We set $\mathscr{G}_t^M = \mathscr{F}_t^M \vee \mathscr{H}_t$ and $\mathbb{G}^M = \left(\mathscr{G}_t^M\right)_{t\in[0,T]}$.

Insurance contracts can now be considered on the combined filtered probability space

$$\left(\Omega, \mathscr{G}, \mathscr{Q}, \mathbb{G} = (\mathscr{G}_t)_{t\in[0,T]}\right),$$

where $\Omega = \Omega^M \times \Omega^F$, $\mathscr{G} = \mathscr{F}^F \vee \mathscr{G}^M$, $\mathscr{G}_t = \mathscr{F}_t^F \vee \mathscr{G}_t^M$, and $\mathscr{Q} = \mathscr{Q}^F \otimes \mathscr{P}^M$ is the product measure of independent *financial* and *biometric* events. We further let $\mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}$, where $\mathscr{F}_t = \mathscr{F}_t^F \vee \mathscr{F}_t^M$. A slight extension of the results by Lando ([34], Prop. 3.1) now yields that for an $\mathscr{F}_t$-measurable payment C_t, we have for $u \leq t$[3]

$$\begin{aligned} &B_u\, \mathbb{E}^{\mathscr{Q}}\left[B_t^{-1} C_t\, \mathbb{1}_{\{T_x>t\}} \middle| \mathscr{G}_u \right] \\ &= \mathbb{1}_{\{T_x>u\}}\, B_u\, \mathbb{E}^{\mathscr{Q}}\left[B_t^{-1} C_t \exp\left\{ -\int_u^t \mu(x+s, Y_s)\, ds \right\} \middle| \mathscr{F}_u \right], \end{aligned}$$

which can be readily applied to the valuation of insurance contracts. For notational convenience, we introduce the *realized survival probabilities*

$$\begin{aligned} {}_tp_x^{(t)} &:= \mathbb{E}^{\mathscr{Q}}\left[\mathbb{1}_{\{T_x>t\}} \middle| \mathscr{F}_t \vee \mathscr{H}_0 \right] = \exp\left\{ -\int_0^t \mu(x+s, Y_s^M)\, ds \right\}, \\ {}_{t-u}p_{x+u}^{(t)} &:= \frac{{}_tp_x^{(t)}}{{}_up_x^{(u)}} = \exp\left\{ -\int_u^t \mu(x+s, Y_s^M)\, ds \right\}, 0 \leq u \leq t, \end{aligned}$$

as well as the corresponding one-year realized death probability

$$q_{x+t-1}^{(t)} = 1 - p_{x+t-1}^{(t)} = 1 - {}_1p_{x+t-1}^{(t)}.$$

While $\mathscr{Q}^F$ is specified as some given equivalent martingale measure, there is some flexibility in the choice of $\mathscr{P}^M$. In a complete financial market, i.e. if $\mathscr{Q}^F$ is unique, with a deterministic evolution of mortality and under the assumption of risk-neutrality of an insurer with respect to mortality risk (cf. [2]), Møller ([39]) points out that if $\mathscr{P}^M$ denotes the physical measure, $\mathscr{Q}$ as defined above is the so-called Minimal Martingale Measure (see [44]). This result can be extended to incomplete market settings when choosing $\mathscr{Q}^F$ to be the Minimal Martingale Measure for the financial market (see e.g. [42]). However, in [22], Delbaen and Schachermayer quote "*the use of mortality tables in insurance*" as "*an example that this technique* [change of measure] *in fact has a long history*" in actuarial sciences, indicating that the assumption of risk-neutrality with respect to mortality risk may not be adequate. Then, the measure choice depends on the availability of adequate mortality-linked securities traded in the market (see [21] for a particular example and [13] for a survey on mortality-linked securities) and/or the insurer's

[3] In what follows, we write $\mu(x+t, Y_t) := \mu(x+t, Y_t^M)$ and $r(t, Y_t) := r\left(t, Y_t^F\right)$, where $(Y_t)_{t\in[0,T]} := \left(Y_t^F, Y_t^M\right)_{t\in[0,T]}$ is the *state process*.

preferences (see [10] or [40]). In what follows, we assume that the insurer has chosen a measure $\mathscr{P}^M$ for valuation purposes, so that a particular choice for the valuation measure $\mathscr{Q}$ is given.

To obtain a model for our generic life insurance contract, we analyze the way such contracts are administrated in an insurance company. An important observation is that cash flows, such as premium payments, benefit payments, or withdrawals, are often not generated continuously but only at discrete points in time. For the sake of simplicity, we assume that these discrete points in time are the anniversaries $\nu \in \{0,\dots,T\}$ of the contract.

Therefore, the value V_ν of some life insurance contract at time ν by the risk-neutral valuation formula is:

$$V_\nu = B_\nu \sum_{\mu=\nu}^{T} \mathbb{E}^{\mathscr{Q}}\left[B_\mu^{-1} C_\mu \middle| \mathscr{F}_\nu\right],$$

where C_μ is the cash flow at time μ, $0 \le \mu \le T$.

Since the value of our contract under the assumption that the insured in view is alive at time t only depends on the evolution of mortality and the financial market, and as these again only depend on the evolution of $(Y_s)_{s\in[0,t]}$, we can write:

$$V_t = \tilde{V}(t, Y_s, s \in [0,t]).$$

But saving the entire history of the state process is cumbersome and, fortunately, unnecessary: Within the bookkeeping system of an insurance company, a life insurance contract is usually managed (or represented) by several accounts saving relevant information about the history of the contract, such as the account value, the cash-surrender value, the current death benefit, etc. Therefore, we introduce $m \in \mathbb{N}$ virtual accounts $(D_t)_{t\in[0,T]} = (D_t^{(1)},\dots,D_t^{(m)})_{t\in[0,T]}$, the so-called *state variables*, to store the relevant history. In this way, we obtain a Markovian structure since the relevant information about the past at time t is contained in (Y_t, D_t). Furthermore, we observe that these virtual accounts are usually not updated continuously, but adjustments, such as crediting interest or guarantee updates, are often only made at certain key dates. Also, policyholders' decisions, such as withdrawals, surrenders, or changes to the insured amount, often only become effective at predetermined dates. To simplify notation, we again assume that these dates are the anniversaries of the contract. Therefore, to determine the contract value at time t if the insured in view is alive, it is sufficient to know the current state of the stochastic drivers and the values of the state variables at $\lfloor t \rfloor = \max\{n \in \mathbb{N} | n \le t\}$, i.e. the value of the generic life insurance contract can be described as follows:

$$V_t = V(t, Y_t, D_t) = V(t, Y_t, D_{\lfloor t \rfloor}), \quad t \in [0,T].$$

We denote the set of all possible values of (Y_t, D_t) by Θ_t.

Most models for the market-consistent valuation of life insurance contracts presented in literature fit into this framework. For example, in [14], Brennan and Schwartz price equity-linked life insurance policies with an asset value guarantee. Here, the value of the contract at time t only depends on the value of the underlying asset which is modeled by a geometric Brownian motion, i.e. we have an insurance contract which can be described by a one-dimensional state processes and no state variables.

Participating life insurance contracts are characterized by an interest rate guarantee and some bonus distribution rules, which provide the possibility for the policyholder to participate in the earnings of the insurance company. Furthermore, these contracts usually contain a surrender option, i.e. the policyholder is allowed to lapse the contract at time $\nu \in \{1,\dots,T\}$. Such contracts are, for instance, considered in [15,25,38]. All these models can be represented within our framework. Moreover, the setup is not restricted to the valuation of one "entire" insurance contracts, but, on one hand, it can also be used to determine the value of multiple contracts at a time (see [26]) or, on the other hand, parts of insurance contracts, such as embedded options. Clearly, we can determine the value of an arbitrary option by computing the value of the same

contract in- and excluding that option, ceteris paribus. The difference in value of the two contracts is the value of the option. For example, the generic model can be used in this way to analyze paid-up and resumption options within participating life insurance contracts such as in [23] or exchange options such as in [41]. Alternatively, the value of a certain embedded option may be determined by isolating the cash-flows corresponding to the considered guarantee (see [8]).

In [9], Bauer et al. consider Variable Annuities including so-called Guaranteed Minimum Death Benefits (GMDBs) and/or Guaranteed Minimum Living Benefits (GMLBs). Again, their model structure fits into our framework. Bauer et al. use one stochastic driver to model the asset process and eight state variables to specify the contract.

3 A survey of numerical methods

The contracts under consideration are often relatively complex, path-dependent derivatives, and in most cases, analytical solutions to the valuation problems cannot be found. Hence, one has to resort to numerical methods. In this section, we present different possibilities to numerically tackle these valuation problems.

3.1 Monte Carlo simulations

Monte Carlo simulations are a simple and yet useful approach to the valuation of insurance contracts provided that the considered contract does not contain any early exercise features, i.e. policyholders cannot change or (partially) surrender the contract during its term. We call such contracts *European*.

In this case, we can simulate K paths of the state process $(Y_t)_{t\in[0,T]}$, say $(Y_t^{(k)})_{t\in[0,T]}$, $k = 1,\dots,K$, and compute the numéraire process, the realized survival probabilities as well as the state variables at each anniversary of the contract. Then, the "value" of the contract for path k, $V_0^{(k)}$, $1 \le k \le K$, is given as the sum of discounted cash flows in path k, and, by the Law of Large Numbers (LLN), the risk-neutral value of the contract at inception V_0 may be estimated by the sample mean for K sufficiently large.

However, if the contract includes early exercise features, the problem is more delicate since the value of the option or guarantee in view depends on the policyholder's actions.

The question of how to incorporate policyholder behavior does not have a straight-forward answer. From an economic perspective, one could assume that policyholders will maximize their personal utility, which would lead to a non-trivial control problem similar as for the valuation of employee stock options (see [16], [28], or references therein). However, the assumption of homogenous policyholders does not seem proximate. In particular, the implied assertion that options within contracts with the same characteristics are exercised at the same time does not hold in practice, and it is not clear how to include heterogeneity among policyholders.

Alternatively, it is possible to assess the exercise behavior empirically. For such an approach, our framework provides a convenient setup: A regression of historical exercise probabilities on the state variables could yield coherent estimates for future exercise behavior. However, aside from problems with retrieving suitable data, when adopting this methodology insurers will face the risk of systematically changing policyholder behavior, which has had severe consequences in the past. For example, the UK-based mutual life insurer *Equitable Life*, the world's oldest life insurance company, was closed to new business due to solvency problems arising from a misjudgment of policyholders' exercise behavior of guaranteed annuity options within individual pension policies.

Hence, in compliance with ideas from the new solvency and financial reporting regulations, we take a different approach and consider a valuation of embedded options as if they were traded in the financial market. While from the insurer's perspective, the resulting "value" may exceed the actual or realized value, it is a unique "supervaluation" in the sense that policyholders have

the possibility (or the *option*) to exercise optimally with respect to the financial value of their contract. Moreover, the resulting "superhedging" strategy for attainable embedded options is unique in the same sense.

In order to determine this value, we need to solve an optimal stopping problem. To illustrate it, let us consider a life insurance contract with surrender option. The option is most valuable if the policyholder behaves "financially rational", i.e.

$$V_0 = \sup_{\tau \in \Upsilon_0} \mathbb{E}^{\mathcal{Q}} \left[(B_\tau)^{-1} \, {}_\tau p_x^{(\tau)} C(\tau, Y_\tau, D_\tau) + \sum_{\nu=0}^{\tau-1} (B_{\nu+1})^{-1} \, {}_\nu p_x^{(\nu)} q_{x+\nu}^{(\nu+1)} f(\nu+1, Y_{\nu+1}, D_\nu) \middle| \mathscr{F}_0 \right]$$

where $C(\nu, y_\nu, d_\nu)$ is the surrender value at time ν and $f(\nu, y_\nu, d_{\nu-1})$ is the death benefit upon death in $[\nu-1, \nu)$ if the state process and the state variables take values y_ν and d_ν ($d_{\nu-1}$ at $t = \nu - 1$), respectively, and Υ_0 denotes the set of all stopping times in $\{1, \ldots, T\}$. Clearly, maximizing the exercise value over each single sample path and computing the sample mean, as e.g. pursued in [23] and [33] for different types of contracts, overestimates this value.

To determine a Monte Carlo approximation of this value, which we refer to as *the contract value* in what follows, we need to rely on so-called *"nested simulations"*. We do not allow for surrenders at inception of the contract, so we define $C(0, y_0, d_0) := 0$. By the Bellman equation (see e.g. [11] for an introduction to dynamic programming and optimal control) the contract value at time ν, $\nu \in \{0, \ldots, T-1\}$, is the maximum of the exercise value and the continuation value. The latter is the weighted sum of the discounted expectation of the contract value given the information $(y_\nu, d_\nu) \in \Theta_\nu$, i.e.

$$\begin{aligned} & V(\nu, y_\nu, d_\nu) \\ = \max \Big\{ & C(\nu, y_\nu, d_\nu), B_\nu \, \mathbb{E}^{\mathcal{Q}} \left[B_{\nu+1}^{-1} \, p_{x+\nu}^{(\nu+1)} V(\nu+1, Y_{\nu+1}, D_{\nu+1}) \middle| (Y_\nu, D_\nu) = (y_\nu, d_\nu) \right] \\ & + B_\nu \, \mathbb{E}^{\mathcal{Q}} \left[B_{\nu+1}^{-1} \, q_{x+\nu}^{(\nu+1)} f(\nu+1, Y_{\nu+1}, D_\nu) \middle| (Y_\nu, D_\nu) = (y_\nu, d_\nu) \right] \Big\}. \end{aligned}$$

We now generate a tree with T time steps and $b \in \mathbb{N}$ branches out of each node. We start with initial value Y_0 and then generate b independent successors $Y_1^1, \ldots, Y_1^b$. From each node we generate again b successors and so on. To simplify notation, let $X_\nu^{l_1 l_2 \ldots l_\nu} = (Y_\nu^{l_1 l_2 \ldots l_\nu}, D_\nu^{l_1 l_2 \ldots l_\nu})$. With this notation, an estimator for V_ν, $\nu \in \{0, \ldots, T\}$, at node $Y_\nu^{l_1 \ldots l_\nu}$ is

$$\hat{V}_\nu^{l_1 \ldots l_\nu} := \begin{cases} \max \Big\{ C(\nu, X_\nu^{l_1 \ldots l_\nu}), \frac{B_\nu^{l_1 \ldots l_\nu}}{b} \sum_{l=1}^b (B_{\nu+1}^{l_1 \ldots l_\nu l})^{-1} \, p_{x+\nu}^{l_1 \ldots l_\nu l} \, \hat{V}_{\nu+1}^{l_1 \ldots l_\nu l} \\ \quad + \frac{B_\nu^{l_1 \ldots l_\nu}}{b} \sum_{l=1}^b (B_{\nu+1}^{l_1 \ldots l_\nu l})^{-1} q_{x+\nu}^{l_1 \ldots l_\nu l} \, f(\nu+1, Y_{\nu+1}^{l_1 \ldots l_\nu l}, D_\nu^{l_1 \ldots l_\nu}) \Big\} & , \nu \in \{0, \ldots, T-1\} \\ C(\nu, X_\nu^{l_1 \ldots l_\nu}) & , \nu = T \end{cases}$$

where $B_\nu^{l_1 \ldots l_\nu}$ and $p_{x+\nu-1}^{l_1 \ldots l_\nu}$ ($q_{x+\nu-1}^{l_1 \ldots l_\nu}$) denote the values of the bank account and the one-year survival (death) probability at $t = \nu$ in sample path $(Y_0, Y_1^{l_1}, \ldots, Y_\nu^{l_1 \ldots l_\nu})$, respectively.

Using K replications of the tree, we determine the sample mean $\bar{V}_0(K, b)$, and by the LLN we get $\bar{V}_0(K, b) \to \mathbb{E}^{\mathcal{Q}}[\hat{V}_0]$ as $K \to \infty$ almost surely. Hence, fixing b, we can construct an asymptotically valid $(1-\delta)$ confidence interval for $\mathbb{E}^{\mathcal{Q}}\left[\hat{V}_0\right]$. But this estimator for the risk-neutral value $V_0 = V(0, X_0)$ is biased high (see [24], p. 433), i.e.

$$\mathbb{E}^{\mathcal{Q}}\left[\hat{V}_0\right] \geq V(0, X_0),$$

where, in general, we have a sharp inequality. However, under some integrability conditions the estimator is asymptotically unbiased and hence, we can reduce the bias by increasing the number of branches b in each node.

In order to construct a confidence interval for the contract value $V(0, X_0)$, following [24], we introduce a second estimator. It differs from the estimator introduced above in that all but one replication are used to decide whether to exercise the option or not, whereas in case exercising

is not decided to be optimal, the last replication is employed. More precisely, we define for $\nu \in \{0,\dots,T-1\}$

$$\hat{v}_{\nu u}^{l_1\dots l_\nu} := \begin{cases} C(\nu, X_\nu^{l_1\dots l_\nu}) \text{ , if } & \frac{B_\nu^{l_1\dots l_\nu}}{b-1} \sum_{l=1,l\neq u}^{b} (B_{\nu+1}^{l_1\dots l_\nu l})^{-1} p_{x+\nu}^{l_1\dots l_\nu l} \hat{v}_{\nu+1}^{l_1\dots l_\nu l} \\ & + \frac{B_\nu^{l_1\dots l_\nu}}{b} \sum_{l=1,l\neq u}^{b} (B_{\nu+1}^{l_1\dots l_\nu l})^{-1} q_{x+\nu}^{l_1\dots l_\nu l} f(\nu+1, Y_{\nu+1}^{l_1\dots l_\nu l}, D_\nu^{l_1\dots l_\nu}) \\ & \leq C(\nu, X_\nu^{l_1\dots l_\nu}) \\ B_\nu^{l_1\dots l_\nu} (B_{\nu+1}^{l_1\dots l_\nu u})^{-1} p_{x+\nu}^{l_1\dots l_\nu u} \hat{v}_{\nu+1}^{l_1\dots l_\nu u} & \text{, otherwise} \\ + B_\nu^{l_1\dots l_\nu} (B_{\nu+1}^{l_1\dots l_\nu u})^{-1} q_{x+\nu}^{l_1\dots l_\nu u} f(\nu+1, Y_{\nu+1}^{l_1\dots l_\nu u}, D_\nu^{l_1\dots l_\nu}) & \end{cases}.$$

Then, averaging over all b possibilities of leaving out one replication, we obtain

$$\hat{v}_\nu^{l_1\dots l_\nu} := \begin{cases} \frac{1}{b}\sum_{u=1}^{b} \hat{v}_{\nu u}^{l_1\dots l_\nu}. & ,\nu \in \{0,\dots,T-1\} \\ C(\nu, X_\nu^{l_1\dots l_\nu}) & ,\nu = T \end{cases}.$$

Again using K replications of the tree, we obtain a second estimator for the contract value by the sample mean $\bar{v}_0(K,b)$ which is now biased low, and we can construct a second asymptotically valid $(1-\delta)$ confidence interval, this time for $\mathbb{E}^{\mathscr{Q}}[\hat{v}_0]$.

Taking the upper bound from the first confidence interval and the lower bound from the second one, we obtain an asymptotically valid $(1-\delta)$-confidence interval for V_0:

$$\left(\bar{v}_0(K,b) - z_{\frac{\delta}{2}} \frac{s_v(K,b)}{\sqrt{K}}, \bar{V}_0(K,b) + z_{\frac{\delta}{2}} \frac{s_V(K,b)}{\sqrt{K}}\right),$$

where $z_{\frac{\delta}{2}}$ is the $\frac{\delta}{2}$-quantile of the standard normal distribution. $s_V(K,b)$ and $s_v(K,b)$ denote the sample standard deviations of the K replications for the two estimators.

The drawback for non-European insurance contracts is that the number of necessary simulation steps increases exponentially in time. Since insurance contracts are usually long-term investments, the computation of the value using "nested simulations" is therefore rather extensive and time-consuming. Moreover, for different options with several (or even infinitely many) admissible actions, such as withdrawals within variable annuities, the complexity will increase dramatically.

3.2 A PDE approach

P(I)DE methods bear certain advantages in comparison to the Monte Carlo approach. On one hand, they include the calculation of certain sensitivities (the so-called "Greeks"; see e.g. [27], Chapter 13), which are useful for hedging purposes. On the other hand, they often present a more efficient method for the valuation of non-European insurance contracts. The idea for this algorithm is based on solving the corresponding control problem on a discretized state space and, for special insurance contracts, was originally presented in [25] and [45].

The value V_t of our generic insurance contract depends on t, the state process Y_t, and the state variables D_t. By arbitrage arguments, it can be shown that the value function is almost surely left-continuous at all policy anniversaries $\nu \in \{1,\dots,T\}$ (see [45]), i.e.

$$V_t \overset{a.s.}{\to} V_\nu \quad \text{as } t \to \nu\text{- }.$$

Between two policy anniversaries, the evolution of the value function V depends on t and Y_t only, since the state variables remain constant. Consequently, given the state variables $D_{\nu-1} =$

$d_{\nu-1}$ and the value function at some time $t_0 \in [\nu-1,\nu)$, V_{t_0}, depending on whether the insured in view is alive at time t_0 or not, the value function on the interval $[\nu-1,t_0]$ is

$$V_t = \underbrace{\mathbb{E}^{\mathscr{Q}}\left[\exp\left\{-\int_t^{t_0} r_s+\mu(x+s,Y_s)\,ds\right\}V_{t_0}(\text{“alive”})\,\middle|\,\mathscr{F}_t\right]}_{=:F(t,Y_t)}$$
$$+\mathbb{E}^{\mathscr{Q}}\left[\exp\left\{-\int_t^{t_0} r_s\,ds\right\}\left(1-\exp\left\{-\int_t^{t_0}\mu(x+s,Y_s)\,ds\right\}\right)V_{t_0}(\text{“dead”})\,\middle|\,\mathscr{F}_t\right].$$

Applying Itô's formula for Lévy processes (see e.g. Prop. 8.18 in [18]), we obtain

$$dF(t,Y_t) = k(t,Y_{t-},F(t,Y_{t-}))\,dt + d\,M_t,$$

with drift term $k(t,Y_{t-},F(t,Y_{t-}))$ and local martingale part M_t. Since, by construction,

$$\left(\exp\left(-\int_0^t r_s+\mu(x+s,Y_s)\,ds\right)F(t,Y_t)\right)_{t\in[\nu-1,t_0]}$$

is a $\mathscr{Q}$-local martingale, under certain technical conditions, the drift needs to be zero $\mathscr{Q}$-almost surely. This is a standard technique akin to the well-known Feynman-Kac formula. We thus obtain a P(I)DE for the function $F:(t,y)\mapsto F(t,y)$:

$$-r(t,y)F(t,y)-\mu(t,y)F(t,y)+k(t,y,F(t,y))=0 \tag{1}$$

with terminal condition

$$F(t_0,y) = V(t_0,y,d_{\nu-1},\text{“alive”}).$$

Since the value function at maturity T is known for all $(y,d)\in\Theta_T$, we can construct a backwards algorithm to obtain the value function on the whole interval $[0,T]$:

For $t=T-u$, $u\in\{1,\dots,T\}$, evaluate the P(I)DE (1) for "all possible" d_{T-u} with terminal condition

$$F(T-u+1,y_{T-u+1}) = \sup_{\varphi_{T-u+1}\in\Phi_{T-u+1}} V(T-u+1,h_{\varphi_{T-u+1}}(y_{T-u+1},d_{T-u})), \tag{2}$$

where Φ_ν is the set of all options that may be exercised at $t=\nu$, and $h_{\varphi_\nu}:\Theta_\nu\to\Theta_\nu$ denotes the transition function which describes how the state variables change at $t=\nu$ if option φ_ν is exercised. Then, set

$$\begin{aligned}&V(T-u,y_{T-u},d_{T-u})\\ &= F(T-u,y_{T-u})\\ &\quad+\mathbb{E}^{\mathscr{Q}}\left[\exp\left\{-\int_{T-u}^{T-u+1} r_s\,ds\right\}q_{x+T-u}^{(T-u+1)}f(T-u+1,y_{T-u+1},d_{t-u})\,\middle|\,\mathscr{F}_{T-u}\right].\end{aligned}$$

In order to apply the algorithm, the state spaces Θ_ν, $\nu=0,\dots,T$, are discretized and interpolation methods are employed to determine the right-hand sides of (2) if the arguments are off the grid. In particular, it is necessary to solve the P(I)DE for all state variables on the grid separately, so that the efficiency of the algorithm highly depends on the evaluation of the P(I)DEs.

In [45], the classical Black-Scholes model and a deterministic evolution of mortality are assumed. In this case, the resulting PDE is the well-known Black-Scholes PDE, which can be transformed into a one-dimensional heat equation, from which an integral representation can be derived when the terminal condition is given. If a modified Black-Scholes model with stochastic interest rates is assumed as in [47], the situation gets more complex: The PDE is no longer analytically solvable and one has to resort to numerical methods.

For a general exponential Lévy process driving the financial market, PIDEs with non-local integral terms must be solved. Several numerical methods have been proposed for the solution, e.g. based on finite difference schemes (see e.g. [4,19]), based on wavelet methods ([37]), or Fourier transform based methods ([30,36]).

While, in comparison to Monte Carlo simulations, the complexity does not increase exponentially in time, the high number of P(I)DEs needing to be solved may slow down the algorithm considerably.

3.3 A least-squares Monte Carlo approach

The least-squares Monte Carlo (LSM) approach by Longstaff and Schwartz ([35]) was originally presented for pricing American options but has recently also been applied to the valuation of insurance contracts (see e.g. [5] and [41]). We present the algorithm for life insurance contracts with a simple surrender option. Subsequently, problems for the application of this method to more general embedded options as well as potential solutions are identified.

As pointed out by Clément et al. ([17]), the algorithm consists of two different types of approximations. Within the first approximation step, the continuation value function is replaced by a finite linear combination of certain "basis" functions. As the second approximation, Monte Carlo simulations and least-squares regression are employed to approximate the linear combination given in step one.

Again, let $C(\nu, y_\nu, d_\nu)$ be the payoff at time $\nu \in \{1,\dots,T\}$ if the stochastic drivers and the state variables take values y_ν and d_ν, respectively, and the option is exercised at this time. Furthermore, let $C(s,\nu,y_s,d_s)$, $\nu < s \leq T$ describe the cash flow at time s given the state process y_s and the state variables d_s, conditional on the option not being exercised prior or at time ν, and the policyholder following the optimal strategy according to the algorithm at all possible exercise dates $s \in \{\nu+1,\dots,T\}$ assuming that the policyholder is alive at time ν.

The continuation value $g(\nu, Y_\nu, D_\nu)$ at time ν is the sum of all expected future cash flows discounted back to time ν under the information given at time ν, i.e.

$$g(\nu, Y_\nu, D_\nu) = \mathbb{E}^{\mathcal{Q}}\left[\sum_{s=\nu+1}^{T} \exp\left\{-\int_\nu^s r_u du\right\} C(s,\nu,Y_s,D_s)\,\middle|\,\mathcal{F}_\nu\right].$$

To determine the optimal strategy at time $t=\nu$, i.e. to solve the optimal stopping problem, it is now sufficient to compare the surrender value to the continuation value and choose the greater one. Hence, we obtain the following discrete valued stopping time $\tau = \tau_1$:

$$\begin{cases} \tau_T = T \\ \tau_\nu = \nu \mathbb{1}_{\{C(\nu,Y_\nu,D_\nu) \geq g(\nu,Y_\nu,D_\nu)\}} + \tau_{\nu+1}\mathbb{1}_{\{C(\nu,Y_\nu,D_\nu) < g(\nu,Y_\nu,D_\nu)\}}, \quad 1 \leq \nu \leq T-1 \end{cases} \tag{3}$$

and the contract value can be described as

$$\begin{aligned} V(0,Y_0,D_0) = {} & \mathbb{E}^{\mathcal{Q}}\left[\exp\left(-\int_0^\tau r_u du\right) {}_\tau p_x^{(\tau)} C(\tau, Y_\tau, D_\tau)\,\middle|\,\mathcal{F}_0\right] \\ & + \mathbb{E}^{\mathcal{Q}}\left[\sum_{\nu=0}^{\tau-1} \exp\left\{-\int_0^{\nu+1} r_u du\right\} {}_\nu p_x^{(\nu)} q_{x+\nu}^{(\nu+1)} f(\nu+1, Y_{\nu+1}, D_\nu)\,\middle|\,\mathcal{F}_0\right]. \end{aligned} \tag{4}$$

Following [17], we assume that the sequence $(L_j(Y_\nu,D_\nu))_{j\geq 0}$ is total in $L^2(\sigma((Y_\nu,D_\nu)))$, $\nu = 1,\dots,T-1$, and satisfies a linear independence condition (cf. condition A_1 and A_2 in [17]), such that $g(\nu,Y_\nu,D_\nu)$ can be expressed as

$$g(\nu,Y_\nu,D_\nu) = \sum_{j=0}^{\infty} \alpha_j(\nu) L_j(Y_\nu,D_\nu), \tag{5}$$

for some $\alpha_j(\nu) \in \mathbb{R}$, $j \in \mathbb{N}\cup\{0\}$.

For the first approximation, we replace the infinite sum in (5) by a finite sum of the first J basis function. We call this approximation $g^{(J)}$.

Similarly to (3) and (4), we can now define a new stopping time $\tau^{(J)}$ and a first approximation $V^{(J)}$ for the contract value by replacing g by $g^{(J)}$.

However, in general the coefficients $(\alpha_j(\nu))_{j=0}^{J-1}$ are not known and need to be estimated. We use $K \in \mathbb{N}$ replications of the path (Y_ν, D_ν), $0 \leq \nu \leq T$, and denote them by $(Y_\nu^{(k)}, D_\nu^{(k)})$, $1 \leq k \leq K$. The coefficients are then determined by a least-squares regression. We assume that the optimal strategy for $s \geq \nu+1$ is already known and hence, for each replication the cash flows

$C(s,\nu,Y_s^{(k)},D_s^{(k)})$, $s\in\{\nu+1,\dots,T\}$, are known. Under these assumptions, the least-squares estimator for the coefficients is

$$\hat{\alpha}^{(K)}(\nu)=\arg\min_{\alpha(\nu)\in\mathbb{R}^J}\left\{\sum_{k=1}^{K}\left[\sum_{s=\nu+1}^{T}\exp\left(-\int_\nu^s r_u^{(k)}\,du\right)C(s,\nu,Y_s^{(k)},D_s^{(k)})\right.\right.$$
$$\left.\left.-\sum_{j=0}^{J-1}\alpha_j(\nu)L_j(Y_\nu^{(k)},D_\nu^{(k)})\right]^2\right\}.$$

Replacing $(\alpha_j(\nu))_{j=0}^{J-1}$ by $(\hat{\alpha}_j^{(K)}(\nu))_{j=0}^{J-1}$, we obtain the second approximation $g^{(J,K)}$ and again, we define the stopping time $\tau^{(J,K)}$ and another approximation $V^{(J,K)}$ of the value function by replacing g by $g^{(J,K)}$.

With the help of these approximations, we can now construct a valuation algorithm for our insurance contract:

First, simulate K paths of the state process up to time T and compute the state variables under the assumption that the surrender option is not exercised at any time. Since the contract value, and hence, the cash flow at maturity T is known for all possible states, define the following cash flows:

$$\begin{aligned}&C(T,T-1,Y_{T-1}^{(k)},D_{T-1}^{(k)})\\&=p_{x+T-1}^{(T)}C(T,Y_T^{(k)},D_T^{(k)})+q_{x+T-1}^{(T)}f(T,Y_T^{(k)},D_{T-1}^{(k)}),\quad 1\le k\le K.\end{aligned}$$

For ν=$T-u$, $u\in\{1,\dots,T-1\}$, compute $g^{(J,K)}$ as described above and determine the optimal strategy in each path by comparing the surrender value to the continuation value. Then, determine the new cash flows.[4] For $s\in\{T-u+1,\dots,T\}$, we have

$$\begin{aligned}&C(s,T-u-1,Y_s^{(k)},D_s^{(k)})\\&=\begin{cases}0 & \text{, if the option is exercised}\\ & \text{ at } T-u\\ p_{x+T-u-1}^{(T-u)}\cdot C(s,T-u,Y_s^{(k)},D_s^{(k)}) & \text{, otherwise}\end{cases},\end{aligned}$$

and for $s=T-u$ we set

$$\begin{aligned}&C(T-u,T-u-1,Y_{T-u}^{(k)},D_{T-u}^{(k)})\\&=\begin{cases}p_{x+T-u-1}^{(T-u)}\cdot C(T-u,Y_{T-u}^{(k)},D_{T-u}^{(k)}) & \text{, if the option is exercised}\\ +q_{x+T-u-1}^{(T-u)}\cdot f(T+u,Y_{T-u}^{(k)},D_{T-u-1}^{(k)}) & \text{ at } T-u\\ q_{x+T-u-1}^{(T-u)}\cdot f(T+u,Y_{T-u}^{(k)},D_{T-u-1}^{(k)}) & \text{, otherwise}\end{cases}.\end{aligned}$$

At time ν=0, discount the cash flows in each path and average over all K paths, i.e.

$$V^{(J,K)}(0,Y_0,D_0):=\frac{1}{K}\sum_{k=1}^{K}V_0^{(J,K,k)}(0,Y_0,D_0)$$

with

$$V^{(J,K,k)}(0,Y_0,D_0):=\sum_{s=1}^{T}C(s,0,Y_s^{(k)},D_s^{(k)})\exp\left(-\int_0^s r_u^{(k)}du\right).$$

[4] Note that we do not use the estimated continuation value but the actual cash flows for the next regression. Otherwise the estimator will be biased (cf. [35]).

The two convergence results in Sect. 3 of [17] ensure that, under weak conditions, the algorithm gives a good approximation of the actual contract value when choosing J and K sufficiently large.

The LSM algorithm can be conveniently implemented for insurance contracts containing a simple surrender option since the new future cash flows can be easily determined: If the surrender option is exercised at $\nu_0 \in \{1,\dots,T-1\}$, the cash flow $C(\nu_0,\nu_0-1,Y_{\nu_0},D_{\nu_0})$ equals the surrender value and all future cash flows are zero.

If we have more complex early exercise features, the derivation of the future cash flows could be more involved since the contract may not be terminated. For example, if a withdrawal option in a contract including a Guaranteed Minimum Withdrawal Benefit is exercised, this will change the states variables at that time. However, the future cash flows for the new state variables will not be known from the original sample paths, i.e. it is necessary to determine the new future cash flows up to maturity T. This may be very tedious if it is a long-term insurance contract and the option is exercised relatively early. In particular, if the option can be exercised at every anniversary and if the withdrawal is not fixed but arbitrary with certain limits, this may increase the complexity of the algorithm considerably.

A potential solution to this problem could be employing the discounted estimated conditional expectation for the regression instead of the discounted future cash flows. However, this will lead to a biased estimator (see [35], Sect. 1). But even if this bias is accepted, another problem regarding the quality of the regression function may occur. In the LSM algorithm, we determine the coefficients of the regression function with the help of sample paths that are generated under the assumption that no option is exercised at any time, i.e. the approximation of the continuation value will be good for values which are "close" to the used regressors. But $g^{(J,K)}$ may not be a good estimate for contracts with, e.g., high withdrawals because withdrawals reduce the account balance, and hence, the new state variables will not be close to the regressors. An idea of how to resolve this problem might be the application of different sampling techniques: For each period, we could determine a certain number of different initial values, simulate the development for one period, compute the contract value at the end of this period and use the discounted contract value as the regressand. However, determining these initial values, again, is not straight-forward. We leave the further exploration of this issue to future research.

Aside from these problems, the LSM approach bears profound advantages in comparison to the other approaches: On one hand, the number of simulation steps increases linearly in time and, on the other hand, it avoids solving a large number of P(I)DEs. Also, in contrast to the P(I)DE approach, the LSM approach is independent of the underlying asset model: The only part that needs to be changed in order to incorporate a new asset model is the Monte Carlo simulation.

4 Example: A participating life insurance contract

In this section, we compare the results obtained with the three different numerical approaches for a German participating life insurance contract including a surrender option.

4.1 The contract model

We consider the participating term-fix contract from [8] and [47]. While this contract is rather simple and, in particular, does not depend on biometric events, it presents a convenient example to illustrate advantages and disadvantages of the presented approaches and to compare them based on numerical experiments.

We use a simplified balance sheet to model the insurance company's financial situation (see Table 1). Here, A_t denotes the market value of the insurer's asset portfolio, L_t is the policyholder's account balance, and $R_t = A_t - L_t$ is the bonus reserve at time t. Disregarding any charges, the policyholder's account balance at time zero equals the single up-front premium P,

Table 1 Simplified balance sheet

Assets	Liabilities
A_t	L_t
	R_t
A_t	A_t

that is $L_0 = P$. During its term, the policyholder may surrender her contract: If the contract is lapsed at time $\nu_0 \in \{1,\dots,T\}$, the policyholder receives the current account balance L_{ν_0}. Furthermore, we assume that dividends are paid to shareholders at the anniversaries in order to compensate them for adopted risk.

As in [8] and [47], we use two different bonus distribution schemes, which describe the evolution of the liabilities: The MUST-case describes what insurers are obligated to pass on to policyholders according to German regulatory and legal requirements, whereas the IS-case models the typical behavior of German insurance companies in the past; this distribution rule was first introduced by Kling et al. (see [32]).

4.1.1 The MUST-case

In Germany, insurance companies are obligated to guarantee a minimum rate of interest g on the policyholder's account, which is currently fixed at 2.25%. Furthermore, according to the regulation about minimum premium refunds in German life insurance, a minimum participation rate δ of the earnings on book values has to be passed on to the policyholders. Since earnings on book values usually do not coincide with earnings on market values due to accounting rules, we assume that earnings on book values amount to a portion y of earnings on market values. The earnings on market values equal $A_\nu^- - A_{\nu-1}^+$, where A_ν^- and $A_\nu^+ = A_\nu^- - d_\nu$ describe the market value of the asset portfolio shortly before and after the dividend payments d_ν at time ν, respectively. Therefore, we have

$$L_\nu = (1+g)\,L_{\nu-1} + \left[\delta y\left(A_\nu^- - A_{\nu-1}^+\right) - gL_{\nu-1}\right]^+, \quad 1 \le \nu \le T. \tag{6}$$

Assuming that the remaining part of earnings on book values is paid out as dividends, we have

$$\begin{aligned} d_\nu = {} & (1-\delta)y\left(A_\nu^- - A_{\nu-1}^+\right)\mathbb{1}_{\left\{\delta y\left(A_\nu^- - A_{\nu-1}^+\right) > gL_{\nu-1}\right\}} \\ & + \left[y\left(A_\nu^- - A_{\nu-1}^+\right) - gL_{\nu-1}\right]\mathbb{1}_{\left\{\delta y\left(A_\nu^- - A_{\nu-1}^+\right) \le gL_{\nu-1} \le y\left(A_\nu^- - A_{\nu-1}^+\right)\right\}}. \end{aligned} \tag{7}$$

4.1.2 The IS-case

In the past, German insurance companies have tried to grant their policyholders stable but yet competitive returns. In years with high earnings, reserves are accumulated and passed on to policyholders in years with lower earnings. Only if the reserves dropped beneath or rose above certain limits would the insurance companies decrease or increase the bonus payments, respectively.

In the following, we give a brief summary of the bonus distribution introduced by Kling et al. ([32]), which models this behavior.

The reserve quota x_ν is defined as the ratio of the reserve and the policyholder's account, i.e. $x_\nu = \frac{R_\nu}{L_\nu} = \frac{A_\nu^+ - L_\nu}{L_\nu} = \frac{A_\nu^- - d_\nu - L_\nu}{L_\nu}$. Let $z \in [0,1]$ be the target interest rate of the insurance company and $\alpha \in [0,1]$ be the proportion of the remaining surplus after the guaranteed interest rate is credited to the policyholder's account that is distributed to the shareholders. Whenever the target interest rate z leads to a reserve quota between specified limits a and b with

$$\begin{aligned} L_\nu &= (1+z)L_{\nu-1} \\ d_\nu &= \alpha\,(z-g)\,L_{\nu-1} \\ A_\nu^+ &= A_\nu^- - d_\nu \\ R_\nu &= A_\nu^+ - L_\nu, \end{aligned}$$

then exactly the target interest rate z is credited to the policyholder's account.

If the reserve quota drops below a or exceeds b when crediting z to the policyholder's account, then the rate is chosen such that it exactly results in a reserve quota of a or b, respectively. However, (6) needs to be fulfilled in any case. Hence, by combining all cases and conditions, we obtain (see [47]):

$$\begin{aligned} L_v = (1+g)L_{v-1} + \max\Bigg\{ & \left[\delta y \left(A_v^- - A_{v-1}^+\right) - gL_{v-1}\right]^+, \\ & (z-g)L_{v-1}\mathbb{1}_{\left\{((1+a)(1+z)+\alpha(z-g))L_{v-1} \le A_v^- \le ((1+b)(1+z)+\alpha(z-g))L_{v-1}\right\}} \\ & + \frac{1}{1+a+\alpha}\left[A_v^- - (1+g)(1+a)L_{v-1}\right] \\ & \qquad \mathbb{1}_{\left\{(1+a)(1+g)L_{v-1} < A_v^- < ((1+a)(1+z)+\alpha(z-g))L_{v-1}\right\}} \\ & + \frac{1}{1+b+\alpha}\left[A_v^- - (1+g)(1+b)L_{v-1}\right]\mathbb{1}_{\left\{((1+b)(1+z)+\alpha(z-g))L_{v-1} < A_v^-\right\}}\Bigg\}, \end{aligned}$$

and

$$\begin{aligned} d_v = \max\Bigg\{ & \alpha\left[\delta y \left(A_v^- - A_{v-1}^+\right) - gL_{v-1}\right]^+, \\ & \alpha(z-g)L_{v-1}\mathbb{1}_{\left\{((1+a)(1+z)+\alpha(z-g))L_{v-1} \le A_v^- \le ((1+b)(1+z)+\alpha(z-g))L_{v-1}\right\}} \\ & + \frac{\alpha}{1+a+\alpha}\left[A_v^- - (1+g)(1+a)L_{v-1}\right] \\ & \qquad \mathbb{1}_{\left\{(1+a)(1+g)L_{v-1} < A_v^- < ((1+a)(1+z)+\alpha(z-g))L_{v-1}\right\}} \\ & + \frac{\alpha}{1+b+\alpha}\left[A_v^- - (1+g)(1+b)L_{v-1}\right]\mathbb{1}_{\left\{((1+b)(1+z)+\alpha(z-g))L_{v-1} < A_v^-\right\}}\Bigg\}. \end{aligned}$$

4.2 Asset models (I)

We consider two different asset models, namely a geometric Brownian motion with deterministic interest rate (constant short rate r), and a geometric Brownian motion with stochastic interest rates given by a Vasicek model (see [46]).

In the first case, we have the classical Black-Scholes (BS) setup, so the asset process under the risk-neutral measure $\mathcal{Q}$ evolves according to the SDE:

$$dA_t = rA_t\,dt + \sigma_A A_t\,dW_t, \quad A_0 = P(1+x_0),$$

where r is the constant short rate, $\sigma_A > 0$ denotes the volatility of the asset process A, and W is a standard Brownian motion under $\mathcal{Q}$. Since we allow for dividend payments at each anniversary of the contract, we obtain

$$A_t^- = A_{t-1}^+ \exp\left(r - \frac{\sigma_A^2}{2} + \sigma_A\left(W_t - W_{t-1}\right)\right).$$

In the second case, we have a generalized Black-Scholes model with

$$\begin{aligned} dA_t &= r_t A_t\,dt + \rho\sigma_A A_t\,dW_t + \sqrt{1-\rho^2}\sigma_A A_t\,dZ_t, \quad A_0 = P(1+x_0), \\ dr_t &= \kappa\left(\xi - r_t\right)dt + \sigma_r\,dW_t, \quad r_0 > 0, \end{aligned}$$

where $\rho \in [-1,1]$ describes the correlation between the asset process A and the short rate r, σ_r is the volatility of the short rate process, and W and Z are two independent Brownian motions. ξ and κ are constants. Hence,

$$A_t^- = A_{t-1}^+ \exp\left(\int_{t-1}^t r_s ds - \frac{\sigma_A^2}{2} + \int_{t-1}^t \rho\sigma_A dW_s + \int_{t-1}^t \sqrt{1-\rho^2}\sigma_A dZ_s\right).$$

We refer to this model as the extended Black-Scholes (EBS) model.

According to the risk-neutral valuation formula, the value for our participating life insurance contract including a surrender option is given by:[5]

$$V_0^{NON-EUR} = \sup_{\tau \in \Upsilon_0} \mathbb{E}^{\mathcal{Q}} \left[\exp\left\{ -\int_0^\tau r_u \, du \right\} L_\tau \,\middle|\, \mathcal{F}_0 \right]. \tag{8}$$

For a discussion of the problems occurring when implementing a suitable hedging strategy as well as potential solutions, we refer to [8] and [47].

4.3 Choice of parameters and regression function

To compare results, we use the same parameters as in [8] and [47]. We let the guaranteed minimum interest rate $g = 3.5\%$,[6] the minimum participation rate $\delta = 90\%$, and the minimal proportion of market value earnings that has to be identified as book value earnings in the balance sheet $y = 50\%$. Moreover, the reserve corridor is defined to be $[a,b] = [5\%, 30\%]$, the proportion of earnings that is passed on to the shareholders is fixed at $\alpha = 5\%$, and the volatility of the asset portfolio is assumed to be $\sigma_A = 7.5\%$. The correlation between asset returns and money market returns is set to $\rho = 0.05$. We consider a contract with maturity $T = 10$ years. The initial investment is $P = 10,000$, the insurer's initial reserve quota is $x_0 = 10\%$, and the initial (or constant) interest rate $r_0 = r$ is set to 4%. In the Vasicek model, the volatility of the short rate process σ_r is chosen to be 1%, the mean reversion rate is $\kappa = 0.14$, and the mean reversion level $\xi = 4\%$.

A crucial point in the LSM approach for non-European contracts is the choice of the regression function as a function of the state process and the state variables. Clearly, in the current setup, the state processes are $(A_t)_{t\in[0,T]}$ and $(A_t, r_t)_{t\in[0,T]}$ for the BS and the EBS model, respectively, and the state variables can be represented by $(D_\nu)_{\nu\in\{1,\dots,T-1\}} = (A_\nu^+, A_\nu, L_\nu)_{\nu\in\{1,\dots,T-1\}}$ in both models. Using a top down scheme, we found that a regression function with eight different terms is sufficient; more terms do not make a significant difference. We estimate the continuation value with the help of the following regression function:[7]

$$\begin{aligned} g^{(8)}(D_\nu) = g^{(8)}(A_\nu^+, A_\nu, L_\nu) = {} & \alpha_0(\nu) + \alpha_1(\nu) A_\nu^+ + \alpha_2(\nu) L_\nu + \alpha_3(\nu) x_\nu + \alpha_4(\nu) x_\nu^2 \\ & + \alpha_5(\nu) e^{r_\nu} + \alpha_6(\nu)(e^{r_\nu})^2 + \alpha_7(\nu) r_\nu, \end{aligned}$$

where $x_\nu = \frac{A_\nu^+ - L_\nu}{L_\nu}$ is the reserve quota and $\alpha_0(\nu), \dots, \alpha_7(\nu) \in \mathbb{R}$. We use the same regression function at all times $\nu \in \{1, \dots, T-1\}$ but, of course, the coefficients may vary. Note that in this particular setup, the dimensionality of the problem can be further reduced by modifying the state process: Here, it is sufficient to consider

$$\tilde{A}_t = A_{\nu-1}^+ \cdot \frac{A_t}{A_{\nu-1}}$$

and r_t as above in the corresponding one period problem, i.e. for $t \in [\nu-1, \nu]$, and consequently, it suffices to save (A_ν^+, L_ν) as the state variables in both asset models (cf. [8,47]).

Note that we do not have to specify a regression function for the European contract case: Here, the LSM approach trivially coincides with the simple Monte Carlo approach.

[5] Υ_0 is the set of all stopping times in $\{1, \dots, T\}$.

[6] The largest German insurer "Allianz Lebensversicherungs-AG" reports an average guaranteed interest rate of approximately 3.5% in 2006 (see [3], p. 129).

[7] Note that in the BS model, the last three terms may be discarded.

4.4 Numerical experiments

The valuation of European contracts, i.e. contracts without surrender option, is simple. Here, Monte Carlo simulations provide a fast and accurate valuation methodology. Therefore, we focus on the valuation of non-European contracts.

We start by analyzing the valuation via "Nested Simulations". Table 2 shows our results for 5,000 trees with 1 to 7 paths per node in the MUST-case and the BS setting. Aside from the two estimators $\hat{V}_0$ and $\hat{v}_0$, the (real) times for the procedures are displayed.[8] The difference between

Table 2 "Nested Simulations" for a non-European contract

paths per node	$\hat{V}_0$	$\hat{v}_0$	**time**
1	10523.4	10055.0	$<$ 1 sec.
2	10411.0	10051.6	15 sec.
3	10395.6	10100.3	11 min. 37 sec.
4	10388.8	10169.8	2 h 47 min.
5	10380.2	10228.6	$\approx$ 1 day
6	10378.6	10270.3	$\approx$ 6 days
7	10375.6	10293.8	$\approx$ 30 days

the two estimators is relatively large even for 7 paths per node. In particular, this means that resulting confidence intervals are relatively wide. Hence, although Monte-Carlo simulations are the only considered approach where confidence intervals may be produced, the computational effort to produce results in a reliable range is enormous.

Within the PDE and LSM approach, on the other hand, we find that for both asset models, the contract values resulting from the two approaches differ by less than 0.2% of the initial investment (see Table 3). However, the PDE approach is more sensitive to discretization errors and takes significantly more time: In the BS model, in the current computation environment, it takes approximatively 10 minutes to compute the non-European contract value with the PDE approach, whereas with the LSM approach, we obtain the result in approximately 24 seconds. The difference is even more pronounced in the EBS model. Here we have about 40 hours with the PDE approach compared to about 30 seconds with the LSM approach.

Table 3 Contract values in the two asset models[9]

	MUST BS		**IS BS**		**MUST EBS**		**IS EBS**	
	PDE	LSM	PDE	LSM	PDE	LSM	PDE	LSM
NON-EUR	10360.4	10361.2	10919.1	10920.0	10619.1	10603.9	11103.0	11088.9
EUR	10360.4	10361.2	10919.1	10920.0	10449.9	10452.0	11020.7	11022.9
SUR	0	0	0	0	169.2	151.9	82.3	66.0

While clearly all results depend on the particular implementations and the contract in view, due to the magnitude of the differences, we conclude that for the valuation of non-European insurance contracts for financial reporting within Solvency II and/or IFRS 4, the LSM approach appears to be the superior choice for determining the risk-neutral value. However, if additional sensitivities need to be computed for risk management purposes ("the Greeks"), the PDE method may still present a valuable alternative.

[8] All numerical experiments were carried out on a Linux machine with a Pentium IV 2.40 Ghz CPU and 2.0 GB RAM, with no other user processes running.

[9] The values for the PDE approach are taken from [8] and [47], respectively. Here, the PDE approach is only used to valuate the surrender option. The European contract values are calculated using Monte Carlo techniques to obtain a higher accuracy. The difference of the European contract values is due to Monte Carlo errors.

4.5 Influence of the surrender option

While values for the surrender option within this particular contract model have been calculated before in [8] and [47] via PDE approaches, no detailed sensitivity analyses are presented due to the high computational effort. However, the LSM approach allows for such analyses. We fix the parameters as indicated above (cf. Sec. 4.3), but as in the latter part of [47] choose an alternative value for the volatility parameter. For the *S&P 500* index, Schoutens ([43]) finds an implied volatility of 18.12%, but since insurers' asset portfolios contain a limited proportion of risky assets only,[10] we choose $\sigma = 0.03624$, which approximately corresponds to a portfolio consisting of 20% S&P 500 and 80% short maturity bonds.

Table 4 Value of the surrender option in the BS model

MUST	g=2.25%	g=3.5%	g=4.0%
NON-EUR	9885.3	9966.8	10065.8
EUR	8976.0	9687.8	10065.8
SUR	909.3	279.0	0.0

IS	g=2.25%	g=3.5%	g=4.0%
NON-EUR	10326.3	10450.8	10565.9
EUR	10177.8	10419.5	10565.9
SUR	148.5	31.3	0.0

Table 4 presents the values of the surrender option in the BS model for three different choices of the guaranteed rate g. We find that for low g in the MUST-case, the surrender option is of significant value. However, the non-European contract values are almost equal to the initial premium of 10,000, so this is clearly due to the possibility of surrendering the contract early in its term. The surrender option is considerably less valuable in the IS-case since the target interest rate exceeds the riskless interest rate, and therefore, in most cases, it is advantageous not to exercise. Moreover, for a high guaranteed rate g, the rationale for surrendering decreases in both case since the contract is close to the riskless asset with an additional option feature.

Figure 1 illustrates the influence of g on the value of the surrender option in the EBS model. We observe the same effects as in the BS model. The option value is smaller in the IS-case than in the MUST-case, and it is decreasing in g. However, in this case the value of the option is positive even for guaranteed interest rates exceeding 4% because interest rates could increase over the term of the contract.

All in all, we find that even though in many cases the influence of the surrender option is not very pronounced, the value for some parameter combinations is significant. In particular, this means that in changing environments, as e.g. increasing interest rates, the option adds significantly to the value of the contract and, hence, should not be disregarded by insurance companies. Moreover, for different kinds of non-European options and/or contracts, the influence may be significantly more pronounced (see e.g. [9] for Guaranteed Minimum Benefits within Variable Annuities).

4.6 Asset models (II)

Although the Black-Scholes model is still very popular in practice, numerous empirical studies suggest that it is not adequate to describe many features of financial market data. Exponential Lévy models present one possible alternative and have become increasingly popular. To assess the influence of model risk on our example contract, we introduce a third asset model with a normal inverse Gaussian (NIG) process driving the asset process $(A_t)_{t\in[0,T]}$. This model better represents the statistical properties of empirical log returns. Similar exponential Lévy models have been applied to the valuation of insurance contracts by different authors (see e.g. [6] or [31]).

[10] By the regulation on investments ([1]), German insurers are obligated to keep the proportion of stocks within their asset portfolio below 35%. For example, the German "Allianz Lebensversicherungs-AG" reports a proportion of 21% stocks in 2006 (see [3], p. 32).

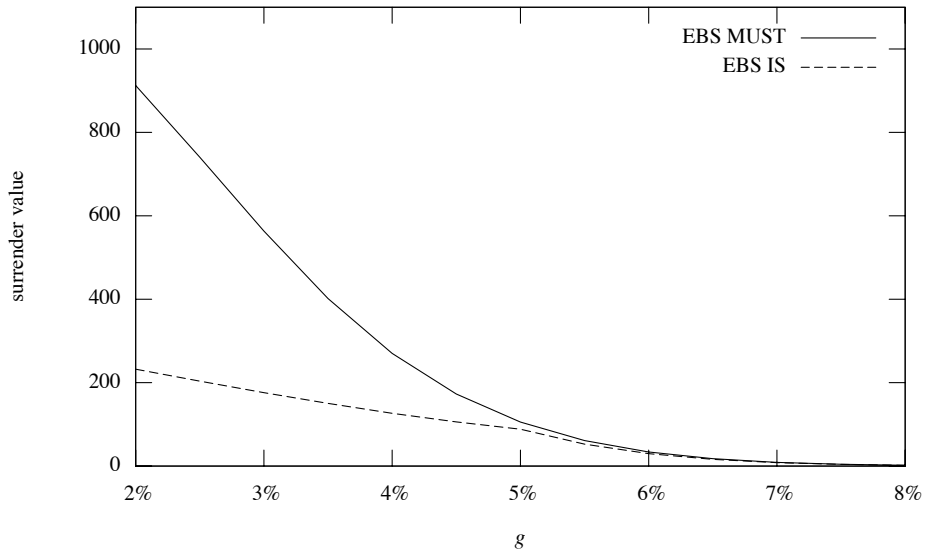

Fig. 1 Influence of g on the value of the surrender option in the EBS model

The probability density function of an NIG(α,β,δ,m) distribution is given by

$$\Phi_{NIG}(x,\alpha,\beta,\delta,m) = \frac{\alpha\delta}{\pi}\exp\left(\delta\sqrt{\alpha^2-\beta^2}+\beta(x-m)\right)\frac{K_1\left(\alpha\sqrt{\delta^2+(x-m)^2}\right)}{\sqrt{\delta^2+(x-m)^2}},$$

where K_1 denotes the modified Bessel function of the third kind with index 1, and an NIG process is defined as a Lévy process $(X_t)_{t\in[0,T]}$ at zero with $X_t \sim \text{NIG}(\alpha,\beta,\delta\cdot t, m\cdot t)$ (see [7] or [43] for more details).

As in the classical BS model, we assume a constant short rate r and define our exponential Lévy (NIG) model by

$$A_t = A_0 e^{X_t},$$

where $X_t \sim \text{NIG}(\alpha,\beta,\delta\cdot t, m\cdot t)$ under "a" risk-neutral measure $\mathscr{Q}$: Financial markets driven by Lévy processes are generally not complete, and hence, the equivalent martingale measure is not unique. There are different methods of how to choose a valuation measure, e.g. by the so-called Esscher transform or the mean correcting method. As in [31], we use the mean correcting method. Here, the parameters α, β, and δ are calibrated to observed option prices, and the parameter m is chosen such that the discounted price process is a martingale under $\mathscr{Q}$, i.e.

$$m = r + \delta\left(\sqrt{\alpha^2-(\beta+1)^2}-\sqrt{\alpha^2-\beta^2}\right).$$

Hence, under the risk-neutral measure $\mathscr{Q}$, for this asset model we have

$$A_t^- = A_{t-1}^+ e^{X_t - X_{t-1}}, \quad X_t \sim \text{NIG}(\alpha,\beta,\delta\cdot t, m\cdot t).$$

Again following [31], we choose the parameters resulting from the calibration procedure for the *S&P 500* index from [43] based on call options prices, i.e. $\alpha = 6.1882$, $\beta = -3.8941$, and $\delta = 0.1622$, where the volatility from [43] is adapted according to our assumptions on the asset portfolio.

4.7 Model risk: The BS model vs. the NIG model

While one may expect that the contract increases in value when changing the asset process from a geometric Brownian motion to an exponential Lévy model, Table 5 illustrates that this is not always the case.[11] At least in the MUST-case, the question of whether the European contract value is higher for the BS or the NIG model depends on the guaranteed minimum interest rate. This can be explained by considering the different shapes of the two corresponding density

Table 5 European contract values in the BS and NIG model

	g=2.25%		g=3.5%		g=4.0%	
	BS	NIG	BS	NIG	BS	NIG
MUST	8976.0	9040.0	9687.8	9671.4	10065.8	10042.5
IS	10177.8	10279.0	10419.5	10490.0	10565.9	10613.3

functions. In the MUST-case, for a European contract with maturity $T = 1$, we have

$$L_1 = (1+g)P + \left[\delta y \left(A_1^- - A_0^+\right) - gP\right]^+ = (1+g)P + \left[P\left(\delta y (1+x_0)\left(e^{X_1}-1\right) - g\right)\right]^+,$$

where X_1 is Normal and NIG distributed in the BS and the NIG model, respectively, and thus

$$V_0^i = \mathbb{E}^{\mathcal{Q}}\left[L_1 e^{-r}\right] = e^{-r}\left[(1+g)P + \int_c^\infty P\left[\delta y (1+x_0)\left(e^u - 1\right) - g\right]\phi^i(u)du\right], i \in \{\text{BS}, \text{NIG}\}. \tag{9}$$

where $c = \log\left(\frac{g}{(1+x_0)\delta y} + 1\right)$ and ϕ^{BS} (ϕ^{NIG}) is the corresponding density of the log returns within the BS (NIG) model.

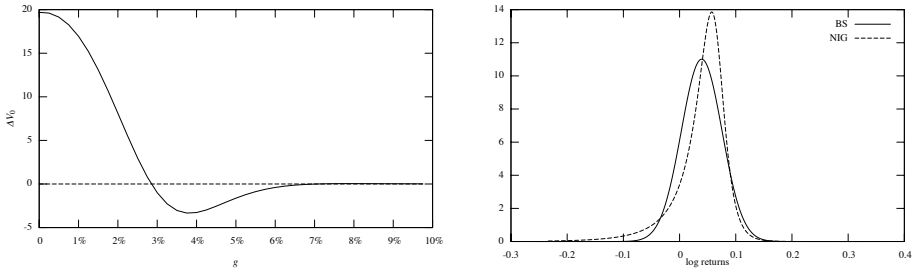

Fig. 2 Influence of g on the difference in the contract values (MUST-case, $T = 1$)

The left-hand side of Figure 2 now illustrates the difference in values for the two models

$$\Delta V_0 = V_0^{\text{NIG}} - V_0^{\text{BS}},$$

and we find that for g smaller than approximately 3%, the contract is worth more in the NIG model, whereas for g between 3% and 7%, the BS model yields higher contract values. If the guaranteed interest rate is unrealistically high ($\geq 7\%$), the difference is comparatively negligible. In order to analyze this behavior, in view of (9) it is now sufficient to compare both density functions with parameters fitted to the data as described above (see the right-hand side of Figure 2): For low values of g, the interest rate guarantee is worth more within the NIG model due to the increased kurtosis of the corresponding distribution. However, for an increasing level of g,

[11] Since the differences in contract values for the two asset models are consistent for European and non-European contracts, we only present results for European contracts.

this influence vanishes and the option in the BS model becomes more valuable due to the skewness of the NIG distribution. In contrast, in the IS-case the contract is worth more in the NIG model since the target rate z is credited unless very extreme outcomes occur, which are clearly "more likely" under the NIG distribution.

But not only the guaranteed minimum interest rate g influences this relationship. The left-hand side of Figure 3 illustrates the influence of the stock proportion within the insurer's asset portfolio on the difference in contract values in the two models for the MUST-case, $g = 3.5\%$, and $T = 10$. For small proportions, the difference of the two contract values is negative, i.e. the value in the BS model is higher, due to the afore-mentioned effect. However, a higher stock proportion increases the volatility, and in the NIG model, the tails "fatten" faster than in the BS model. From a stock proportion of about 30%, this leads to a higher value for the NIG driven model.

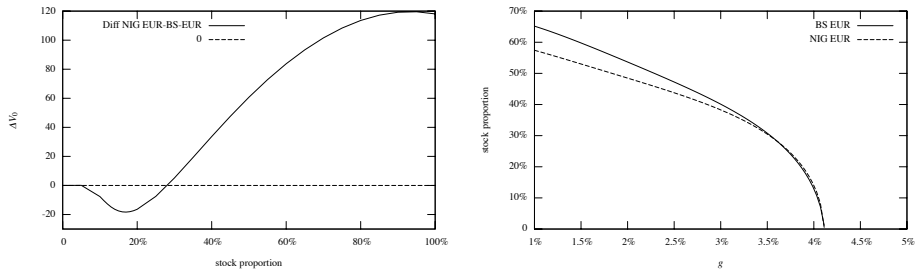

Fig. 3 Influence of the stock proportion on the difference in the contract values (MUST-case, $T = 10$)

The right-hand side of Figure 3 shows combinations of g and the stock proportion that result in a "fair" contract,[12] i.e. the contract value equals the initial investment of 10,000. For low g and the standard parameters, the contract values lie below 10,000. Hence, the stock proportion needs to be increased in order to yield a fair contract. Since for small g, the NIG model leads to higher contract values than the BS model, the increase in the stock proportion is comparatively lower in the NIG model. In contrast to this, in the IS-case we cannot find any realistic fair parameter combinations at all.

All in all, our analyses show that for our example contract and a realistic range for the parameters, the influence of the asset model on the contract value is rather small and it depends on the particular parameter choice which model leads to the higher value. However, clearly the influence may be a lot more pronounced for different embedded options and/or contracts.

5 Conclusion

In this paper, we construct a generic valuation model for life insurance benefits and give a survey on existing valuation approaches. Firstly, we explain how to use Monte Carlo simulations for the valuation. The Monte Carlo approach yields fast results for European contracts, i.e. contracts without any early exercise features, but it is inefficient for the valuation of long-term non-European contracts: In this case, the number of necessary simulation steps to obtain accurate results may be extremely high. Secondly, we present a discretization approach based on the consecutive solution of certain partial (integro-)differential equations (PDE approach). This approach is more apt for the valuation of long-term non-European contracts and allows for the calculation of the "Greeks", but depending on the model specifications solving the P(I)DEs can be very complex and can slow down the algorithm considerably.

[12] For a discussion of the notion "fairness" we refer to [8].

Lastly, we discuss the so-called least-squares Monte Carlo approach. It combines the advantages of the Monte Carlo and the PDE approach: On one hand, it is a backward iterative scheme such that early exercise features can be readily considered and, on the other hand, it remains efficient even if the dimension of the state space becomes larger as the valuation is carried out by Monte Carlo simulations rather than the numerical solution of P(I)DEs.

We apply all algorithms to the valuation of participating life insurance contracts and initially consider two asset models, namely the classical Black-Scholes setup and a generalized Black-Scholes model with stochastic interest rates. Our numerical experiments show that the differences in the computational time needed for the valuation of non-European contracts is enormous.

Furthermore, again based on the example of participating life insurance contracts, we analyze the influence of the "early exercise feature", i.e. a surrender option, as the difference between the non-European and European contract. We find that for many scenarios, the surrender option is (almost) worthless in this particular case. However, we demonstrate that the sensitivities of European and non-European contract values with respect to key contract parameters differ considerably, so that disregarding this contract feature may be misleading.

Finally, we study the impact of model risk on our example contract by additionally introducing an exponential Lévy (NIG) model for the asset side. Comparing the NIG model to the classical Black-Scholes model, we find that for realistic parameter combinations, the influence is not very pronounced. In particular, it depends on the parameter choice which model yields higher contract values.

All in all, this article provides a framework for the market-consistent valuation of life insurance contracts and a survey as well as a discussion of different numerical methods for applications in practice and academia. Our numerical experiments give insights on the effectiveness of the different methods and show that the influence of early exercise features should be analyzed.

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