What is the Impact of Stock Market Contagion on an Investor's Portfolio Choice?

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Abstract

Stocks are exposed to the risk of sudden downward jumps, and a crash in one stock (or index) may increase the risk of a crash for other stocks (or indices). This may have a crucial impact on investors' portfolio choices, since it reduces their ability to diversify their portfolios. Allowing the economy to be in either of two regimes (calm, contagion), we explicitly take contagion risk into account and study its impact on the portfolio decision of a CRRA investor both in a complete and in an incomplete market. We find that the investor significantly adjusts his portfolio when contagion is more likely to occur. Capturing the time dimension of contagion, i.e. the time difference between the downward jump in the first and in the second stock, is thus of first-order importance when analyzing portfolio decisions. An investor ignoring contagion completely or accounting for contagion while ignoring its time dimension suffers a large and economically significant utility loss. This loss is larger in a complete than in an incomplete market, and the investor might be better off if he does not trade derivatives at all.

Keywords: Asset Allocation, Jumps, Contagion, Model Risk

JEL: G12, G13

1 Introduction and Motivation

The notion of contagion in financial markets refers to a phenomenon where losses in one asset, one asset class, or one country increase the risk of subsequent losses in other assets, other asset classes, or other countries. Contagion may arise due to economic relations, e.g. when one firm is the main customer of another firm, due to the exposure to a common macroeconomic risk factor, or due to psychological reasons, when e.g. problems for one financial institution increase the risk of a bank run for other financial institutions. One example for such a situation is the recent subprime crisis that has been threatening the financial markets all over the world: When real estate prices in the US started to decrease, homeowners who had borrowed heavily against the equity in their homes were suddenly realizing that they could no longer afford to keep up their mortgage payments. An estimate from December 2007 states that "subprime borrowers will probably default on 220 billion – 450 billion of mortgages".[1] This threat has had a significant effect on the markets for structured credit contracts like Collateralized Debt Obligations (CDOs) leading to huge losses that the banks have now started to report. All along the way, the fear has extended into equity markets:

> "Fears about an end to the leveraged buy-out boom triggered heavy selling of global equities yesterday, leading to the FTSE 100's worst one-day slide for more than four years. [...] The FTSE 100 fell more than 200 points, or 3.2%, to 6.251,2; its biggest drop since March 2003 in the run-up to the Iraq war. [...] By early afternoon in New York, the Dow Jones Industrial Average was down more than 300 points, or 2.4%." (FT, July 27, 2007)

> "'In this sort of climate it is all about sentiment, not about the numbers at all, and sentiment at present is all about fear and nervousness,' said Kevin Gardiner, head of global equity strategy at HSBC." (WSJ, July 27, 2007)

or as catchily summarized:

> "The grievous experience of two centuries of financial busts is that when the banking system is in difficulties the mess spreads." (Economist, Dec 19, 2007)

These examples show how losses in one part of the economy or in one country can spread out into other parts of the economy or other countries.

[1] See Economist, Dec 19 2007.

Our paper concentrates on contagion effects occurring in stock markets. We study the optimal portfolio decision of an investor who is exposed to these effects. The stocks in our economy follow a jump-diffusion process where the jumps are downward jumps. Contagion is built into the model by allowing for a dependence between these downward jumps of the stock prices. Das and Uppal (2004) study an economy where downward jumps in stock prices always happen simultaneously. The dependence between stocks is thus driven by the (perfect) correlation of the jumps. We generalize this idea and focus on the probabilities that jumps happen. More precisely, we model contagion using a Markov chain with two states, a calm state and a contagion state. In the calm state, the probability of losses is rather low, while it increases significantly when the economy enters the contagion state which is therefore much more risky. In both states, there are occasional (downward) jumps in the stock prices. Some of these jumps in the calm state do not only lead to a loss in one of the stocks, but also trigger a jump of the economy into the contagion state and thus increase the overall riskiness of stocks. Subsequently, the economy can jump back into the calm state, and this kind of jumps occurs without a jump in stock prices.

Our approach allows us to capture two stylized facts at the same time: Firstly, contagion is not a "one time event" in the sense that it occurs, leads to immediate losses in several stocks, but has no longer-lasting impact. Usually, the probability for subsequent crashes remains higher for some time. This time-dimension of contagion implies that the investor can adjust his portfolio when the threat of contagion becomes apparent. Secondly, contagion is usually triggered by an initial crash in a particular market, i.e. the jump into the contagion state occurs when some stock prices drop.

Our paper is related to the literature on (continuous-time) portfolio choice starting with Merton (1969, 1971). There are two approaches to deal with contagion effects in portfolio problems. One strand of the literature models contagion as joint Poisson jumps. Papers in this area include Das and Uppal (2004) and Kraft and Steffensen (2008) which, however, disregard the time dimension of contagion. In particular, the probability of subsequent jumps remains the same after a joint jump has happened, since these papers do not allow for regime shifts. These frameworks therefore do not allow to study how an investor hedges against subsequent losses given that contagion effects become apparent in the market. The second strand of the literature are so-called regime-switching models. Papers in this area include Ang and Bekaert (2002) and Guidolin and Timmermann (2007a,b). Although these models capture the time dimension of contagion, regime shifts are not triggered by jumps in asset prices, but occur independently of crashes in the stock market. Buraschi, Porchia, and Trojani (2007) study a model with stochastic correlations between assets, but do not allow for jumps. The relevance of contagion is also empirically documented in Bae, Karolyi, and Stulz (2003) and Boyson, Stahel, and Stulz (2006).

In the paper, we address the following points: Firstly, we solve for the optimal stock demand in the calm and in the contagion state both in a complete and in an incomplete market. We show that there is a hedging demand for those jumps that lead to a different state. The sign of this hedging demand depends on the investment opportunities in both states and on the risk aversion of the investor relative to the log investor. We then analyze whether and how the investor adjusts his portfolio when the economy enters or leaves the contagion state. These portfolio revisions turn out to be significant, and they are the larger the more the calm and contagion state differ. Whether the investor increases or decreases his holdings of the risky assets depends on the changes in the market prices of risk and on whether the market is complete or incomplete.

Secondly, we analyze the utility loss an investor suffers from if he ignores contagion or if he ignores the time dimension of contagion. We show that the utility loss due to model mis-specification can be significant. This is particularly true when the market is complete and the investor uses derivatives. In this case, an investor with a rather low risk aversion of 1.5 might lose more than 20% a year when he bases his decision on an incorrect model. If the calm and contagion state differ significantly, the utility loss is largest if the investor ignores contagion completely. For smaller differences, the utility losses are largest if he only ignores the time dimension of contagion. This latter model also results in the largest losses if the market is incomplete, even if these losses are much smaller than in a complete market, where the investor does not only suffer from basing his portfolio decision on an incorrect model, but also from implementing his (seemingly) optimal strategy using an incorrect pricing model for the derivatives. The utility loss from this second mistake might even be so large that it more than offsets the utility gain from having access to derivatives, resulting in a situation where the investor is better off if he does not trade derivatives at all.

The remainder of the paper is structured as follows. In Section 2, we present the model and the portfolio planning problem. The optimal portfolios both in complete and incomplete markets are derived in Section 3. In Section 4, we analyze two benchmark models where the investor either completely ignores contagion or just its time dimension. Section 5 provides some numerical examples, discusses the impact of model mis-specification, and provides some robustness checks. Section 6 concludes.

2 Model Setup

2.1 The Economy

We consider an economy with two stocks A and B. The interest rate r is assumed to be constant. The stocks are driven by jump-diffusion processes, the dynamics of stock i ($i \in \{A, B\}$) are

$$\frac{dS_i(t)}{S_i(t)} = \mu_i^{Z(t)} dt + \sigma_i^{Z(t)} dW_i(t) - \sum_{k \neq Z(t)} L_i^{Z(t),k} dN^k(t).$$

W_A and W_B are two Brownian motions with correlation $\rho^{Z(t)}$ which capture normal stock price movements. Sudden large changes in the stock prices are driven by the Poisson processes N^k, and the loss if a jump happens is given by $L_i^{Z(t),k}$, where we assume that the loss sizes are constant. Note that in our notation $L > 0$ corresponds to a loss.

The dynamics of the stock prices depend on some state of the economy Z. We interpret these states as calm and contagion states and assume that these states mainly differ with respect to the jump intensities. While the jump intensities are rather low in a calm state, they increase significantly if the economy enters a contagion state. In a contagion state, the probability that there will be several downward jumps in stock prices in a given time interval is thus much larger than in a calm state.

Formally, contagion is modeled using a Markov chain. In general, the Markov chain jumps from state i to state j ($j \neq i$) with intensity $\lambda^{i,j}$, and the process N^j counts the number of jumps into state j. The current state is denoted by $Z(t)$. We use a Markov chain with eight states $k \in \{cont_{A1}, cont_{A2}, cont_{B1}, cont_{B2}, calm_{A1}, calm_{A2}, calm_{B1}, calm_{B2}\}$ which is illustrated in Figure 1. The first subscript of the state indicates the stock in which the last jump took place, the second subscript is due to the technical reason that there cannot be any jumps from a state into itself.[2] In the calm states, there may be a jump in any of the two stocks, and this jump may (but needs not) trigger contagion. The intensity of a jump in stock i that does not trigger contagion is $\lambda_i^{calm,calm}$, and the corresponding loss in stock i is $L_i^{calm,calm}$ (the loss in the other stock is zero). When such a jump takes place, the Markov chain goes from $calm_{\cdot 1}$ to $calm_{i2}$ or from $calm_{\cdot 2}$ to $calm_{i1}$. The intensity of a jump in stock i that triggers contagion is $\lambda_i^{calm,cont}$, the loss of stock i for such a jump is $L_i^{calm,cont}$, and the Markov chain goes from $calm_{\cdot j}$ to $cont_{ij}$. If the economy is in a contagion state, the intensity for a loss in stock i is $\lambda_i^{cont,cont}$, and the corresponding loss size

[2] If there are, e.g., several successive jumps in stock A in the calm state, then the Markov chain changes between the states $calm_{A1}$ and $calms_{A2}$. Without these different calm states, the Markov chain would have to jump from the (unique) calm state into the calm state, which is not possible technically.

is $L_i^{cont,cont}$. If such a jump happens, the Markov chain goes from $cont_{\cdot 1}$ to $cont_{i2}$ or from $cont_{\cdot 2}$ to $cont_{i1}$. Eventually, the economy will go back to the calm state. The intensity for this event is $\lambda^{cont,calm}$, and we assume that this event does not induce any losses in the stocks, i.e. $L_i^{cont,calm} \equiv 0$ ($i \in \{A, B\}$). The Markov chain goes from $cont_{ij}$ to $calm_{ij}$. The intensities for all other jumps are equal to zero.

The Markov chain has four contagion states and four calm states. We assume that the model parameters coincide in all calm state and in all contagion states, respectively. The future behavior of the stock prices thus only depends on whether the economy is in a calm state or in a contagion state, but not on which specific calm or contagion state is realized. This implies that optimal portfolios, indirect utilities, and other economic quantities we are interested in also depend only on whether we are in a calm or contagion state. The use of four contagion and four calm states thus does not have any economic implications, but is for technical reasons only.

Finally, we have to specify the drift and the risk premia of the stocks. The drift of stock i is equal to

$$\mu_i^{Z(t)} = r + \phi_i^{Z(t)} + \sum_{k \neq Z(t)} L_i^{Z(t),k} \lambda^{Z(t),k}$$

where the last term is the compensator for the jump processes. In general, the risk premium on the stock is

$$\phi_i^{Z(t)} = \sigma_i^{Z(t)} \eta_i^{Z(t)} + \sum_{k \neq Z(t)} L_i^{Z(t),k} \lambda^{Z(t),k} \eta^{Z(t),k}$$

where η_i^j is the premium for diffusion risk W_i when the economy is in state j, and $\eta^{j,k}$ is the premium for jumps from j to k. The intensity for a jump from j to k under the risk neutral measure is thus $(1 + \eta^{j,k})$ times the intensity under true measure.

With our definition of the Markov chain, the risk premium depends only on whether the economy is in one of the calm or in one of the contagion states. The risk premia on stock i are

$$\begin{aligned} \phi_i^{calm} &= \sigma_i^{calm} \eta_i^{calm} + L_i^{calm,calm} \lambda_i^{calm,calm} \eta_i^{calm,calm} + L_i^{calm,cont} \lambda_i^{calm,cont} \eta_i^{calm,cont} \\ \phi_i^{cont} &= \sigma_i^{cont} \eta_i^{cont} + L_i^{cont,cont} \lambda_i^{cont,cont} \eta_i^{cont,cont}. \end{aligned}$$

Besides the stocks and the money market account, the investor might also have access to derivatives. We assume that there are either no derivatives at all, or enough derivatives to complete the market. The exposure of the derivatives to the risk factors can be calculated using Ito's lemma.

2.2 The Investor

We consider a representative investor with CRRA-utility

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

where $\gamma > 0$ denotes his relative risk aversion. The planning horizon of the investor is T, and he derives utility from terminal wealth only.

The indirect utility at time t and in state j is defined as

$$G^j(t, X_t) = \max_{X_T \in \mathcal{A}^j(t,X_t)} \left\{ E\left[u(X_T)|Z(t) = j\right] \right\}$$

where $\mathcal{A}^j(t, X_t)$ is the set of all wealth levels at T that meet the budget restriction and can be financed with a current wealth level X_t. More details on this set will be given later on.

3 Asset Allocation

3.1 Complete Market

In a complete market, the investor can choose the optimal exposures to the risk factors first, and then implement these exposures by some appropriate trading strategy, as explained e.g. in Liu and Pan (2003). We follow this ansatz, and the budget restriction for the investor is

$$\begin{aligned} \frac{dX(t)}{X(t)} &= rdt + \theta_A^{Z(t)}(t)\left[dW_A(t) + \eta_A^{Z(t)}dt\right] + \theta_B^{Z(t)}(t)\left[dW_B(t) + \eta_B^{Z(t)}dt\right] \\ &+ \sum_{k \neq Z(t), \lambda^{Z(t),k} \neq 0} \theta^{Z(t),k}(t)\left[dN^k(t) - \lambda^{Z(t),k}dt - \eta^{Z(t),k}\lambda^{Z(t),k}dt\right] \end{aligned} \tag{1}$$

where $\theta_i^j(t)$ is the exposure of wealth to diffusion risk W_i in state j and $\theta^{j,k}$ is the exposure to a jump from state j to state k. In the calm state, we have to choose the four exposures to jumps in stock A and stock B that (do not) induce contagion, and we denote these exposures by $\theta_i^{calm,cont}$ ($\theta_i^{calm,calm}$). In the contagion state, we have to choose the three exposures to jumps in stock A, jumps in stock B, and jumps back from the contagion to the calm state. These exposures are denoted by $\theta_i^{cont,cont}$ and $\theta^{cont,calm}$. The portfolio planning problem of the investor is

$$G^j(t, X_t) = \max_{\{\theta_A^j(s), \theta_B^j(s), \theta^{j,k}(s), t \leq s < T\}} E\left[u(X_T)|Z(t) = j\right]$$

subject to the budget restriction (1).

Proposition 1 (Contagion, Complete Market) *In an economy with contagion, the optimal exposures to the risk factors are*

$$\theta_A^j = \frac{\eta_A^j - \rho^j \eta_B^j}{\gamma(1-(\rho^j)^2)} \qquad \theta_B^j = \frac{\eta_B^j - \rho^j \eta_A^j}{\gamma(1-(\rho^j)^2)}$$

$$\theta_A^{calm,calm} = (1+\eta_A^{calm,calm})^{-\frac{1}{\gamma}} - 1 \qquad \theta_B^{calm,calm} = (1+\eta_B^{calm,calm})^{-\frac{1}{\gamma}} - 1$$

$$\theta_A^{calm,cont} = (1+\eta_A^{calm,cont})^{-\frac{1}{\gamma}} \frac{f^{cont}}{f^{calm}} - 1 \qquad \theta_B^{calm,cont} = (1+\eta_B^{calm,cont})^{-\frac{1}{\gamma}} \frac{f^{cont}}{f^{calm}} - 1$$

$$\theta_A^{cont,cont} = (1+\eta_A^{cont,cont})^{-\frac{1}{\gamma}} - 1 \qquad \theta_B^{cont,cont} = (1+\eta_B^{cont,cont})^{-\frac{1}{\gamma}} - 1$$

$$\theta^{cont,calm} = (1+\eta^{cont,calm})^{-\frac{1}{\gamma}} \frac{f^{calm}}{f^{cont}} - 1.$$

The indirect utility function of the investor is

$$G^j(t) = \frac{x^{1-\gamma}}{1-\gamma} \left(f^j(t)\right)^{\gamma}$$

where

$$\begin{pmatrix} f^{calm}(t) \\ f^{cont}(t) \end{pmatrix} = \exp\left\{ \begin{pmatrix} C^{calm,calm} & C^{calm,cont} \\ C^{cont,calm} & C^{cont,cont} \end{pmatrix} (T-t) \right\} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

with

$$\begin{aligned}
C^{calm,calm} &= \frac{1-\gamma}{\gamma} \Bigg[r + \frac{(\eta_A^{calm})^2 + (\eta_B^{calm})^2 - 2\rho^{calm}\eta_A^{calm}\eta_B^{calm}}{2\gamma(1-(\rho^{calm})^2)} \\
&\qquad + \left(1+\eta_A^{calm,calm} - \frac{1}{1-\gamma}\right)\lambda_A^{calm,calm} + \left(1+\eta_B^{calm,calm} - \frac{1}{1-\gamma}\right)\lambda_B^{calm,calm} \\
&\qquad + \left(1+\eta_A^{calm,cont} - \frac{1}{1-\gamma}\right)\lambda_A^{calm,cont} + \left(1+\eta_B^{calm,cont} - \frac{1}{1-\gamma}\right)\lambda_B^{calm,cont} \Bigg] \\
&\quad + \left(1+\eta_A^{calm,calm}\right)^{1-\frac{1}{\gamma}} \lambda_A^{calm,calm} + \left(1+\eta_B^{calm,calm}\right)^{1-\frac{1}{\gamma}} \lambda_B^{calm,calm} \\
C^{calm,cont} &= \left(1+\eta_A^{calm,cont}\right)^{1-\frac{1}{\gamma}} \lambda_A^{calm,cont} + \left(1+\eta_B^{calm,cont}\right)^{1-\frac{1}{\gamma}} \lambda_B^{calm,cont} \\
C^{cont,cont} &= \frac{1-\gamma}{\gamma} \Bigg[r + \frac{(\eta_A^{cont})^2 + (\eta_B^{cont})^2 - 2\rho^{cont}\eta_A^{cont}\eta_B^{cont}}{2\gamma(1-(\rho^{cont})^2)} \\
&\qquad + \left(1+\eta_A^{cont,cont} - \frac{1}{1-\gamma}\right)\lambda_A^{cont,cont} + \left(1+\eta_B^{cont,cont} - \frac{1}{1-\gamma}\right)\lambda_B^{cont,cont} \\
&\qquad + \left(1+\eta^{cont,calm} - \frac{1}{1-\gamma}\right)\lambda^{cont,calm} \Bigg] \\
&\quad + \left(1+\eta_A^{cont,cont}\right)^{1-\frac{1}{\gamma}} \lambda_A^{cont,cont} + \left(1+\eta_B^{cont,cont}\right)^{1-\frac{1}{\gamma}} \lambda_B^{cont,cont} \\
C^{cont,calm} &= \left(1+\eta^{cont,calm}\right)^{1-\frac{1}{\gamma}} \lambda^{cont,calm}.
\end{aligned}$$

The proof is given in Appendix A.1.

Following Merton (1971), the optimal exposures can be decomposed into a speculative demand and a hedging demand. The demand for diffusion risk is purely speculative, since diffusion risk does not have any impact on the investment opportunity set. It depends on the risk premia (and the correlations) only. The optimal exposure to jump risk is more involved. The speculative demand for a jump from state *old* to state *new* (where the two states might coincide) is given by

$$(1+\eta^{old,new})^{-\frac{1}{\gamma}} - 1.$$

If the market price of jump risk $\eta^{old,new}$ is positive, jumps are more likely under the risk-neutral measure than under the true measure, and the optimal exposure to this kind of jumps is negative. In line with intuition, it increases in absolute terms in the risk premium, and it decreases in absolute terms in risk aversion. The second part of the demand for jump risk is the hedging demand, which is given by

$$(1+\eta^{old,new})^{-\frac{1}{\gamma}} \left(\frac{f^{new}}{f^{old}} - 1 \right).$$

It differs from zero only if the old and new state are not equal, i.e. if the economy changes from calm to contagion or vice versa. In this case, the investor takes changes in the investment opportunity set into account, where his reaction to these changes depends on whether he is more or less risk-averse than the log-investor, as explained e.g. in Kim and Omberg (1996), Liu and Pan (2003) or Liu, Longstaff, and Pan (2003). For $f^{new} > f^{old}$, the induced hedging demand is positive. If $\gamma > 1$, $f^{new} > f^{old}$ implies that investment opportunities are worse in the new state than in the old state. The investor is more risk-averse than the log investor, he cares about hedging, and he wants to have more wealth in those states of the world where investment opportunities are bad. This results in a positive hedging demand. If $\gamma < 1$, $f^{new} > f^{old}$ implies that investment opportunities are better in the new state than in the old state. The investor is less risk-averse than the log investor and he speculates on changes in the investment opportunity set. He thus wants to have more wealth in the good new state, and the induced 'hedging demand' is positive.

To assess how good the investment opportunities in state j are, we rely on the certainty equivalent return (CER). It is defined by

$$G^j(t,x) = \frac{\left(x e^{CER^j(t,x)(T-t)}\right)^{1-\gamma}}{1-\gamma}.$$

The CER gives the deterministic return on wealth that would result in the same indirect utility as the optimal investment in the risky assets.

When the economy changes from the calm state to the contagion state (or vice versa), the indirect utility of the investor changes due to two reasons. First, his wealth changes, where the loss or gain depends on his exposure towards the jump. Second, the investment opportunity set and thus the CER changes. Consider e.g. the case where the optimal exposure to a jump from the calm to the contagion state is negative. If the investment opportunities are worse in the contagion state, the investor will be worse off after the jump. If, on the other hand, the investment opportunities are better in the contagion state, the overall impact on the indirect utility depends on the trade-off between the lower wealth and the higher CER.

3.2 Incomplete Market

If the investor can only trade in the two stocks and in the money market account, the market is incomplete. The budget restriction becomes

$$\frac{dX(t)}{X(t)} = \pi_A^{Z(t)}(t)\frac{dS_A(t)}{S_A(t)} + \pi_B^{Z(t)}(t)\frac{dS_B(t)}{S_B(t)} + \left(1 - \pi_A^{Z(t)}(t) - \pi_B^{Z(t)}(t)\right) r dt$$

where $\pi_i^j(t)$ is the proportion of wealth invested in stock i ($i = A, B$) at time t and in state j. The optimal portfolio strategy is given in

Proposition 2 (Contagion, Incomplete Market) *In an economy with contagion where only the two stocks and the money market account are traded, the indirect utility of the investor in state $j \in \{calm, cont\}$ is*

$$G^j(t,x) = \frac{x^{1-\gamma}}{1-\gamma} f^j(t)$$

where f^j solves the ordinary differential equations

$$\begin{aligned}
0 \;=\;& f_t^{calm} + (1-\gamma)\left[r + \pi_A^{calm}(\mu_A^{calm} - r) + \pi_B^{calm}(\mu_B^{calm} - r)\right] f^{calm} && (2)\\
& - 0.5\gamma(1-\gamma)\left[(\pi_A^{calm}\sigma_A^{calm})^2 + (\pi_B^{calm}\sigma_B^{calm})^2 + 2\pi_A^{calm}\pi_B^{calm}\sigma_A^{calm}\sigma_B^{calm}\rho^{calm}\right] f^{calm} \\
& + \lambda_A^{calm,cont}\left[(1-\pi_A^{calm}L_A)^{1-\gamma} f^{cont} - f^{calm}\right] + \lambda_A^{calm,calm}\left[(1-\pi_A^{calm}L_A)^{1-\gamma} - 1\right] f^{calm} \\
& + \lambda_B^{calm,cont}\left[(1-\pi_B^{calm}L_B)^{1-\gamma} f^{cont} - f^{calm}\right] + \lambda_B^{calm,calm}\left[(1-\pi_B^{calm}L_B)^{1-\gamma} - 1\right] f^{calm} \\
0 \;=\;& f_t^{cont} + (1-\gamma)\left[r + \pi_A^{cont}(\mu_A^{cont} - r) + \pi_B^{cont}(\mu_B^{cont} - r)\right] f^{cont} && (3)\\
& - 0.5\gamma(1-\gamma)\left[(\pi_A^{cont}\sigma_A^{cont})^2 + (\pi_B^{cont}\sigma_B^{cont})^2 + 2\pi_A^{cont}\pi_B^{cont}\sigma_A^{cont}\sigma_B^{cont}\rho^{cont}\right] f^{cont} \\
& + \lambda_A^{cont,cont}\left[(1-\pi_A^{cont}L_A)^{1-\gamma} - 1\right] f^{cont} + \lambda_B^{cont,cont}\left[(1-\pi_B^{cont}L_B)^{1-\gamma} - 1\right] f^{cont} \\
& + \lambda^{cont,calm}(f^{calm} - f^{cont}).
\end{aligned}$$

and where the optimal portfolio weights solve

$$
\begin{aligned}
\mu_A^{calm} - r - \gamma(\sigma_A^{calm})^2\pi_A^{calm} - \gamma\pi_B^{calm}\sigma_A^{calm}\sigma_B^{calm}\rho^{calm} & \\
-L_A\lambda_A^{calm,cont}(1-\pi_A^{calm}L_A)^{-\gamma}\frac{f^{cont}}{f^{calm}} - L_A\lambda_A^{calm,calm}(1-\pi_A^{calm}L_A)^{-\gamma} &= 0 \quad (4)\\
\mu_B^{calm} - r - \gamma(\sigma_B^{calm})^2\pi_B^{calm} - \gamma\pi_A^{calm}\sigma_A^{calm}\sigma_B^{calm}\rho^{calm} & \\
-L_B\lambda_B^{calm,cont}(1-\pi_B^{calm}L_B)^{-\gamma}\frac{f^{cont}}{f^{calm}} - L_B\lambda_B^{calm,calm}(1-\pi_B^{calm}L_B)^{-\gamma} &= 0 \quad (5)\\
\mu_A^{cont} - r - \gamma(\sigma_A^{cont})^2\pi_A^{cont} - \gamma\pi_B^{cont}\sigma_A^{cont}\sigma_B^{cont}\rho^{cont} - L_A\lambda_A^{cont,cont}(1-\pi_A^{cont}L_A)^{-\gamma} &= 0 \quad (6)\\
\mu_B^{cont} - r - \gamma(\sigma_B^{cont})^2\pi_B^{cont} - \gamma\pi_A^{cont}\sigma_A^{cont}\sigma_B^{cont}\rho^{cont} - L_B\lambda_B^{cont,cont}(1-\pi_B^{cont}L_B)^{-\gamma} &= 0. \quad (7)
\end{aligned}
$$

The proof is given in Appendix A.2.

Equations (2), (3), (4) and (5) form a system of so-called differential-algebraic equations which can only be solved numerically.

As compared to the complete market, the investor can in general no longer achieve the optimal exposures, since he is restricted to the package of exposures offered by the two stocks, as e.g. pointed out in Liu and Pan (2003). As we will show in some numerical examples in Section 5, his exposure to some risk factors will thus be too high, while the exposure to some other risk factors will be too low. The exposure to jumps from the contagion to the calm state plays a special role. Since the exposure of both stocks to this jump is assumed to be zero, the investor has no exposure to this jump at all, and he cannot even approximately implement his hedging demand.

The indirect utility of the investor is lower in the incomplete market than in the complete market. The size of the utility loss due to market incompleteness can be measured by the difference in the certainty equivalent returns.

4 Benchmark Cases

We consider two benchmark cases. In the first case ('no contagion'), the investor ignores contagion completely. The stocks jump independently of each other, and the jump intensities are constant over time. In the second case ('joint jumps'), studied e.g. by Das and Uppal (2004), the investor takes contagion into account by assuming that stock price jumps can only happen simultaneously.

Our model is in between these extreme cases in two respects. First, we assume that some jumps are normal jumps which do not trigger contagion, while some other jumps induce contagion. Second, we allow for a time dimension of contagion. If the economy enters into the contagion state, the investor can adjust his portfolio and take a smaller (or larger) position in the risky assets. In the benchmark model with joint jumps, on the other hand, the jumps happen simultaneously, and the investor cannot react to the event of contagion any more.

4.1 No Contagion: Independent Downward Jumps

In the first benchmark case, there is no contagion at all, and downward jumps in the stocks happen independently of each other. The dynamics of stock i are

$$\frac{dS_i(t)}{S_i(t-)} = \Big[\underbrace{r + \phi_i + L_i\lambda_i}_{\mu_i}\Big] dt + \sigma_i dW_i(t) - L_i dN_i(t).$$

The Wiener processes W_A and W_B are correlated with correlation ρ. N_i is a Poisson process with intensity λ_i. The risk premium on the stock is

$$\phi_i = \sigma_i \eta_i^{diff} + L_i \lambda_i \eta_i^{jump}$$

where η_i^{diff} is the premium for diffusion risk and η_i^{jump} is the premium for jumps.

In a complete market, the investor can again choose the exposures to the risk factors. The budget restriction becomes

$$\begin{aligned}\frac{dX(t)}{X(t)} &= rdt + \theta_A^{diff}(t)\Big[dW_A(t) + \eta_A^{diff} dt\Big] + \theta_B^{diff}(t)\Big[dW_B(t) + \eta_B^{diff} dt\Big] \\ &\quad + \theta_A^{jump}(t)\left[dN_A(t) - \lambda_A dt - \eta_A^{jump}\lambda_A dt\right] \\ &\quad + \theta_B^{jump}(t)\left[dN_B(t) - \lambda_B dt - \eta_B^{jump}\lambda_B dt\right]\end{aligned}$$

where θ_i^{diff} is the exposure to diffusion risk W_i, and θ_i^{jump} is the exposure to jumps in stock i. The optimal portfolio is given in

Proposition 3 (No Contagion, Complete Market) *If there are no contagion effects in the market, the optimal exposures to the risk factors are*

$$\theta_A^{diff} = \frac{\eta_A^{diff} - \rho\eta_B^{diff}}{\gamma(1-\rho^2)} \qquad \theta_B^{diff} = \frac{\eta_B^{diff} - \rho\eta_A^{diff}}{\gamma(1-\rho^2)}$$
$$\theta_A^{jump} = (1 + \eta_A^{jump})^{-\frac{1}{\gamma}} - 1 \qquad \theta_B^{jump} = (1 + \eta_B^{jump})^{-\frac{1}{\gamma}} - 1.$$

The indirect utility function of the investor is

$$G(t,x) = \frac{x^{1-\gamma}}{1-\gamma}\exp\left\{\gamma\, C^{nc,c}\cdot(T-t)\right\}$$

where

$$
\begin{aligned}
C^{nc,c} \quad = \quad & \frac{1-\gamma}{\gamma}\Bigg[r + \frac{(\eta_A^{diff})^2 + (\eta_B^{diff})^2 - 2\rho\eta_A^{diff}\eta_B^{diff}}{2\gamma(1-\rho^2)} \\
& \qquad\qquad + \left(1+\eta_A^{jump}\right)\lambda^A + \left(1+\eta_B^{jump}\right)\lambda^B - \frac{1}{1-\gamma}(\lambda^A + \lambda^B)\Bigg] \\
& + \left(1+\eta_A^{jump}\right)^{1-\frac{1}{\gamma}}\lambda^A + \left(1+\eta_B^{jump}\right)^{1-\frac{1}{\gamma}}\lambda^B.
\end{aligned}
$$

The proof is given in Appendix B.1.

The investment opportunity set is constant. There is thus speculative demand only. Both for diffusion risk and for jump risk, this speculative demand has the same structure as in the contagion model discussed in Section 3 and is driven by the risk premia (and the diffusion correlation) only.

The certainty equivalent return is given by $\frac{\gamma}{1-\gamma}\, C^{nc,c}$. It captures how good the investment opportunities are. In a complete market, it does not depend on asset specific parameters like stock price volatilities and loss sizes, but only on economy-wide variables like the risk premia and the jump intensities. Obviously, the certainty equivalent return is increasing in the risk premia. Furthermore, it is increasing in the jump intensities λ_A and λ_B, which is formally shown in Appendix B.2. To get the intuition, note that the risk premium the investor earns on his optimal portfolio is increasing in the optimal exposure to jumps (i.e. the loss in case of a jump), the market prices of jump risk, and the jump intensities (i.e. the overall amount of jump risk in the market). The CER is thus increasing in these three variables, too.

In the incomplete market, the investor chooses the optimal weights of the two stocks, which are given in the next proposition.

Proposition 4 (No Contagion, Incomplete Market) *If there are no contagion effects in the market and only the money market account and the two stocks are traded, the indirect utility of the investor is given by*

$$
G(t,x) = \frac{x^{1-\gamma}}{1-\gamma}\exp(C^{nc,ic}\cdot(T-t))
$$

where

$$
\begin{aligned}
C^{nc,ic} \quad = \quad & (1-\gamma)\Big[r + \pi_A(\mu_A - r) + \pi_B(\mu_B - r) - \frac{\gamma}{2}(\pi_A^2\sigma_A^2 + \pi_B^2\sigma_B^2 + 2\pi_A\pi_B\sigma_A\sigma_B\rho)\Big] \\
& + \lambda_A\left[(1-\pi_A L_A)^{1-\gamma} - 1\right] + \lambda_B\left[(1-\pi_B L_B)^{1-\gamma} - 1\right]
\end{aligned}
$$

and where the optimal portfolio weights are given as the unique solution of

$$\begin{aligned}\mu_A - r - \gamma\sigma_A^2\pi_A - \gamma\pi_B\sigma_A\sigma_B\rho - L_A\lambda_A(1-\pi_A L_A)^{-\gamma} &= 0\\ \mu_B - r - \gamma\sigma_B^2\pi_B - \gamma\pi_A\sigma_B\sigma_A\rho - L_B\lambda_B(1-\pi_B L_B)^{-\gamma} &= 0.\end{aligned}$$

The proof is given in Appendix B.3.

Just as in our contagion model, the investor can in general no longer achieve the optimal exposures as compared to the complete market, since he is restricted to the package of exposures offered by the two stocks, as also pointed out by Liu and Pan (2003) in a model with jump risk, but one stock only. Again, his exposure to some risk factors will be too high, while the exposure to some other risk factors will be too low. Since the investment opportunity set is constant, the investor does not need to implement a hedging demand in the incomplete market, either.

The indirect utility of the investor is lower in the incomplete market than in the complete market. The size of the utility loss due to market incompleteness can be measured by the difference in the certainty equivalent returns.

4.2 Joint Downward Jumps

In the second benchmark case, the investor takes contagion into account by assuming that stock price jumps always happen simultaneously. The dynamics for stock i are

$$\frac{dS_i(t)}{S_i(t-)} = \Big[\underbrace{r + \phi_i + L_i\lambda_{joint}}_{\mu_i}\Big]dt + \sigma_i dW_i(t) - L_i dN_{joint}(t)$$

and the risk premium on the stock is

$$\phi_i = \sigma_i\eta_i^{diff} + L_i\lambda_{joint}\eta_{joint}^{jump}.$$

We want the behavior of the individual stocks to be the same in both benchmark cases, so that only the joint behavior differs. Consequently, we assume that the parameters for the individual stocks are the same as in Section 4.1, and we set $\lambda_{joint} = \lambda_A = \lambda_B$ and $\eta_{joint}^{jump} = \eta_A^{jump} = \eta_B^{jump}$.

In the complete market, the solution to the portfolio planning problem is given in the next proposition.

Proposition 5 (Joint Downward Jumps, Complete Market) *If there are joint downward jumps, the optimal exposures to the risk factors are*

$$\theta_A^{diff} = \frac{\eta_A^{diff} - \rho\eta_B^{diff}}{\gamma(1-\rho^2)} \qquad \theta_B^{diff} = \frac{\eta_B^{diff} - \rho\eta_A^{diff}}{\gamma(1-\rho^2)}$$
$$\theta_{joint}^{jump} = (1+\eta_{joint}^{jump})^{-\frac{1}{\gamma}} - 1.$$

The indirect utility function of the investor is

$$G(t,x) = \frac{x^{1-\gamma}}{1-\gamma} \exp\left\{\gamma\, C^{jj,c} \cdot (T-t)\right\}$$

where

$$C^{jj,c} = \frac{1-\gamma}{\gamma}\left[r + \frac{(\eta_A^{diff})^2 + (\eta_B^{diff})^2 - 2\rho\eta_A^{diff}\eta_B^{diff}}{2\gamma(1-\rho^2)} + \left(1+\eta_{joint}^{jump}\right)\lambda_{joint} - \frac{1}{1-\gamma}\lambda_{joint}\right] + \left(1+\eta_{joint}^{jump}\right)^{1-\frac{1}{\gamma}}\lambda_{joint}.$$

The optimal exposures depend on the market prices of risk (and on the correlation) only. With identical parameters for the behavior of the individual stocks, they are thus the same as in the case of independent jumps. If a jump happens, the investor loses exactly the same amount of money, no matter whether he assumes independent jumps or joint jumps. What differs, however, is the optimal portfolio held by the investor. If there are joint jumps, the portfolio that is optimal with independent jumps would have a jump risk exposure that is twice as high as optimal. With joint jumps, the investor is thus more conservative.

The CER is lower with joint jumps than with independent jumps. To get the intuition, note that the market prices of risk are identical, while the average number of jumps is twice as large in the case of independent jumps as in the case of joint jumps. Since the CER increases in the jump intensity and thus in the average number of jumps, it is indeed smaller with joint jumps.

In the incomplete market, the investor is again restricted to the package of exposures offered by the stocks. The optimal portfolio is given in the next proposition.

Proposition 6 (Joint Downward Jumps, Incomplete Market) *If there are joint downward jumps and only the money market account and the two stocks are traded, the indirect utility of the investor is given by*

$$G(t,x) = \frac{x^{1-\gamma}}{1-\gamma} \exp\{C^{jj,ic} \cdot (T-t)\}$$

where

$$\begin{aligned} C^{jj,ic} &= (1-\gamma)\left[r+\pi_A(\mu_A-r)+\pi_B(\mu_B-r)-\frac{\gamma}{2}\left(\pi_A^2\sigma_A^2+\pi_B^2\sigma_B^2+2\pi_A\pi_B\sigma_A\sigma_B\rho\right)\right] \\ &\quad +\lambda_{joint}\left[(1-\pi_A L_A-\pi_B L_B)^{1-\gamma}-1\right] \end{aligned}$$

and where the optimal portfolio weights are given as the unique solutions of

$$\begin{aligned} \mu_A - r - \gamma\sigma_A^2\pi_A - \gamma\pi_B\sigma_A\sigma_B\rho - L_A\lambda_{joint}(1-\pi_A L_A-\pi_B L_B)^{-\gamma} &= 0 \\ \mu_B - r - \gamma\sigma_B^2\pi_B - \gamma\pi_A\sigma_B\sigma_A\rho - L_B\lambda_{joint}(1-\pi_A L_A-\pi_B L_B)^{-\gamma} &= 0. \end{aligned}$$

Just as in the model setup without contagion, the investment opportunity set is constant and the investor does not have a hedging demand in the incomplete market, either.

5 Numerical Results

5.1 Parameter Choice and Model Calibration

We consider a CRRA-investor with a relative risk aversion of $\gamma = 3$ and a planning horizon of 20 years. The interest rate is set to $r = 0.01$. The two stocks are assumed to follow identical processes. We choose the parameters such that they approximately fit the behavior of the S&P500 over the last 25 years, where we rely on the parameter estimates of Eraker, Johannes, and Polson (2003) and Broadie, Chernov, and Johannes (2007). Since we want to focus on the impact of contagion, which is reflected in the difference between the jump intensities in the calm and in the contagion state, all other parameters are assumed to be equal in both states.

The diffusion volatility σ is set to 0.15, and the Wiener processes driving the stock price dynamics are correlated with $\rho = 0.5$. Both these parameters do not depend on the current state. The jump intensity in the benchmark models is set to 1.5, and we calibrate the jump intensities in our contagion model such that the average long-run jump intensity is equal to 1.5, too. More details on this step of the calibration will be given below. The loss in case of a jump in one of the stocks is assumed to be constant and set equal to -0.05, which is slightly higher than the estimate provided in models that also include stochastic volatility. Remember that the loss for a jump back from the contagion to the calm state is equal to zero.

The market price for diffusion risk is assumed to be equal to 0.35 in both states. Jumps from the contagion state back to the calm state are not priced. For the other market prices of jump risk, we consider two extreme cases. In the first case (parametrization 1), we assume that they are identical in all states. This implies a rather high drift of the stocks in the contagion state. In the second case (parametrization 2), we assume that the expected excess returns of the stocks are equal in both states, which results in larger market prices of risk in the calm state and lower ones in the contagion state. We calibrate the market prices of jump risk such that the average expected excess return of the stocks is equal to 8.25% for both parametrizations.

The two benchmark models without contagion and with joint jumps are calibrated such that the behavior of the stock prices in the benchmark model is as similar as possible to the behavior in our model. Therefore, we set the local moments in the benchmark models equal to the long run averages of the local moments in our model. The stationary probability of the calm and contagion state is

$$\begin{aligned} p^{calm} &= \tfrac{\lambda^{cont,calm}}{\lambda^{cont,calm}+\lambda_A^{calm,cont}+\lambda_B^{calm,cont}} \\ p^{cont} &= \tfrac{\lambda_A^{calm,cont}+\lambda_B^{calm,cont}}{\lambda^{cont,calm}+\lambda_A^{calm,cont}+\lambda_B^{calm,cont}}, \end{aligned}$$

and we know from the ergodic theorem for Markov chains[3] that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t g(Z(s))ds = g(calm)p^{calm} + g(cont)p^{cont}$$

where g is some state-dependent function.

Firstly, we want the stocks to have the same risk in the contagion model and in the benchmark models. We thus equate the variance of the stock, which gives

$$\begin{aligned} (\sigma_i)^2 + L_i^2\lambda_i &= p^{calm}\left[\left(\sigma_i^{calm}\right)^2 + \left(L_i^{calm,calm}\right)^2\lambda_i^{calm,calm} + \left(L_i^{calm,cont}\right)^2\lambda_i^{calm,cont}\right] \\ &\quad + p^{cont}\left[\left(\sigma_i^{cont}\right)^2 + \left(L_i^{cont,cont}\right)^2\lambda_i^{cont,cont}\right] \\ &\quad + p^{calm}p^{cont}\Big[\sigma_i^{calm}\eta_i^{calm} + L_i^{calm,calm}\lambda_i^{calm,calm}\eta_i^{calm,calm} \\ &\qquad\qquad + L_i^{calm,cont}\lambda_i^{calm,cont}\eta_i^{calm,cont} \\ &\qquad\qquad - \sigma_i^{cont}\eta_i^{cont} - L_i^{cont,cont}\lambda_i^{cont,cont}\eta_i^{cont,cont}\Big]^2. \end{aligned}$$

We also equate the jump intensity (for those jumps that result in a loss) and the average

[3] See, e.g., Brémaud (2001).

jump size

$$\begin{aligned}
\lambda_i &= p^{calm}\left(\lambda_i^{calm,calm}+\lambda_i^{calm,cont}\right)+p^{cont}\lambda_i^{cont,cont} \qquad (8)\\
L_i &= p^{calm}\left[\frac{\lambda_i^{calm,calm}}{\lambda_i^{calm,calm}+\lambda_i^{calm,cont}}\cdot L_i^{calm,calm}+\frac{\lambda_i^{calm,cont}}{\lambda_i^{calm,calm}+\lambda_i^{calm,cont}}\cdot L_i^{calm,cont}\right]\\
&\quad + p^{cont}\cdot L_i^{cont,cont}
\end{aligned}$$

Secondly, we want the stocks to have the same expected excess returns. Since the investor might deal differently with jump and diffusion risk, we also equate the risk premia earned on stock diffusion risk and stock jump risk. This gives two additional restrictions

$$\begin{aligned}
\sigma_i\eta_i^{diff} &= p^{calm}\sigma_i^{calm}\eta_i^{calm}+p^{cont}\sigma_i^{cont}\eta_i^{cont}\\
L_i\lambda_i\eta_i^{jump} &= p^{calm}\left(L_i^{calm,calm}\lambda_i^{calm,calm}\eta_i^{calm,calm}+L_i^{calm,cont}\lambda_i^{calm,cont}\eta_i^{calm,cont}\right)\\
&\quad + p^{cont}L_i^{cont,cont}\lambda_i^{cont,cont}\eta_i^{cont,cont}.
\end{aligned}$$

The jump intensities and the loss sizes in the benchmark models are identical for both parameterizations. This also holds for the jump risk premia, which coincide with the state-independent jump risk premia of parametrization 1. The diffusion volatility in the benchmark models is identical to that in our model with parametrization 2. It is slightly larger and depends on the jump intensities if the market prices of risk are equal (parametrization 1), which accounts for the slightly larger variance in this case. As a consequence, the market price for diffusion risk is slightly lower in this case.

The different jump intensities in our model are chosen such that the average number of jumps per year, which follows from Equation (8), is equal to the benchmark value of 1.5. Since we want to focus on contagion, we explicitly control for its severeness and thus for the wedge driven between the two states. The difference between the jump intensities in the calm and contagion state is captured by $\xi \geq 1$:

$$\lambda_i^{cont,cont} = \xi_i\left(\lambda_i^{calm,calm}+\lambda_i^{calm,cont}\right) \qquad i\in\{A,B\}.$$

The conditional probability that a loss in a stock actually triggers contagion is given by the parameter α:

$$\lambda_i^{calm,cont} = \alpha_i\left(\lambda_i^{calm,calm}+\lambda_i^{calm,cont}\right) \qquad i\in\{A,B\},$$

and the average time the economy stays in the contagion state depends on ψ:

$$\lambda^{cont,calm} = \psi\left(\lambda_A^{cont,cont}+\lambda_B^{cont,cont}\right).$$

Given ξ, α, and ψ and the average jump intensity of 1.5, all other jump intensities can be calculated. In the base case calibration, we set $\xi = 4$, $\alpha = 0.5$ and $\psi = 0.25$. The resulting parameters are given in Table 1. Table 2 shows the resulting conditional equity risk premia and variances of stock returns for both parameterizations and in the benchmark models as well as their decomposition into diffusion and jump components. Several other combinations of the parameters we have considered in robustness checks are given in Table 3, where we use $\xi \in [1, 10]$, $\alpha \in [0.2, 0.5]$ and $\psi \in [0.2, 2/3]$.

5.2 Optimal Exposures and Optimal Portfolios

Table 4 gives the solution to the portfolio planning problem for the base-case parameters from Table 1 both for the complete and the incomplete market. We discuss the case of complete markets first, where the investor can achieve any desired payoff profile.

The demand for diffusion risk is driven by the speculative component only. It is identical in the calm and in the contagion state and for both parametrizations, because the market prices of diffusion risk are identical by assumption. In the benchmark models, the optimal demand is the same for parametrization 2 and slightly lower for parametrization 1, which can be attributed to the lower market price of diffusion risk in this case.

The demand for jump risk can be decomposed into a speculative component and – for those jumps that change the state – a hedging component. The speculative demand depends on the market prices of jump risk. Since jumps from the contagion state back to the calm state are not priced by assumption, the speculative demand is zero. For the other jumps, which all lead to a loss in stock prices, jump risk is priced, and there is a negative speculative demand. If the market prices of risk are identical in all states (parametrization 1), the speculative demand does not depend on the state and coincides with the speculative demand in the two benchmark models. If equity risk premia are constant (parametrization 2), on the other hand, the market price of risk is lower in the contagion state than in the calm state, and consequently, the speculative demand is lower in absolute terms in the contagion state, too. The investor is thus more aggressive in the calm state, and less aggressive in the contagion state, as compared to parametrization 1. The market price of risk in the benchmark models is in between the market prices of risk in the calm and the contagion state, and the speculative demand is in between those from the contagion model, too.

The sign of the hedging demand depends on which of the two states is the better one. The right panel of Figure 2 shows the certainty equivalent returns in both states. If the market prices of risk are constant (parametrization 1), the investment opportunity set is better in the contagion state where jumps happen more often than in the calm state. Given that $\gamma > 1$, the hedging demand for jumps from the calm to the (better) contagion state is negative, which implies that the investor takes a more aggressive position in jump risk in the calm state. In the contagion state, on the other hand, his optimal exposure to jumps back to the (worse) calm state is positive. If the expected returns are equal (parametrization 2), the calm state is better than the contagion state. Then, the hedging demand in the (better) calm state is positive, reducing the demand for jump risk in this state, while the hedging demand in the (worse) contagion state is negative.

The optimal exposures are different in the calm and in the contagion state, and the investor will adjust his portfolio when the state of the economy changes. He thus profits from the time dimension of contagion captured in our model. His trading desire due to contagion is much more pronounced for the case of equal equity risk premia (parametrization 2), where trading is induced by changes in the market prices of risk and in the hedging demand, than for the case of identical market prices of risk, where trading is induced by changes in the hedging demand only.

If the market is incomplete, the investor cannot implement the overall optimal exposures. As can be seen in Table 4, the realized exposures will be somewhere in between the optimal exposures from the complete case. For the given parameters, the exposure to diffusion risk is too high, and the exposure to jump risk is too low in absolute terms both in our model and in the benchmark models. The position in risky assets is larger in the state in which investment opportunities are better, that is in the calm state in case of equal equity risk premia and in the contagion state in case of equal market prices of risk.

In the benchmark models, the investor does not distinguish between calm and contagion states. If he ignores contagion completely, the optimal position in stocks is somewhere in between the optimal positions in the calm and in the contagion state. If the investor assumes that there are joint jumps, he is more conservative and reduces his optimal position in stocks significantly.

The certainty equivalent returns in our model and in the two benchmark models are shown in the left panel Figure 2. As expected, the utility loss due to market incompleteness is largest in our contagion model since the investor fails to implement the optimal myopic demand as well as the intertemporal hedging demand, whereas a hedging demand does not exist in both benchmark models. In the benchmark models, the utility loss is larger in case of no contagion than in case of joint jumps for our parametrizations (in particular, for our choice of the relation between diffusion risk and jump risk in stocks). To get the intuition, note that the actual exposures in the incomplete market are much closer to the optimal exposures in the complete market for the case of joint jumps, so that the utility loss is smaller in this case. In absolute numbers, however, the joint jumps model gives the lowest utility both in an incomplete and in a complete market, since the average number of jumps is cut in half compared to the other models.

Comparing both parametrizations (equal market prices of risk and equal equity risk premia), the utility loss caused by market incompleteness is roughly equal. The overall size of the utility loss is thus determined rather by a certain model specification than by the assumptions on the market prices of risk. The utility differences between the calm and the contagion state, however, are more pronounced with equal market prices of risk (parametrization 1) due to the extreme changes of the conditional equity risk premium.

Robustness checks show that the results do not change qualitatively when we vary ξ, α and ψ, i.e. the overall size of contagion, the risk of entering the contagion state, and the duration of the contagion state. In line with intuition, a larger difference between the calm and contagion state, i.e. a larger value of ξ, leads to more extreme results. This effect is most pronounced for equal market prices of risk (parametrization 1), where the investment opportunities in the contagion state become very attractive and can induce the investor to take a highly levered position in stocks. With equal equity risk premia (parametrization 2), this effect is much smaller. For both parametrizations, the utility of the investor is increasing in ξ in a complete market, but is not monotonous in an incomplete market. Therefore, utility losses due to market incompleteness also tend to increase in ξ. The probability α of entering the contagion state does not have much impact on the results. On the other hand, the smaller ψ, i.e. the longer the economy stays in the contagion state once it has entered this state, the more extreme the portfolio weights, exposures and utility functions.

5.3 Model Mis-Specification

If the investor relies on a benchmark model instead of the true model from Section 2.1, the investor will not hold the optimal portfolio. In this section, we analyze the utility loss he suffers from due to this suboptimal behavior.

5.3.1 Incomplete Market

In the incomplete market, the investor can only invest into the two stocks and into the money market account. In case of model mis-specification, he (incorrectly) uses one of the benchmark models to determine the optimal portfolio. For both these models, the optimal portfolio weights are constant over time. The indirect utility derived from this strategy is given in the next proposition.

Proposition 7 (Model Mis-Specification, Incomplete Market) *In an economy with contagion where only the two stocks and the money market account are traded and for an investor who uses the portfolio weights $\widehat{\pi}_A, \widehat{\pi}_B$, the indirect utility in state $j \in \{calm, cont\}$ is*

$$G^j(t,x) = \frac{x^{1-\gamma}}{1-\gamma}\widehat{f}^j(t)$$

where $\widehat{f}^j$ is given by

$$\begin{pmatrix} \widehat{f}^{calm}(t) \\ \widehat{f}^{cont}(t) \end{pmatrix} = \exp\left\{ \begin{pmatrix} \widehat{C}^{calm,calm} & \widehat{C}^{calm,cont} \\ \widehat{C}^{cont,calm} & \widehat{C}^{cont,cont} \end{pmatrix} (T-t) \right\} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

where

$$\begin{aligned}
\widehat{C}^{calm,calm} &= (1-\gamma)\left[r + \widehat{\pi}_A(\mu_A^{calm} - r) + \widehat{\pi}_B(\mu_B^{calm} - r)\right] \\
&\quad - 0.5\gamma(1-\gamma)\left[(\widehat{\pi}_A\sigma_A^{calm})^2 + (\widehat{\pi}_B\sigma_B^{calm})^2 + 2\widehat{\pi}_A\widehat{\pi}_B\sigma_A^{calm}\sigma_B^{calm}\rho^{calm}\right] \\
&\quad - \lambda_A^{calm,cont} + \lambda_A^{calm,calm}\left[(1-\widehat{\pi}_A L_A)^{1-\gamma} - 1\right] \\
&\quad - \lambda_B^{calm,cont} + \lambda_B^{calm,calm}\left[(1-\widehat{\pi}_B L_B)^{1-\gamma} - 1\right] \\
\widehat{C}^{calm,cont} &= \lambda_A^{calm,cont}(1-\widehat{\pi}_A L_A)^{1-\gamma} + \lambda_B^{calm,cont}(1-\widehat{\pi}_B L_B)^{1-\gamma} \\
\widehat{C}^{cont,calm} &= \lambda^{cont,calm} \\
\widehat{C}^{cont,cont} &= (1-\gamma)\left[r + \widehat{\pi}_A(\mu_A^{cont} - r) + \widehat{\pi}_B(\mu_B^{cont} - r)\right] \\
&\quad - 0.5\gamma(1-\gamma)\left[(\widehat{\pi}_A\sigma_A^{cont})^2 + (\widehat{\pi}_B\sigma_B^{cont})^2 + 2\widehat{\pi}_A\widehat{\pi}_B\sigma_A^{cont}\sigma_B^{calm}\rho^{cont}\right] \\
&\quad - \lambda^{cont,calm} + \lambda_A^{cont,cont}\left[(1-\widehat{\pi}_A L_A)^{1-\gamma} - 1\right] + \lambda_B^{cont,cont}\left[(1-\widehat{\pi}_B L_B)^{1-\gamma} - 1\right].
\end{aligned}$$

The proof is given in Appendix C.1.

The upper panels of Figure 3 and 4 show the certainty equivalent returns in case of model mis-specification for equal market prices of risk and equal equity risk premia, respectively. For the base case parametrization, the investor looses up to 20 basis points a year if he relies on an incorrect model. The losses are larger for equal market prices of risk (parametrization 1) than for equal equity risk premia (parametrization 2), since the differences in the optimal portfolios between the states which the investor fails to pick up are larger in the first case. Surprisingly, the investor is (slightly) worse off if he assumes joint jumps and thus only ignores the time dimension of contagion than if he ignores contagion completely. And again, the results, i.e. the utility losses, increase in the difference between the calm and contagion state as measured by ξ.

5.3.2 Complete Market

Next, we analyze the impact of model mis-specification when the market is complete. To determine whether enough derivatives are traded for market completeness, the investor relies on the benchmark model. In the case of independent jumps, four risky assets are needed, while in the case of joint jumps, three risky assets are enough. We assume that the investor uses the two stocks, an ATM-call on stock A with a time to maturity of 3 months, and – if needed – an identical ATM-call on stock B. These short-term ATM-options are usually among the most liquid contracts. Note however that the choice of contracts will have an impact on the utility loss due to model mis-specification.

The analysis of model mis-specification is more complicated than in case of an incomplete market. In the first step, the investor determines the seemingly optimal exposures in the benchmark model. In the second step, he uses the risky assets and their risk exposure to implement these seemingly optimal exposures, where he (incorrectly) determines the sensitivities of the derivatives in the benchmark model. Given the seemingly optimal portfolio, we (but not the investor) can then use the sensitivities from the true model to determine the realized exposure. Given these realized exposures $\widehat{\theta}$, which are again constant over time, we can then finally calculate the realized indirect utility.

Proposition 8 (Model Mis-Specification, Complete Market) *In a complete market with contagion effects, the utility obtained by an investor who uses the incorrect risk factor exposures $\widehat{\theta}$ is given by*

$$\widehat{G}^j(t,x) = \frac{x^{1-\gamma}}{1-\gamma}\,\widehat{f}^j(t)$$

where $j \in \{calm, cont\}$ and

$$\begin{pmatrix} \widehat{f}^{calm}(t) \\ \widehat{f}^{cont}(t) \end{pmatrix} = \exp\left\{ \begin{pmatrix} \widehat{C}^{calm,calm} & \widehat{C}^{calm,cont} \\ \widehat{C}^{cont,calm} & \widehat{C}^{cont,cont} \end{pmatrix} (T-t) \right\} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

with

$$
\begin{aligned}
\widehat{C}^{calm,calm} &= (1-\gamma)\Big[r + \widehat{\theta}_A^{calm}\eta_A^{calm} + \widehat{\theta}_B^{calm}\eta_B^{calm} \\
&\qquad - \widehat{\theta}_A^{calm,calm}\lambda_A^{calm,calm}(1+\eta_A^{calm,calm}) - \widehat{\theta}_A^{calm,cont}\lambda_A^{calm,cont}(1+\eta_A^{calm,cont}) \\
&\qquad - \widehat{\theta}_B^{calm,calm}\lambda_B^{calm,calm}(1+\eta_B^{calm,calm}) - \widehat{\theta}_B^{calm,cont}\lambda_B^{calm,cont}(1+\eta_B^{calm,cont})\Big] \\
&\quad - 0.5\gamma(1-\gamma)\Big[(\widehat{\theta}_A^{calm})^2 + (\widehat{\theta}_B^{calm})^2 + 2\rho^{calm}\widehat{\theta}_A^{calm}\widehat{\theta}_B^{calm}\Big] \\
&\quad + \lambda_A^{calm,calm}\Big[(1+\widehat{\theta}_A^{calm,calm})^{1-\gamma} - 1\Big] + \lambda_B^{calm,calm}\Big[(1+\widehat{\theta}_B^{calm,calm})^{1-\gamma} - 1\Big] \\
&\quad - \lambda_A^{calm,cont} - \lambda_B^{calm,cont} \\
\widehat{C}^{calm,cont} &= \lambda_A^{calm,cont}(1+\widehat{\theta}_A^{calm,cont})^{1-\gamma} + \lambda_B^{calm,cont}(1+\widehat{\theta}_B^{calm,cont})^{1-\gamma} \\
\widehat{C}^{cont,calm} &= \lambda^{cont,calm}(1+\widehat{\theta}^{cont,calm})^{1-\gamma} \\
\widehat{C}^{cont,cont} &= (1-\gamma)\Big[r + \widehat{\theta}_A^{cont}\eta_A^{cont} + \widehat{\theta}_B^{cont}\eta_B^{cont} \\
&\qquad - \widehat{\theta}_A^{cont,cont}\lambda_A^{cont,cont}(1+\eta_A^{cont,cont}) - \widehat{\theta}_B^{cont,cont}\lambda_B^{cont,cont}(1+\eta_B^{cont,cont}) \\
&\qquad - \widehat{\theta}^{cont,calm}\lambda^{cont,calm}(1+\eta^{cont,calm})\Big] \\
&\quad - 0.5\gamma(1-\gamma)\Big[(\widehat{\theta}_A^{cont})^2 + (\widehat{\theta}_B^{cont})^2 + 2\rho^{cont}\widehat{\theta}_A^{cont}\widehat{\theta}_B^{cont}\Big] \\
&\quad + \lambda_A^{cont,cont}\Big[(1+\widehat{\theta}_A^{cont,cont})^{1-\gamma} - 1\Big] + \lambda_B^{cont,cont}\Big[(1+\widehat{\theta}_B^{cont,cont})^{1-\gamma} - 1\Big] \\
&\quad - \lambda^{cont,calm}
\end{aligned}
$$

The proof is given in Appendix C.2.

The lower panels of Figure 3 and 4 show the certainty equivalent returns when the correct model is used and when one of the benchmark models is used to determine the (seemingly) optimal portfolio. The CER losses are highly economically significant, and they are much higher than in the incomplete market, since the investor now makes an additional mistake. To set up the optimal portfolio, he has to convert the optimal exposures into portfolio weights. While the exposures of the stocks are model independent, the exposures of the derivatives depend on the model, and an investor using an incorrect model for portfolio planning will use the same incorrect model for pricing derivatives, too. As can be seen from the figures, the mistakes in calculating the exposures and in pricing the derivatives do not cancel each other, but rather add up.

Figure 5 compares the utility losses for different values of ξ, where we assume equal equity risk premia in both states. The results for equal market prices of risk (not shown here) are qualitatively similar. As can be seen from the graphs, the difference between the calm and contagion state has a very large impact on the utility losses. They are already far from negligible for a rather low value of $\xi = 2$, and increase to around 10%-15%

a year for $\xi = 10$. For this high level of ξ, the CER can even become negative, and the investor would be better off if he just invested his wealth at the risk-free rate only, ignoring all risky assets. The graphs also show that the utility losses in the calm and contagion state are approximately equal. If the investor relies on one of the benchmark models, the parameters of this model and thus also his (seemingly) optimal portfolio represents kind of an average between the two states. The distance to the truly optimal portfolios is thus approximately equal for both states, and this also holds for the utility losses.

Different from the incomplete market, it now depends on ξ, i.e. on the severeness of contagion, which of the two benchmark models leads to the smaller utility loss. For low values of ξ, for which the differences between the calm and contagion state are rather moderate, the investor is still better off if he ignores contagion completely. For higher values of ξ, however, he is significantly better off if he just ignores the time dimension of contagion. To get the intuition, note that the investor uses two options in case of independent jumps (when he ignores contagion), but only one option in case of joint jumps (when he only ignores the time dimension of contagion). Since the use of derivatives is the main reason for the high utility losses, he is better off the less derivatives he adds to his portfolio, i.e. if he relies on the model with joint jumps. In this model, however, he is too conservative and does not take advantage of the jump risk premia offered in the market. The trade-off between these two arguments depends on the absolute size of the position in derivatives. The larger this position, the larger the relative advantage of the model with joint jumps. Since the position in derivatives increases in the difference between the calm and contagion state, the investor is indeed better off if he uses the model with joint jumps for high values of ξ. This effect is more pronounced for equal market prices of risk (parametrization 1) than for equal equity risk premia (parametrization 2), since the differences in the optimal exposures are larger in the first case.

An investor who relies on the correct model is better off in the complete market. In case of model mis-specification, this may no longer be true, as can be seen in Figure 3 and 4. While an investor who incorrectly bases his decisions on a model with joint jumps is still better off in the complete market, an investor ignoring any contagion might be better off in the incomplete market. In this case, the utility gain from having access to derivatives (and thus more payoff patterns) is more than offset by the utility loss from using the incorrect sensitivities and implementing the seemingly optimal strategy in the wrong way.

We also did a robustness check with respect to α and ψ, which govern the risk of entering the contagion state and the average time the economy stays in the contagion state. As already seen above, the impact of these two parameters is rather small, and the qualitative results do not change.

5.4 Robustness Checks

In the preceding sections, we have shown that contagion has a substantial effect on optimal exposures, optimal portfolio weights, and the investor's expected utility. Furthermore, an investor who uses an incorrect model might suffer large utility losses in particular in a complete market where he also uses derivatives. While we have already discussed the sensitivity of our results with respect to the severeness of contagion, we now do some additional robustness checks with respect to the risk aversion, the size of the losses, and the diffusion correlation between the stocks.

5.4.1 Relative Risk Aversion

The results up to now have been based on a relative risk aversion of $\gamma = 3$. We have redone the analysis for values of γ between 1.5 and 10. In line with intuition, the results become less extreme the higher the risk aversion and the less the investor therefore invests in risky assets. The qualitative results, however, do not change.

While the utility losses due to model mis-specification decrease in γ, they are still highly economically significant even for a high risk aversion of $\gamma = 10$. The investor is much more conservative in this case. Nevertheless, the loss in CER can well exceed 8% in the complete market and is thus far from negligible.

5.4.2 Loss Size

In a second step, we have changed the loss size from $L = 0.05$ to the more moderate value of $L = 0.03$. This has no impact on the results in the complete market, which are independent of the exact losses in the stocks, but depend on only on the intensity of jumps and their market prices of risk. In the incomplete market, however, the smaller loss size decreases the utility of the investor, since the package offered by stock fits the optimal exposure now even worse. Consequently, the utility loss due to market incompleteness increases.

The impact of the loss size on the losses due to model mis-specification is mixed. While the utility loss in the incomplete market and in case the joint jumps model is used decreases with the lower loss size, the opposite is true in a complete market and in case the investor relies on a model with no contagion at all. Overall, however, the results do not change qualitatively when we change the loss size.

5.4.3 Diffusion Correlation

As an additional robustness check, we consider different values for the diffusion correlation parameter ρ, which was set to $\rho = 0.5$ in our base case. We redo the analysis for $\rho = 0$ and $\rho = -0.5$.

When ρ decreases, the overall level of risk decreases in all economies, while the market prices of risk and the equity risk premia on individual stocks, respectively, stay the same. Consequently, the utility of the investor increases both in a complete and in an incomplete market and for all models. The increase in utility is smallest in the benchmark model with joint jumps, where the stocks are also correlated due to joint jumps and where the decrease in diffusion correlation thus is of second-order importance.

The utility loss due to market incompleteness is smallest for $\rho = -0.5$ in our contagion model. This can be explained by the fact that the package offered by the stocks is closest to the overall optimal exposure in this case. The result is specific to the parameters used and will not hold in general. The utility losses due to model mis-specification in an incomplete market may also depend on ρ. While there is hardly any impact if the investor ignores contagion completely, the utility loss increases to around 2% for $\rho = -0.5$ if the investor relies on a model with joint jumps.

Figure 6 shows the CER in case of model mis-specification in a complete market, where we assume equal market prices of risk (parametrization 1). The results are qualitatively similar for equal equity risk premia (parametrization 2). As can be seen from the graphs, it depends on ρ whether the investor is better off if he ignores contagion completely or if he just ignores the time dimension of contagion. To get the intuition, remember that the model with joint jumps leads to a portfolio that is too conservative, but reduces the impact of calculating the incorrect sensitivities. For $\rho = -0.5$, the optimal portfolio includes only a small position in derivatives, so that the model with joint jumps performs worse than the model with no contagion at all. For $\rho = 0.5$, on the other hand, the investor is better off if he uses the model with joint jumps, since the position in derivatives is now significantly larger. Again, the utility loss due to model mis-specification may exceed the utility gain due to market completeness if the differences between the calm and the contagion state are large enough. This again suggests that the investor may be better off if he does not use derivatives at all instead of using them the wrong way.

6 Conclusion

The paper analyses the optimal portfolio in case of contagion risk. Instead of capturing contagion by joint jumps in the stocks, we assume that some large losses in stocks increase the jump intensities significantly, which adds a time dimension to contagion. The investor is thus able to adjust his portfolio when the economy enters the contagion state, and our results show that he indeed uses this possibility. The direction of the portfolio adjustment depends on his relative risk aversion and on the market prices of risk.

If the investor incorrectly uses a simpler model, he suffers a utility loss. In an incomplete market where the investor uses stocks and an investment at the risk-free rate only, the investor's utility loss is larger if he assumes joint jumps (and thus ignores only the time dimension of contagion) than if he ignores contagion completely. If the investor also uses derivatives, on the other hand, the utility loss is larger if the investor ignores all aspects of contagion and if the difference between the calm and the contagion state is rather large. Furthermore, an investor worrying about model mis-specification might be better off if he does not use derivatives at all, since the utility gain from having access to derivatives can be more than offset by the utility loss due to using an incorrect model.

There are several directions for future research. First, one might want to include learning into the model, where the investor can no longer observe the true state of the economy, but has to learn about from observing stock prices. He will then use a filtering approach to update the probabilities of the two states over time. Second, our results show that the assumptions about the market prices of risk have a significant impact on the optimal portfolios. The next step would be a general equilibrium setup in which market prices of risk are determined endogenously in order to show how the investors price the risk of contagion.

A Contagion

A.1 Complete Market - Proof

We solve the portfolio problem in a complete market for a general Markov chain with states $k \in \{1, \ldots, K\}$. The indirect utility function in state j at time t and for a current wealth level of x is denoted by $G^j(t,x)$. The functions G must solve the system of Hamilton-Jacobi-Bellman equations, where we have one equation for each state j:

$$\begin{aligned} 0 \;=\; \max\Bigg\{ & G^j_t + G^j_x x \Bigg[r + \theta^j_A(t)\eta^j_A + \theta^j_B(t)\eta^j_B - \sum_{k\neq j} \theta^{j,k}(t)\lambda^{j,k}\left(1+\eta^{j,k}\right) \Bigg] \\ & + 0.5 G^j_{xx} x^2 \left[\theta^j_A(t)^2 + \theta^j_B(t)^2 + 2\rho^j \theta^j_A(t)\theta^j_B(t)\right] \\ & + \sum_{k\neq j} \left[G^k(t, x(1+\theta^{j,k}(t))) - G^j(t,x)\right]\lambda^{j,k} \Bigg\}. \end{aligned}$$

Subscripts of G denote partial derivatives. We assume constant relative risk aversion, and rely on the usual guess for the indirect utility function

$$G^j(t,x) = \frac{x^{1-\gamma}}{1-\gamma}\left(f^j(t)\right)^{\gamma}.$$

The partial derivatives are

$$\begin{aligned} G^j_t(t,x) &= \frac{x^{1-\gamma}}{1-\gamma}\gamma\left(f^j(t)\right)^{\gamma-1} f^j(t)_t \\ G^j_x(t,x) &= x^{-\gamma}\left(f^j(t)\right)^{\gamma} \\ G^j_{xx}(t,x) &= -\gamma x^{-\gamma-1}\left(f^j(t)\right)^{\gamma}, \end{aligned}$$

and the change in the indirect utility due to a jump is

$$G^k(t, x(1+\theta^{j,k}(t))) - G^j(t,x) = \frac{x^{1-\gamma}}{1-\gamma}\left[\left(f^k(t)\right)^{\gamma}(1+\theta^{j,k}(t))^{1-\gamma} - \left(f^j(t)\right)^{\gamma}\right].$$

Plugging these expressions into the HJB-equations and simplifying gives

$$\begin{aligned} 0 \;=\; \max\Bigg\{ & \gamma\frac{f^j_t}{f^j} + (1-\gamma)\Bigg[r + \theta^j_A(t)\eta^j_A + \theta^j_B(t)\eta^j_B - \sum_{k\neq j} \theta^{j,k}(t)\lambda^{j,k}\left(1+\eta^{j,k}\right) \Bigg] \\ & - 0.5\gamma(1-\gamma)\left[\theta^j_A(t)^2 + \theta^j_B(t)^2 + 2\rho^j\,\theta^j_A(t)\theta^j_B(t)\right] \\ & + \sum_{k\neq j} \left[\left(\frac{f^k}{f^j}\right)^{\gamma}(1+\theta^{j,k}(t))^{1-\gamma} - 1\right]\lambda^{j,k} \Bigg\}. \end{aligned}$$

Solving the first order conditions for the optimal exposures gives

$$\theta_A^j = \frac{\eta_A^j - \rho^j \eta_B^j}{\gamma(1-(\rho^j)^2)} \qquad \theta^{j,k} = (1+\eta^{j,k})^{-\frac{1}{\gamma}} \frac{f^k}{f^j} - 1$$
$$\theta_B^j = \frac{\eta_B^j - \rho^j \eta_A^j}{\gamma(1-(\rho^j)^2)}.$$

We then plug the optimal exposures back into the HJB-equations to get

$$\begin{aligned} 0 &= \gamma \frac{f_t^j}{f^j} + (1-\gamma)\left[r + \frac{(\eta_A^j)^2 + (\eta_B^j)^2 - 2\rho^j \eta_A^j \eta_B^j}{\gamma(1-(\rho^j)^2)}\right] \\ &\quad - (1-\gamma)\sum_{k\neq j}\left[\left(1+\eta^{j,k}\right)^{1-\frac{1}{\gamma}} \lambda^{j,k} \frac{f^k}{f^j} - \lambda^{j,k}\left(1+\eta^{j,k}\right)\right] \\ &\quad - 0.5(1-\gamma)\frac{(\eta_A^j)^2 + (\eta_B^j)^2 - 2\rho^j \eta_A^j \eta_B^j}{\gamma(1-(\rho^j)^2)} \\ &\quad + \sum_{k\neq j}\left[\frac{f^k}{f^j}\left(1+\eta_{j,k}\right)^{1-\frac{1}{\gamma}} - 1\right]\lambda^{j,k}. \end{aligned}$$

The resulting linear system of homogeneous ordinary differential equations for $f^j(t)$ ($j = 0, 1, 2$) with boundary condition $f^j(T) = 1$ is

$$\begin{aligned} 0 &= f_t^j + \frac{1-\gamma}{\gamma}\left[r + \frac{(\eta_A^j)^2 + (\eta_B^j)^2 - 2\rho^j \eta_A^j \eta_B^j}{2\gamma(1-(\rho^j)^2)}\right] f^j \\ &\quad + \frac{1-\gamma}{\gamma}\sum_{k\neq j}\left[\left(1+\eta^{j,k}\right) - \frac{1}{1-\gamma}\right]\lambda^{j,k} f^j + \sum_{k\neq j}\left(1+\eta^{j,k}\right)^{1-\frac{1}{\gamma}} \lambda^{j,k} f^k. \end{aligned}$$

This is equivalent to

$$0 = f_t^j + C^{(j,j)} f^j + \sum_{k\neq j} C^{(j,k)} f^k$$

where the coefficients C depend on the parameters only

$$\begin{aligned} C^{(j,j)} &= \frac{1-\gamma}{\gamma}\left[r + \frac{(\eta_A^j)^2 + (\eta_B^j)^2 - 2\rho^j \eta_A^j \eta_B^j}{2\gamma(1-(\rho^j)^2)}\right] + \frac{1-\gamma}{\gamma}\sum_{k\neq j}\left[\left(1+\eta^{j,k}\right) - \frac{1}{1-\gamma}\right]\lambda^{j,k} \\ C^{(j,k)} &= \left(1+\eta^{j,k}\right)^{1-\frac{1}{\gamma}} \lambda^{j,k}. \end{aligned}$$

The system of ordinary differential equations can thus be written as

$$\begin{pmatrix} f^1 \\ \vdots \\ f^K \end{pmatrix}_t = - \begin{pmatrix} C^{1,1} & C^{1,2} & \cdots & C^{1,K} \\ C^{2,1} & C^{2,2} & \cdots & C^{2,K} \\ \vdots & \vdots & \ddots & \vdots \\ C^{K,1} & C^{K,2} & \cdots & C^{K,K} \end{pmatrix} \begin{pmatrix} f^1 \\ \vdots \\ f^K \end{pmatrix},$$

and its solution is

$$\begin{pmatrix} f^1 \\ \vdots \\ f^K \end{pmatrix} = e^{C\cdot(T-t)} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Proposition 1 then follows by applying this result to our Markov chain.

A.2 Incomplete Market - Proof

In the incomplete market, the investor decides on the portfolio weights π_A^{calm} and π_B^{calm} of the two stocks. The HJB-equation in the calm state is

$$\begin{aligned} 0 \;=\; & \max_{\pi_A^{calm},\pi_B^{calm}} \{G_t^{calm} + x(r + \pi_A^{calm}(\mu_A^{calm} - r) + \pi_B^{calm}(\mu_B^{calm} - r))G_x^{calm} \\ & + 0.5x^2 \left[(\pi_A^{calm}\sigma_A^{calm})^2 + (\pi_B^{calm}\sigma_B^{calm})^2 + 2\pi_A^{calm}\pi_B^{calm}\sigma_A^{calm}\sigma_B^{calm}\rho^{calm}\right] G_{xx}^{calm} \\ & + \lambda_A^{calm,cont}[G^{cont}(t, x(1 - \pi_A^{calm}L_A)) - G^{calm}(t, x)] \\ & + \lambda_B^{calm,cont}[G^{cont}(t, x(1 - \pi_B^{calm}L_B)) - G^{calm}(t, x)] \\ & + \lambda_A^{calm,calm}[G^{calm}(t, x(1 - \pi_A^{calm}L_A)) - G^{calm}(t, x)] \\ & + \lambda_B^{calm,calm}[G^{calm}(t, x(1 - \pi_B^{calm}L_B)) - G^{calm}(t, x)]\} \end{aligned}$$

and the HJB-equation in the contagion state is

$$\begin{aligned} 0 \;=\; & \max_{\pi_A^{cont},\pi_B^{cont}} \{G_t^{cont} + x(r + \pi_A^{cont}(\mu_A^{cont} - r) + \pi_B^{cont}(\mu_B^{cont} - r))G_x^{cont} \\ & + 0.5x^2 \left[(\pi_A^{cont}\sigma_A^{cont})^2 + (\pi_B^{cont}\sigma_B^{cont})^2 + 2\pi_A^{cont}\pi_B^{cont}\sigma_A^{cont}\sigma_B^{cont}\rho^{cont}\right] G_{xx}^{cont} \\ & + \lambda_A^{cont,cont}[G^{cont}(t, x(1 - \pi_A^{cont}L_A)) - G^{cont}(t, x)] \\ & + \lambda_B^{cont,cont}[G^{cont}(t, x(1 - \pi_B^{cont}L_B)) - G^{cont}(t, x)] \\ & + \lambda^{cont,calm}[G^{calm}(t, x) - G^{cont}(t, x)]\}. \end{aligned}$$

With the guess $G^j(t, x) = \frac{x^{1-\gamma}}{1-\gamma} f^j(t)$, the HJB-equation in the calm state becomes

$$\begin{aligned} 0 \;=\; & \max_{\pi_A^{calm},\pi_B^{calm}} \left\{f_t^{calm} + (1-\gamma)\left(r + \pi_A^{calm}(\mu_A^{calm} - r) + \pi_B^{calm}(\mu_B^{calm} - r)\right) f^{calm} \right. \\ & - 0.5\gamma(1-\gamma)\left((\pi_A^{calm}\sigma_A^{calm})^2 + (\pi_B^{calm}\sigma_B^{calm})^2 + 2\pi_A^{calm}\pi_B^{calm}\sigma_A^{calm}\sigma_B^{calm}\rho^{calm}\right) f^{calm} \\ & + \lambda_A^{calm,cont}\left((1 - \pi_A^{calm}L_A)^{1-\gamma} f^{cont} - f^{calm}\right) \\ & + \lambda_B^{calm,cont}\left((1 - \pi_B^{calm}L_B)^{1-\gamma} f^{cont} - f^{calm}\right) \\ & + \lambda_A^{calm,calm}\left((1 - \pi_A^{calm}L_A)^{1-\gamma} f^{calm} - f^{calm}\right) \\ & \left. + \lambda_B^{calm,calm}\left((1 - \pi_B^{calm}L_B)^{1-\gamma} f^{calm} - f^{calm}\right)\right\} \end{aligned}$$

and the HJB-equation in the contagion state becomes

$$\begin{aligned}
0 \;=\; & \max_{\pi_A^{calm},\pi_B^{calm}} \{ f_t^{cont} + (1-\gamma)\left(r + \pi_A^{cont}(\mu_A^{cont} - r) + \pi_B^{cont}(\mu_B^{cont} - r)\right) f^{cont} \\
& -0.5\gamma(1-\gamma)\left((\pi_A^{cont}\sigma_A^{cont})^2 + (\pi_B^{cont}\sigma_B^{cont})^2 + 2\pi_A^{cont}\pi_B^{cont}\sigma_A^{cont}\sigma_B^{cont}\rho^{cont}\right) f^{cont} \\
& + \lambda_A^{cont,cont}\left((1-\pi_A^{cont}L_A)^{1-\gamma} - 1\right) f^{cont} \\
& + \lambda_B^{cont,cont}\left((1-\pi_B^{cont}L_B)^{1-\gamma} - 1\right) f^{cont} \\
& + \lambda^{cont,calm}(f^{calm} - f^{cont})\}.
\end{aligned}$$

The first order conditions for the portfolio weights are

$$\begin{aligned}
\mu_A^{calm} - r - \gamma(\sigma_A^{calm})^2\pi_A^{calm} - \gamma\pi_B^{calm}\sigma_A^{calm}\sigma_B^{calm}\rho^{calm} & \\
-L_A\lambda_A^{calm,cont}(1-\pi_A^{calm}L_A)^{-\gamma}\frac{f^{cont}}{f^{calm}} - L_A\lambda_A^{calm,calm}(1-\pi_A^{calm}L_A)^{-\gamma} &= 0 \quad (9) \\
\mu_B^{calm} - r - \gamma(\sigma_B^{calm})^2\pi_B^{calm} - \gamma\pi_A^{calm}\sigma_A^{calm}\sigma_B^{calm}\rho^{calm} & \\
-L_B\lambda_B^{calm,cont}(1-\pi_B^{calm}L_B)^{-\gamma}\frac{f^{cont}}{f^{calm}} - L_B\lambda_B^{calm,calm}(1-\pi_B^{calm}L_B)^{-\gamma} &= 0 \quad (10) \\
\mu_A^{cont} - r - \gamma(\sigma_A^{cont})^2\pi_A^{cont} - \gamma\pi_B^{cont}\sigma_A^{cont}\sigma_B^{cont}\rho^{cont} - L_A\lambda_A^{cont,cont}(1-\pi_A^{cont}L_A)^{-\gamma} &= 0 \quad (11) \\
\mu_B^{cont} - r - \gamma(\sigma_B^{cont})^2\pi_B^{cont} - \gamma\pi_A^{cont}\sigma_A^{cont}\sigma_B^{cont}\rho^{cont} - L_B\lambda_B^{cont,cont}(1-\pi_B^{cont}L_B)^{-\gamma} &= 0. \quad (12)
\end{aligned}$$

With the optimal portfolio weights, the differential equations become

$$\begin{aligned}
0 \;=\; & f_t^{calm} + (1-\gamma)\left(r + \pi_A^{calm}(\mu_A^{calm} - r) + \pi_B^{calm}(\mu_B^{calm} - r)\right) f^{calm} \qquad (13) \\
& - 0.5\gamma(1-\gamma)\left((\pi_A^{calm}\sigma_A^{calm})^2 + (\pi_B^{calm}\sigma_B^{calm})^2 + 2\pi_A^{calm}\pi_B^{calm}\sigma_A^{calm}\sigma_B^{calm}\rho^{calm}\right) f^{calm} \\
& + \lambda_A^{calm,cont}\left((1-\pi_A^{calm}L_A)^{1-\gamma} f^{cont} - f^{calm}\right) \\
& + \lambda_B^{calm,cont}\left((1-\pi_B^{calm}L_B)^{1-\gamma} f^{cont} - f^{calm}\right) \\
& + \lambda_A^{calm,calm}\left((1-\pi_A^{calm}L_A)^{1-\gamma} - 1\right) f^{calm} \\
& + \lambda_B^{calm,calm}\left((1-\pi_B^{calm}L_B)^{1-\gamma} - 1\right) f^{calm} \\
0 \;=\; & f_t^{cont} + (1-\gamma)\left(r + \pi_A^{cont}(\mu_A^{cont} - r) + \pi_B^{cont}(\mu_B^{cont} - r)\right) f^{cont} \qquad (14) \\
& - 0.5\gamma(1-\gamma)\left((\pi_A^{cont}\sigma_A^{cont})^2 + (\pi_B^{cont}\sigma_B^{cont})^2 + 2\pi_A^{cont}\pi_B^{cont}\sigma_A^{cont}\sigma_B^{cont}\rho^{cont}\right) f^{cont} \\
& + \lambda_A^{cont,cont}\left((1-\pi_A^{cont}L_A)^{1-\gamma} - 1\right) f^{cont} + \lambda_B^{cont,cont}\left((1-\pi_B^{cont}L_B)^{1-\gamma} - 1\right) f^{cont} \\
& + \lambda^{cont,calm}(f^{calm} - f^{cont}).
\end{aligned}$$

Conditions (11) and (12) can be solved numerically for the optimal portfolio weights in the contagion state. Conditions (13),(14),(9) and (10) form a so-called differential-algebraic system for the functions f^{calm}, f^{cont}, π_A^{calm} and π_B^{calm}. This system can be solved numerically using a Runge-Kutta method of order 3, namely the implicit Radau form of order 3, which is for example studied in Hairer, Lubich, and Roche (1989).

B Benchmark Models: Independent Jumps

B.1 Complete Market - Proof

The model with independent jumps can be interpreted as a special case of the model with contagion where the parameters are identical in all states. The indirect utility function is then no longer state dependent. The optimal exposures are

$$\theta_A^{diff} = \frac{\eta_A^{diff} - \rho\eta_B^{diff}}{\gamma(1-\rho^2)} \qquad \theta_A^{jump} = (1+\eta_A^{jump})^{-\frac{1}{\gamma}} - 1$$
$$\theta_B^{diff} = \frac{\eta_B^{diff} - \rho\eta_A^{diff}}{\gamma(1-\rho^2)} \qquad \theta_B^{jump} = (1+\eta_B^{jump})^{-\frac{1}{\gamma}} - 1.$$

The ordinary differential equation for f becomes

$$0 = f_t + C^{nc,c} f$$

where

$$\begin{aligned} C^{nc,c} = \frac{1-\gamma}{\gamma}\Bigg[r + \frac{(\eta_A^{diff})^2 + (\eta_B^{diff})^2 - 2\rho\eta_A^{diff}\eta_B^{diff}}{2\gamma(1-\rho^2)} \\ + \left(1+\eta_A^{jump}\right)\lambda^A + \left(1+\eta_B^{jump}\right)\lambda^B - \frac{1}{1-\gamma}(\lambda^A+\lambda^B)\Bigg] \\ + \left(1+\eta_A^{jump}\right)^{1-\frac{1}{\gamma}}\lambda^A + \left(1+\eta_B^{jump}\right)^{1-\frac{1}{\gamma}}\lambda^B. \end{aligned}$$

The function f can be solved for in closed form:

$$f(t) = \exp\left\{C^{nc,c}\cdot(T-t)\right\}.$$

The indirect utility is

$$G(t,x) = \frac{x^{1-\gamma}}{1-\gamma}\exp\left\{\gamma\, C^{nc,c}\cdot(T-t)\right\}.$$

B.2 Complete Market: Impact of Jump Intensity

Lemma 1 (Independent Jumps, Complete Market: Impact of Jump Intensity) *If there are no contagion effects and if the market is complete, the indirect utility is increasing in λ_A and λ_B.*

Proof: The partial derivative of G w.r.t. λ_i is

$$\frac{\partial G}{\partial \lambda_i} = \frac{w^{1-\gamma}}{1-\gamma}\gamma e^{\gamma C^{nc,c}\cdot(T-t)}\left[(1+\eta_i^{jump})^{1-\frac{1}{\gamma}} - 1 - \left(1-\frac{1}{\gamma}\right)\eta_i^{jump}\right](T-t).$$

The term in square brackets is positive (negative) if $(1+\eta_i^{jump})^{1-\frac{1}{\gamma}}$ is a convex (concave) function of η_i^{jump}, i.e. if $\gamma < 1$ ($\gamma > 1$), since $1+\left(1-\frac{1}{\gamma}\right)\eta_i^{jump}$ is just the first-order Taylor expansion of $(1+\eta_i^{jump})^{1-\frac{1}{\gamma}}$ around 0. The other terms are positive (negative) if $\gamma < 1$ ($\gamma > 1$). Put together, the partial derivative of the indirect utility function with respect to λ_i is positive, and the indirect utility is increasing in the jump intensity λ_i.

B.3 Incomplete Market - Proof

Again, the model can be interpreted as a special case of the model with contagion. The guess for the indirect utility function is

$$G(t,x) = \frac{x^{1-\gamma}}{1-\gamma} f(t)$$

where G does not depend on the state any more. The optimal portfolio weights π_A and π_B satisfy

$$\begin{aligned} \mu_A - r - \gamma\sigma_A^2\pi_A - \gamma\pi_B\sigma_A\sigma_B\rho - L_A\lambda_A(1-\pi_A L_A)^{-\gamma} &= 0 \\ \mu_B - r - \gamma\sigma_B^2\pi_B - \gamma\pi_A\sigma_B\sigma_A\rho - L_B\lambda_B(1-\pi_B L_B)^{-\gamma} &= 0 \end{aligned}$$

which can be solved numerically. The HJB-equation simplifies dramatically, and with the optimal portfolio weights, the differential equation for f is

$$f_t = -C^{nc,ic} f$$

with boundary condition $f(T) = 1$ and

$$\begin{aligned} C^{nc,ic} = & (1-\gamma)\left[r + \pi_A(\mu_A - r) + \pi_B(\mu_B - r) - 0.5\gamma(\pi_A^2\sigma_A^2 + \pi_B^2\sigma_B^2 + 2\pi_A\pi_B\sigma_A\sigma_B\rho)\right] \\ & +\lambda_A\left[(1-\pi_A L_A)^{1-\gamma} - 1\right] + \lambda_B\left[(1-\pi_B L_B)^{1-\gamma} - 1\right]. \end{aligned}$$

The solution is given by $f(t) = \exp\{C^{nc,ic}\cdot(T-t)\}$.

C Model Mis-Specification

C.1 Incomplete Market - Model Mis-Specification

In case of model mis-specification, the optimal portfolios are determined in the benchmark model. With independent jumps, the weights of the stocks are constant over time. The indirect utility functions in the two states are then given by

$$\widehat{G}^j(t,x) = E_t\left[\frac{X_T^{1-\gamma}}{1-\gamma} \mid X_t = x\right]$$

subject to the budget restriction

$$\frac{dX(t)}{X(t)} = \widehat{\pi}_A(t)\frac{dS_A(t)}{S_A(t)} + \widehat{\pi}_B(t)\frac{dS_B(t)}{S_B(t)} + (1-\widehat{\pi}_A(t)-\widehat{\pi}_B(t))\, rdt$$

where $\widehat{\pi}_A$ and $\widehat{\pi}_B$ denote the seemingly optimal portfolio weights. Since the indirect utility $\widehat{G}$ is a martingale, it holds that

$$\begin{aligned}
0 &= \widehat{G}_t^{calm} + x(r+\widehat{\pi}_A(\mu_A^{calm}-r)+\widehat{\pi}_B(\mu_B^{calm}-r))\widehat{G}_x^{calm} \\
&\quad + 0.5x^2\left[(\widehat{\pi}_A\sigma_A^{calm})^2 + (\widehat{\pi}_B\sigma_B^{calm})^2 + 2\widehat{\pi}_A\widehat{\pi}_B\sigma_A^{calm}\sigma_B^{calm}\rho^{calm}\right]\widehat{G}_{xx}^{calm} \\
&\quad + \lambda_A^{calm,cont}[\widehat{G}^{cont}(t,x(1-\widehat{\pi}_A L_A)) - \widehat{G}^{calm}(t,x)] \\
&\quad + \lambda_B^{calm,cont}[\widehat{G}^{cont}(t,x(1-\widehat{\pi}_B L_B)) - \widehat{G}^{calm}(t,x)] \\
&\quad + \lambda_A^{calm,calm}[\widehat{G}^{calm}(t,x(1-\widehat{\pi}_A L_A)) - \widehat{G}^{calm}(t,x)] \\
&\quad + \lambda_B^{calm,calm}[\widehat{G}^{calm}(t,x(1-\widehat{\pi}_B L_B)) - \widehat{G}^{calm}(t,x)]
\end{aligned}$$

and

$$\begin{aligned}
0 &= \widehat{G}_t^{cont} + x(r+\widehat{\pi}_A(\mu_A^{cont}-r)+\widehat{\pi}_B(\mu_B^{cont}-r))\widehat{G}_x^{cont} \\
&\quad + 0.5x^2\left[(\widehat{\pi}_A\sigma_A^{cont})^2 + (\widehat{\pi}_B\sigma_B^{cont})^2 + 2\widehat{\pi}_A\widehat{\pi}_B\sigma_A^{cont}\sigma_B^{cont}\rho^{cont}\right]\widehat{G}_{xx}^{cont} \\
&\quad + \lambda^{cont,calm}[\widehat{G}^{calm}(t,x) - \widehat{G}^{cont}(t,x)] \\
&\quad + \lambda_A^{cont,cont}[\widehat{G}^{cont}(t,x(1-\widehat{\pi}_A L_A)) - \widehat{G}^{cont}(t,x)] \\
&\quad + \lambda_B^{cont,cont}[\widehat{G}^{cont}(t,x(1-\widehat{\pi}_B L_B)) - \widehat{G}^{cont}(t,x)].
\end{aligned}$$

Since the investor has constant relative risk aversion, we can use a separation approach and set

$$\widehat{G}^j(t,x) = \frac{x^{1-\gamma}}{1-\gamma}\,\widehat{f}^j(t).$$

Plugging in and simplifying gives

$$\begin{aligned}
0 &= \widehat{f}_t^{calm} + (1-\gamma)\left(r+\widehat{\pi}_A(\mu_A^{calm}-r)+\widehat{\pi}_B(\mu_B^{calm}-r)\right)\widehat{f}^{calm} \\
&\quad - 0.5\gamma(1-\gamma)\left((\widehat{\pi}_A\sigma_A^{calm})^2 + (\widehat{\pi}_B\sigma_B^{calm})^2 + 2\widehat{\pi}_A\widehat{\pi}_B\sigma_A^{calm}\sigma_B^{calm}\rho^{calm}\right)\widehat{f}^{calm} \\
&\quad + \lambda_A^{calm,cont}\left((1-\widehat{\pi}_A L_A)^{1-\gamma}\widehat{f}^{cont} - \widehat{f}^{calm}\right) \\
&\quad + \lambda_B^{calm,cont}\left((1-\widehat{\pi}_B L_B)^{1-\gamma}\widehat{f}^{cont} - \widehat{f}^{calm}\right) \\
&\quad + \lambda_A^{calm,calm}\left((1-\widehat{\pi}_A L_A)^{1-\gamma}\widehat{f}^{calm} - \widehat{f}^{calm}\right) \\
&\quad + \lambda_B^{calm,calm}\left((1-\widehat{\pi}_B L_B)^{1-\gamma}\widehat{f}^{calm} - \widehat{f}^{calm}\right)
\end{aligned}$$

and

$$
\begin{aligned}
0 \;=\;& \widehat{f}^{cont}_t + (1-\gamma)\left(r + \widehat{\pi}_A(\mu_A^{cont} - r) + \widehat{\pi}_B(\mu_B^{cont} - r)\right)\widehat{f}^{cont} \\
& - 0.5\gamma(1-\gamma)\left((\widehat{\pi}_A\sigma_A^{cont})^2 + (\widehat{\pi}_B\sigma_B^{cont})^2 + 2\widehat{\pi}_A\widehat{\pi}_B\sigma_A^{cont}\sigma_B^{calm}\rho^{cont}\right)\widehat{f}^{cont} \\
& + \lambda^{cont,calm}\left(\widehat{f}^{calm} - \widehat{f}^{cont}\right) \\
& + \lambda_A^{cont,cont}\left((1-\widehat{\pi}_A L_A)^{1-\gamma}\widehat{f}^{cont} - \widehat{f}^{cont}\right) \\
& + \lambda_B^{cont,cont}\left((1-\widehat{\pi}_B L_B)^{1-\gamma}\widehat{f}^{cont} - \widehat{f}^{cont}\right).
\end{aligned}
$$

This results in a system of two linear ordinary differential equations

$$
\begin{pmatrix} \widehat{f}^{calm}_t \\ \widehat{f}^{cont}_t \end{pmatrix} = -\begin{pmatrix} \widehat{C}^{1,1} & \widehat{C}^{1,2} \\ \widehat{C}^{2,1} & \widehat{C}^{2,2} \end{pmatrix}\begin{pmatrix} \widehat{f}^{calm} \\ \widehat{f}^{cont} \end{pmatrix}
$$

where

$$
\begin{aligned}
\widehat{C}^{1,1} \;=\;& (1-\gamma)\left(r + \widehat{\pi}_A(\mu_A^{calm} - r) + \widehat{\pi}_B(\mu_B^{calm} - r)\right) \\
& - 0.5\gamma(1-\gamma)\left((\widehat{\pi}_A\sigma_A^{calm})^2 + (\widehat{\pi}_B\sigma_B^{calm})^2 + 2\widehat{\pi}_A\widehat{\pi}_B\sigma_A^{calm}\sigma_B^{calm}\rho^{calm}\right) \\
& - \lambda_A^{calm,cont} - \lambda_B^{calm,cont} \\
& + \lambda_A^{calm,calm}\left((1-\widehat{\pi}_A L_A)^{1-\gamma} - 1\right) + \lambda_B^{calm,calm}\left((1-\widehat{\pi}_B L_B)^{1-\gamma} - 1\right) \\
\widehat{C}^{1,2} \quad\;& \lambda_A^{calm,cont}(1-\widehat{\pi}_A L_A)^{1-\gamma} + \lambda_B^{calm,cont}(1-\widehat{\pi}_B L_B)^{1-\gamma} \\
\widehat{C}^{2,1} \;=\;& \lambda^{cont,calm} \\
\widehat{C}^{2,2} \;=\;& (1-\gamma)\left(r + \widehat{\pi}_A(\mu_A^{cont} - r) + \widehat{\pi}_B(\mu_B^{cont} - r)\right) \\
& - 0.5\gamma(1-\gamma)\left((\widehat{\pi}_A\sigma_A^{cont})^2 + (\widehat{\pi}_B\sigma_B^{cont})^2 + 2\widehat{\pi}_A\widehat{\pi}_B\sigma_A^{cont}\sigma_B^{calm}\rho^{cont}\right) \\
& - \lambda^{cont,calm} \\
& + \lambda_A^{cont,cont}\left((1-\widehat{\pi}_A L_A)^{1-\gamma} - 1\right) + \lambda_B^{cont,cont}\left((1-\widehat{\pi}_B L_B)^{1-\gamma} - 1\right).
\end{aligned}
$$

The solution for $\widehat{f}$ is

$$
\begin{pmatrix} \widehat{f}^{calm}(t) \\ \widehat{f}^{cont}(t) \end{pmatrix} = \exp\left\{\begin{pmatrix} \widehat{C}^{1,1} & \widehat{C}^{1,2} \\ \widehat{C}^{2,1} & \widehat{C}^{2,2} \end{pmatrix}(T-t)\right\}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.
$$

C.2 Complete Market - Model Mis-Specification

In case of model mis-specification in a complete market setup, the investor does not implement his optimal risk factor exposures $\theta(t)$, but sub-optimal exposures $\widehat{\theta}$ which are constant over time. As in A.1, we solve for the indirect utility function for a general

Markov chain with states $k \in \{1, \ldots, K\}$. The indirect utility functions in the K states are then given by

$$\widehat{G}^j(t,x) = E_t\left[\frac{X_T^{1-\gamma}}{1-\gamma} \mid X_t = x\right]$$

subject to the budget restriction

$$\begin{aligned}\frac{dX(t)}{X(t)} &= rdt + \widehat{\theta}_A^{Z(t)}\left[dW_A(t) + \eta_A^{Z(t)}dt\right] + \widehat{\theta}_B^{Z(t)}\left[dW_B(t) + \eta_B^{Z(t)}dt\right] \\ &\quad + \sum_{k \neq Z(t), \lambda^{Z(t),k} \neq 0} \widehat{\theta}^{Z(t),k}\left[dN^k(t) - \lambda^{Z(t),k}dt - \eta^{Z(t),k}\lambda^{Z(t),k}dt\right].\end{aligned}$$

Since the indirect utility $\widehat{G}$ is a martingale, it holds that

$$\begin{aligned}0 &= \widehat{G}_t^j + \widehat{G}_x^j x\left[r + \widehat{\theta}_A^j\eta_A^j + \widehat{\theta}_B^j\eta_B^j - \sum_{k\neq j}\widehat{\theta}^{j,k}\lambda^{j,k}\left(1+\eta^{j,k}\right)\right] \\ &\quad + 0.5G_{xx}^j x^2\left[(\widehat{\theta}_A^j)^2 + (\widehat{\theta}_B^j)^2 + 2\rho^j\widehat{\theta}_A^j\widehat{\theta}_B^j\right] \\ &\quad + \sum_{k\neq j}\left[G^k(t, x(1+\widehat{\theta}^{j,k})) - G^j(t,x)\right]\lambda^{j,k}.\end{aligned}$$

Since the investor has constant relative risk aversion, we can use a separation approach and set

$$\widehat{G}^j(t,x) = \frac{x^{1-\gamma}}{1-\gamma}\,\widehat{f}^j(t).$$

Plugging in and simplifying gives a system of linear ordinary differential equations

$$\begin{aligned}0 &= \widehat{f}_t^j + (1-\gamma)\left[r + \widehat{\theta}_A^j\eta_A^j + \widehat{\theta_B}^j\eta_B^j - \sum_{k\neq j}\widehat{\theta}^{j,k}\lambda^{j,k}\left(1+\eta^{j,k}\right)\right]\widehat{f}^j \\ &\quad - 0.5\gamma(1-\gamma)\left[(\widehat{\theta}_A^j)^2 + (\widehat{\theta}_B^j)^2 + 2\rho^j\,\widehat{\theta}_A^j\widehat{\theta}_B^j\right]\widehat{f}^j \\ &\quad + \left.\sum_{k\neq j}\left[\widehat{f}^k(1+\widehat{\theta}^{j,k})^{1-\gamma} - \widehat{f}^j\right]\lambda^{j,k}\right\}\end{aligned}$$

whose solution with respect to the boundary conditions $f^j(T) = 1$ becomes in our case

$$\begin{pmatrix}\widehat{f}^{calm}(t) \\ \widehat{f}^{cont}(t)\end{pmatrix} = \exp\left\{\begin{pmatrix}\widehat{C}^{calm,calm} & \widehat{C}^{calm,cont} \\ \widehat{C}^{cont,calm} & \widehat{C}^{cont,cont}\end{pmatrix}(T-t)\right\}\begin{pmatrix}1 \\ 1\end{pmatrix}$$

with

$$
\begin{aligned}
\widehat{C}^{calm,calm} \quad = \quad & (1-\gamma)\Big[r + \widehat{\theta}_A^{calm}\eta_A^{calm} + \widehat{\theta}_B^{calm}\eta_B^{calm} \\
& \qquad - \widehat{\theta}_A^{calm,calm}\lambda_A^{calm,calm}(1+\eta_A^{calm,calm}) - \widehat{\theta}_A^{calm,cont}\lambda_A^{calm,cont}(1+\eta_A^{calm,cont}) \\
& \qquad -\widehat{\theta}_B^{calm,calm}\lambda_B^{calm,calm}(1+\eta_B^{calm,calm}) - \widehat{\theta}_B^{calm,cont}\lambda_B^{calm,cont}(1+\eta_B^{calm,cont})\Big] \\
& - 0.5\gamma(1-\gamma)\Big[(\widehat{\theta}_A^{calm})^2 + (\widehat{\theta}_B^{calm})^2 + 2\rho^{calm}\widehat{\theta}_A^{calm}\widehat{\theta}_B^{calm}\Big] \\
& + \lambda_A^{calm,calm}\Big[(1+\widehat{\theta}_A^{calm,calm})^{1-\gamma} - 1\Big] + \lambda_B^{calm,calm}\Big[(1+\widehat{\theta}_B^{calm,calm})^{1-\gamma} - 1\Big] \\
& - \lambda_A^{calm,cont} - \lambda_B^{calm,cont} \\
\widehat{C}^{calm,cont} \quad = \quad & \lambda_A^{calm,cont}(1+\widehat{\theta}_A^{calm,cont})^{1-\gamma} + \lambda_B^{calm,cont}(1+\widehat{\theta}_B^{calm,cont})^{1-\gamma} \\
\widehat{C}^{cont,calm} \quad = \quad & \lambda^{cont,calm}(1+\widehat{\theta}^{cont,calm})^{1-\gamma} \\
\widehat{C}^{cont,cont} \quad = \quad & (1-\gamma)\Big[r + \widehat{\theta}_A^{cont}\eta_A^{cont} + \widehat{\theta}_B^{cont}\eta_B^{cont} \\
& \qquad - \widehat{\theta}_A^{cont,cont}\lambda_A^{cont,cont}(1+\eta_A^{cont,cont}) - \widehat{\theta}_B^{cont,cont}\lambda_B^{cont,cont}(1+\eta_B^{cont,cont}) \\
& \qquad -\widehat{\theta}^{cont,calm}\lambda^{cont,calm}(1+\eta^{cont,calm})\Big] \\
& - 0.5\gamma(1-\gamma)\Big[(\widehat{\theta}_A^{cont})^2 + (\widehat{\theta}_B^{cont})^2 + 2\rho^{cont}\widehat{\theta}_A^{cont}\widehat{\theta}_B^{cont}\Big] \\
& + \lambda_A^{cont,cont}\Big[(1+\widehat{\theta}_A^{cont,cont})^{1-\gamma} - 1\Big] + \lambda_B^{cont,cont}\Big[(1+\widehat{\theta}_B^{cont,cont})^{1-\gamma} - 1\Big] \\
& - \lambda^{cont,calm}.
\end{aligned}
$$

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		Our model		Benchmark model		
		Parametrization 1	Parametrization 2		Parametrization 1	Parametrization 2
Data-generating process	σ_A^{calm}	**0.15**		σ_A	0.1515	0.15
	σ_A^{cont}	**0.15**				
	ρ^{calm}	**0.50**		ρ	0.50	0.50
	ρ^{cont}	**0.50**				
	$\lambda_A^{calm,calm}$	*0.375*		$\lambda_A/\lambda^{joint}$	**1.50**	**1.50**
	$\lambda_A^{calm,cont}$	*0.375*				
	$\lambda_A^{cont,cont}$	*3.000*				
	$\lambda^{cont,calm}$	*1.500*				
	$L_A^{calm,calm}$	**0.05**		L_A	0.05	0.05
	$L_A^{calm,cont}$	**0.05**				
	$L_A^{cont,cont}$	**0.05**				
	$L_A^{cont,calm}$	**0.00**				
	ξ_A	**4.00**				
	α_A	**0.50**				
	ψ	**0.25**				
Market prices of risk	η_A^{calm}	**0.35**	**0.35**	η_A^{diff}	0.3466	0.35
	η_A^{cont}	**0.35**	**0.35**			
	$\eta_A^{calm,calm}$	**0.40**	0.80	η_A^{jump}	0.40	0.40
	$\eta_A^{calm,cont}$	**0.40**	0.80			
	$\eta_A^{cont,cont}$	**0.40**	0.20			
	$\eta^{cont,calm}$	**0.00**	0.00			

Table 1: Parameters

The table gives the parameters for the stocks under the physical measure (upper part) and the market prices of risk (lower part) for our base case as explained in Section 5.1. The two stocks are assumed to follow identical processes, so that we only give the parameters for stock A. The market prices of risk in our model are chosen such that either the market prices of jump risk are identical in the calm and the contagion state (parametrization 1) or such that the expected excess return on the stock is identical in both states (parametrization 2). The parameters written in bold numbers have been set in line with recent empirical studies. The jump intensities in our model (written in italic numbers) have been set in the second step. All other numbers have been calibrated in a third step such that the average equity risk premium is identical for both parametrizations (market prices of risk) or such that the benchmark models are as close as possible to our model.

		our model		benchmark
		calm	contagion	
Parametrization 1 (identical market prices of risk)	Excess Return	0.0675	0.1125	0.0825
	... from diffusion	0.0525	0.0525	0.0525
	... from jumps	0.0150	0.0600	0.0300
	Variance	0.0244	0.0300	0.0267
	... from diffusion	0.0225	0.0225	0.0230
	... from jumps	0.0019	0.0075	0.0038
Parametrization 2 (identical equity risk premium)	Excess Return	0.0825	0.0825	0.0825
	... from diffusion	0.0525	0.0525	0.0525
	... from jumps	0.0300	0.0300	0.0300
	Variance	0.0244	0.0300	0.0263
	... from diffusion	0.0225	0.0225	0.0225
	... from jumps	0.0019	0.0075	0.0038

Table 2: Conditional Moments

The table gives the conditional expected excess returns and the conditional variances of stock returns in the calm and in the contagion state as well as in the benchmark models for the parameter set from Table 1. Furthermore, we show the contribution of diffusion risk and jump risk to the local moments. For parametrization 1, the market prices of risk are assumed to be equal in the calm and in the contagion state, while for parametrization 2, the expected excess returns are equal across states.

	$\lambda_A/\lambda^{joint}$	contagion parameters ξ_A	α_A	ψ	calibrated jump parameters $\lambda_A^{calm,calm}$	$\lambda_A^{calm,cont}$	$\lambda_A^{cont,cont}$	$\lambda^{cont,calm}$
no contagion	1.50	1	0.50	0.25	0.75000	0.75000	1.50	0.75
	1.50	2	0.50	0.25	0.50000	0.50000	2.00	1.00
base case	1.50	4	0.50	0.25	0.37500	0.37500	3.00	1.50
	1.50	10	0.50	0.25	0.30000	0.30000	6.00	3.00
	1.50	4	1/3	1/3	0.62500	0.31250	3.75	2.50
	1.50	10	1/3	1/3	0.55000	0.27500	8.25	5.50
	1.50	4	0.20	0.20	0.75000	0.18750	3.75	1.50
	1.50	10	0.20	0.20	0.66000	0.16500	8.25	3.30
	1.50	4	0.50	0.50	0.46875	0.46875	3.75	3.75
	1.50	10	0.50	0.50	0.41250	0.41250	8.25	8.25
	1.50	4	1/3	2/3	0.75000	0.37500	4.50	6.00
	1.50	10	1/3	2/3	0.70000	0.35000	10.50	14.00

Table 3: Selection of calibrated jump parameters

Each line of the table shows one possible combination of contagion and jump parameters leading to an 'average' (i.e. benchmark) jump intensity of 1.5. The line marked 'no contagion' describes a situation where the calm and the contagion state equal (since ξ equals 1). The line marked 'base case' shows the parameters for our base case parameter set also described in Table 1.

		our model		benchmark models	
		calm	cont	no cont	joint
Parametrization 1: identical market prices of risk					
complete	Diff-Exposure	0.0778	0.0778	0.0770	0.0770
	Jump-Exposure				
	no change of state	-0.1061	-0.1061	-0.1061	-0.1061
	change of state	-0.1307	0.0284		
	hedging demand	< 0	> 0	0	0
incomplete	π_A	0.6388	0.8934	0.7150	0.6396
	Diff-Exposure	0.0958	0.1340	0.1083	0.0968
	Jump-Exposure				
	no change of state	-0.0319	-0.0447	-0.0358	-0.0640
	change of state	-0.0319	0		
Parametrization 2: identical expected excess returns					
complete	Diff-Exposure	0.0778	0.0778	0.0778	0.0778
	Jump-Exposure				
	no change of state	-0.1779	-0.0590	-0.1061	-0.1061
	change of state	-0.1601	-0.0212		
	hedging demand	> 0	< 0	0	0
incomplete	π_A	0.7671	0.6583	0.7277	0.6495
	Diff-Exposure	0.1151	0.0988	0.1092	0.0974
	Jump-Exposure				
	no change of state	-0.0384	-0.0329	-0.0364	-0.0649
	change of state	-0.0384	0		

Table 4: Optimal Portfolios/Exposures

The table shows the optimal portfolios for our model and for the two benchmark models in a complete and in an incomplete market for a planning horizon of 20 years and for the benchmark parameters of Table 1. For the complete market, we give the optimal exposures to diffusion risk and the optimal exposure to jumps that (do not) induce a change from calm to contagion or vice versa. For the incomplete market, we give the optimal weight of stock A, as well as the induced exposures to the risk factors. Since the weights of stock B and the exposures to risk factors related to stock B coincide with those for stock A, we only show the results for A.

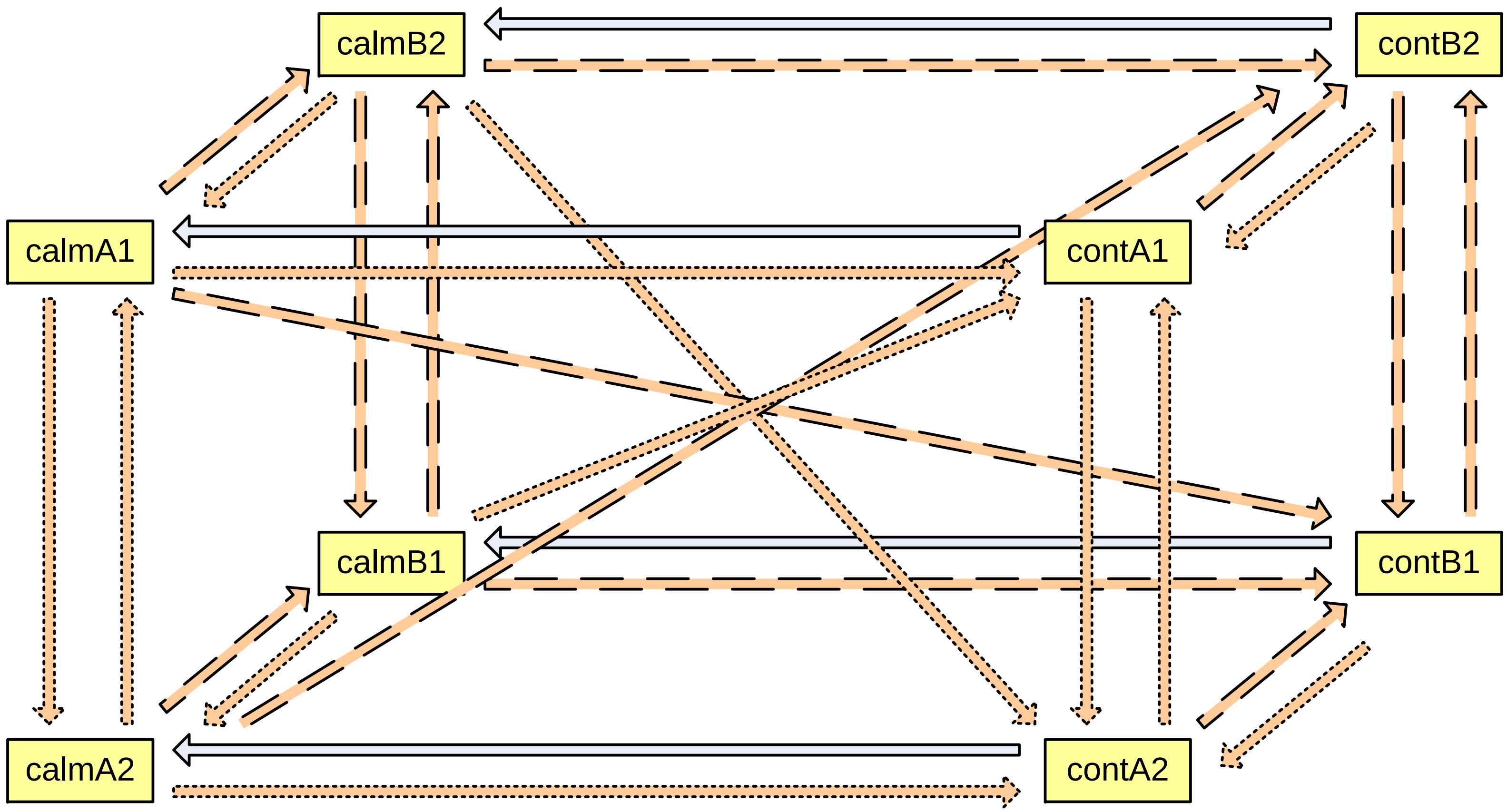


Figure 1: Our Markov Chain

This figure shows the Markov chain used in our model setup. The left states denote the calm states of the economy, the right ones denote those states in which the stocks are affected by contagion. The dotted (orange) arrows indicate a jump event leading to a loss in stock A, the dashed (orange) arrows a jump event leading to a loss in stock B. The solid (blue) arrows denote a change of state without any impact on the stock prices, i.e. a jump from the contagion state back to the calm state.

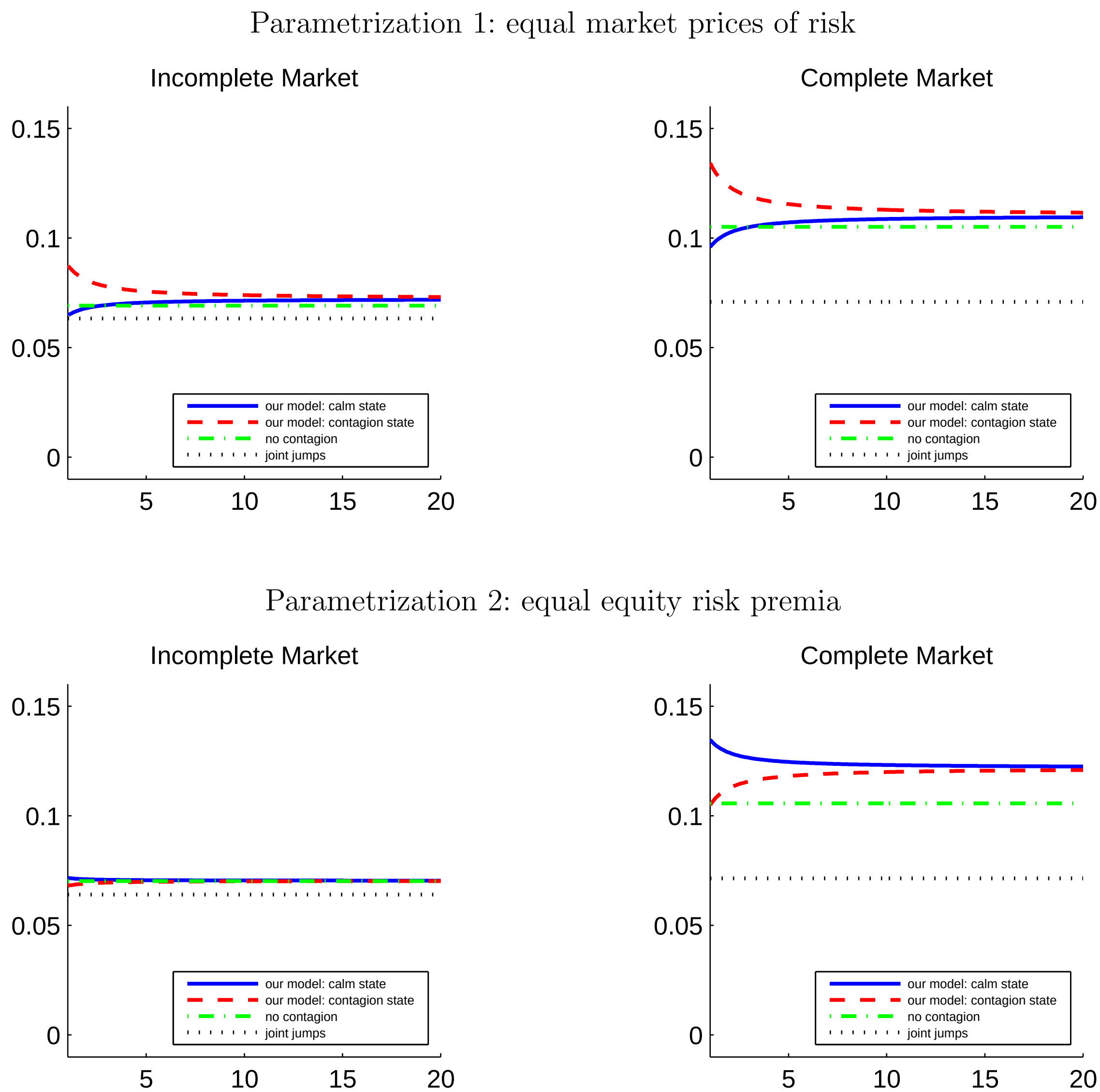
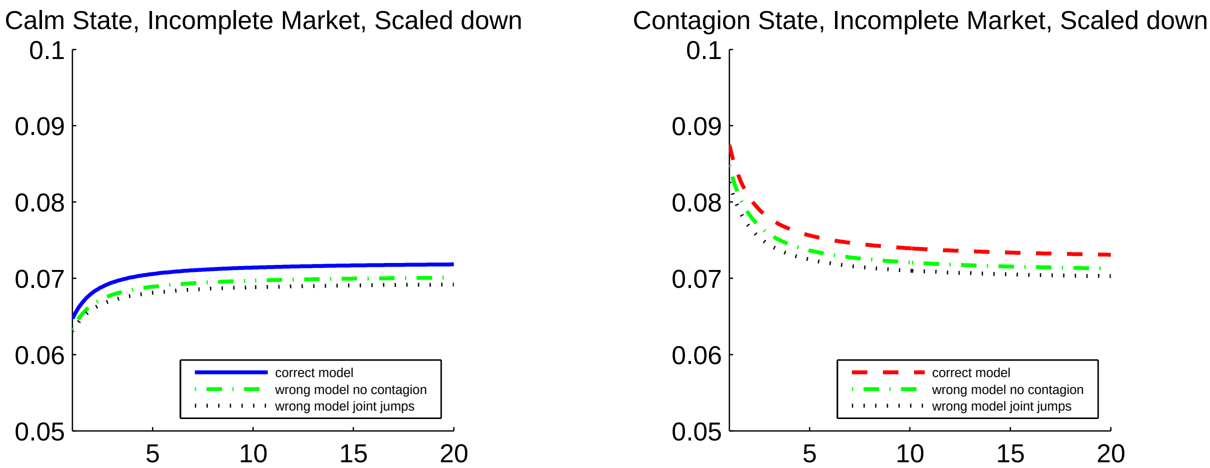


Figure 2: Certainty Equivalent Returns

The figures show the certainty equivalent returns as a function of the planning horizon for the case of equal market prices of risk (upper row) and equal equity risk premia (lower row) in the calm and in the contagion state as well as in the benchmark cases. The results for the incomplete market are given in the left column, the results for the complete one in the right column. The solid blue lines give the certainty equivalent returns in the calm state, the dashed red lines the certainty equivalent returns in the contagion state. The dash-dotted green lines denote the certainty equivalent returns in the benchmark case with no contagion, the dotted black lines the certainty equivalent returns in the model with joint jumps. The results are based on the parameters given in Table 1.

Incomplete Market



Complete Market

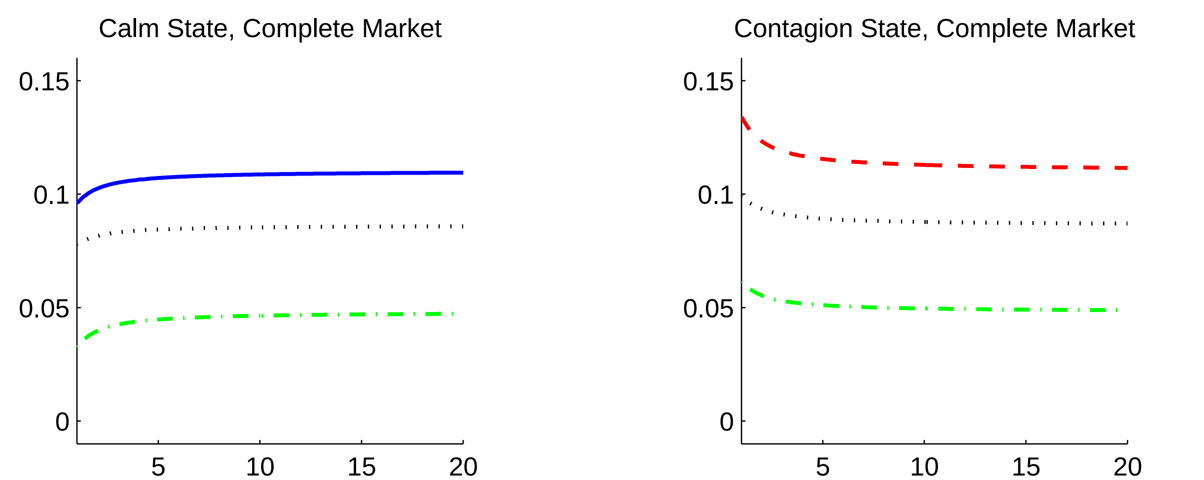
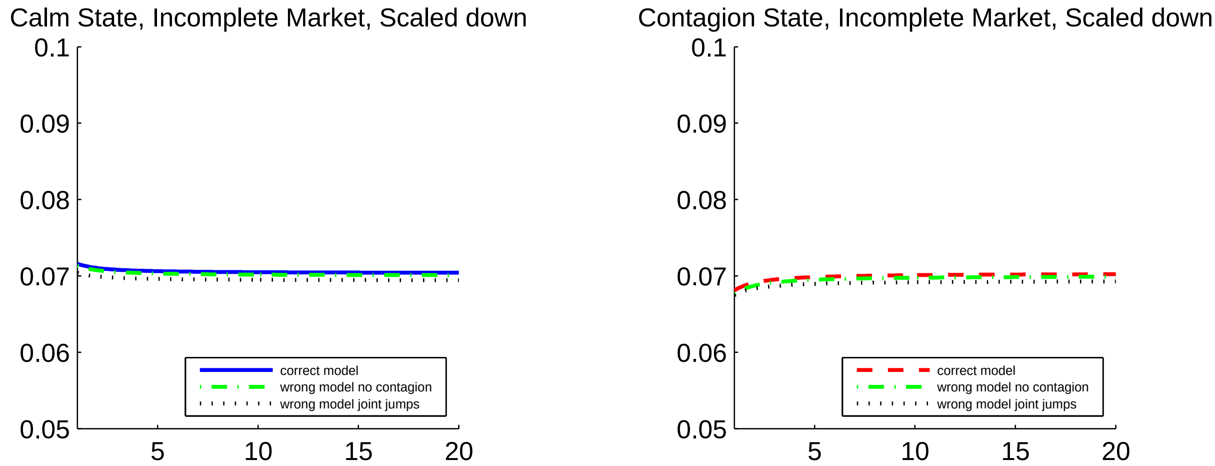


Figure 3: Model Mis-Specification: certainty equivalent returns for equal market prices of risk

The figures show the certainty equivalent returns as a function of the planning horizon for the incomplete (upper panel) and complete market (lower panel) if the economy is in the calm state (left column) and in the contagion state (right column), depending on which model is used for portfolio planning. The solid blue lines and the dashed red lines give the certainty equivalent returns n the calm and contagion state, respectively, if the correct model is used. The dash-dotted green lines indicate the CERs if a model with no contagion is used, the dotted black lines are the CERs if a model with joint jumps is used. The results are based on parametrization 1 from Table 1 for which the market prices of risk are equal in both states.

Incomplete Market



Complete Market

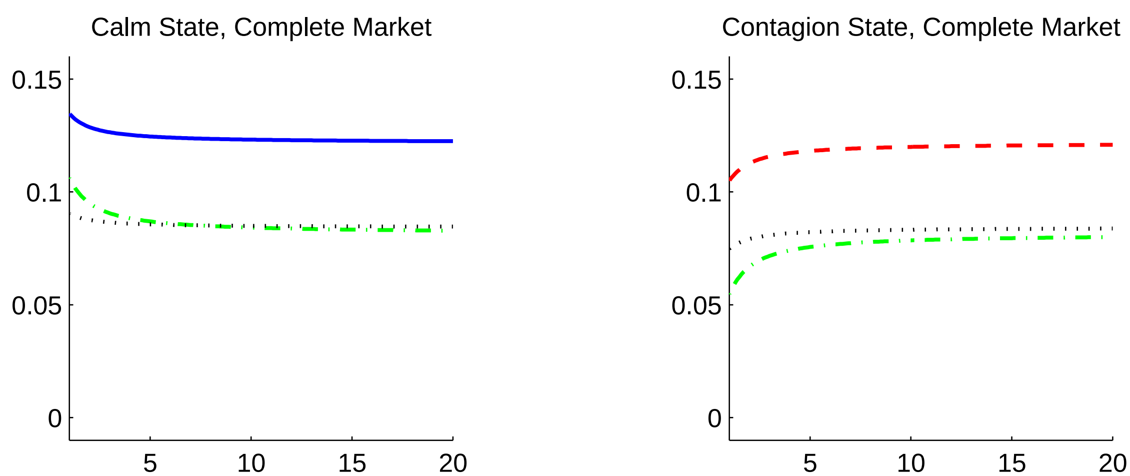


Figure 4: Model Mis-Specification: certainty equivalent returns for equal equity risk premia

The figures show the certainty equivalent returns as a function of the planning horizon for the incomplete (upper panel) and complete market (lower panel) if the economy is in the calm state (left column) and in the contagion state (right column), depending on which model is used for portfolio planning. The solid blue lines and the dashed red lines give the certainty equivalent returns in the calm and contagion state, respectively, if the correct model is used. The dash-dotted green lines indicate the CERs if a model with no contagion is used, the dotted black lines are the CERs if a model with joint jumps is used. The results are based on parametrization 2 from Table 1 for which the equity risk premium is equal in both states.

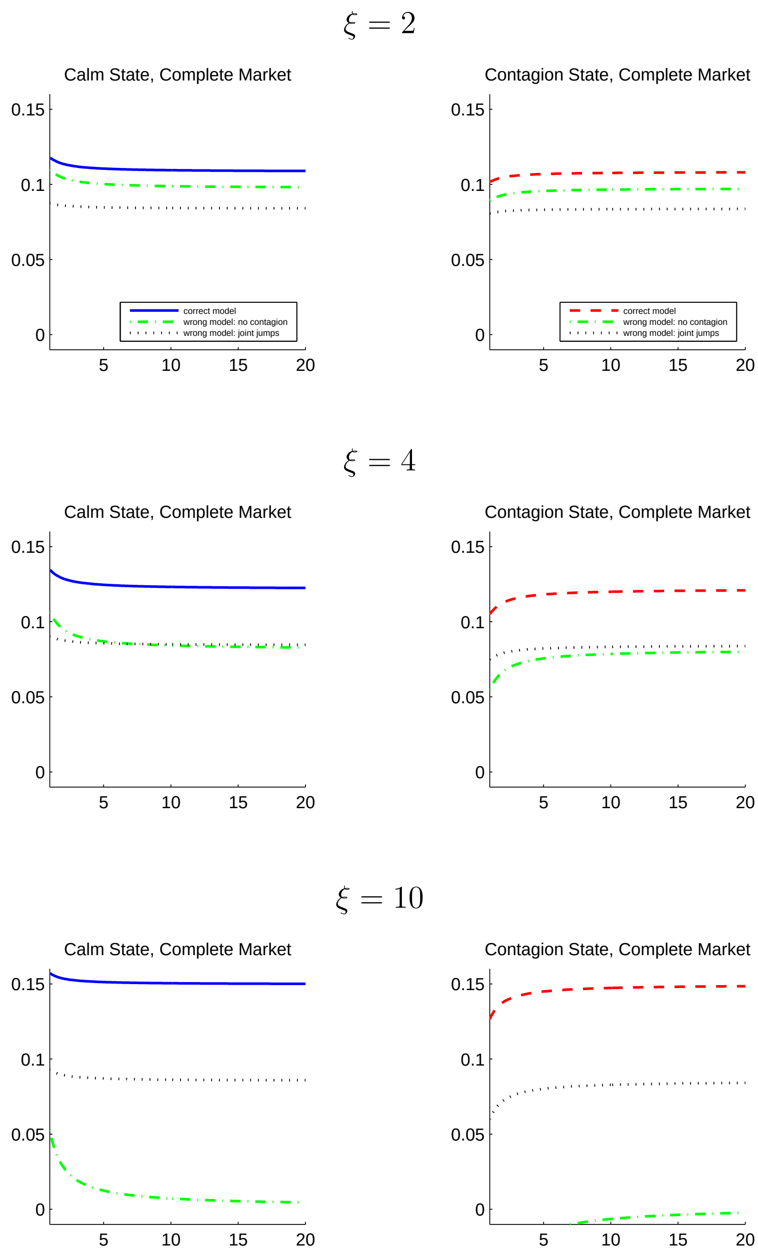


Figure 5: Model Mis-Specification: certainty equivalent returns for different values of ξ

The figures show the certainty equivalent returns for the complete market in case of model mis-specification as a function of the planning horizon for different values of $\xi_A = \xi_B = \xi$. The solid blue lines and the dashed red lines give the certainty equivalent returns in the calm and contagion state, respectively, if the correct model is used. The dash-dotted green lines indicate the CERs if a model with no contagion is used, the dotted black lines are the CERs if a model with joint jumps is used. The results are based on parametrization 2 (equal equity risk premia) from Table 1 where we have chosen $\xi = 2, 4, 10$ and thus changed the jump intensities according to Table 3.

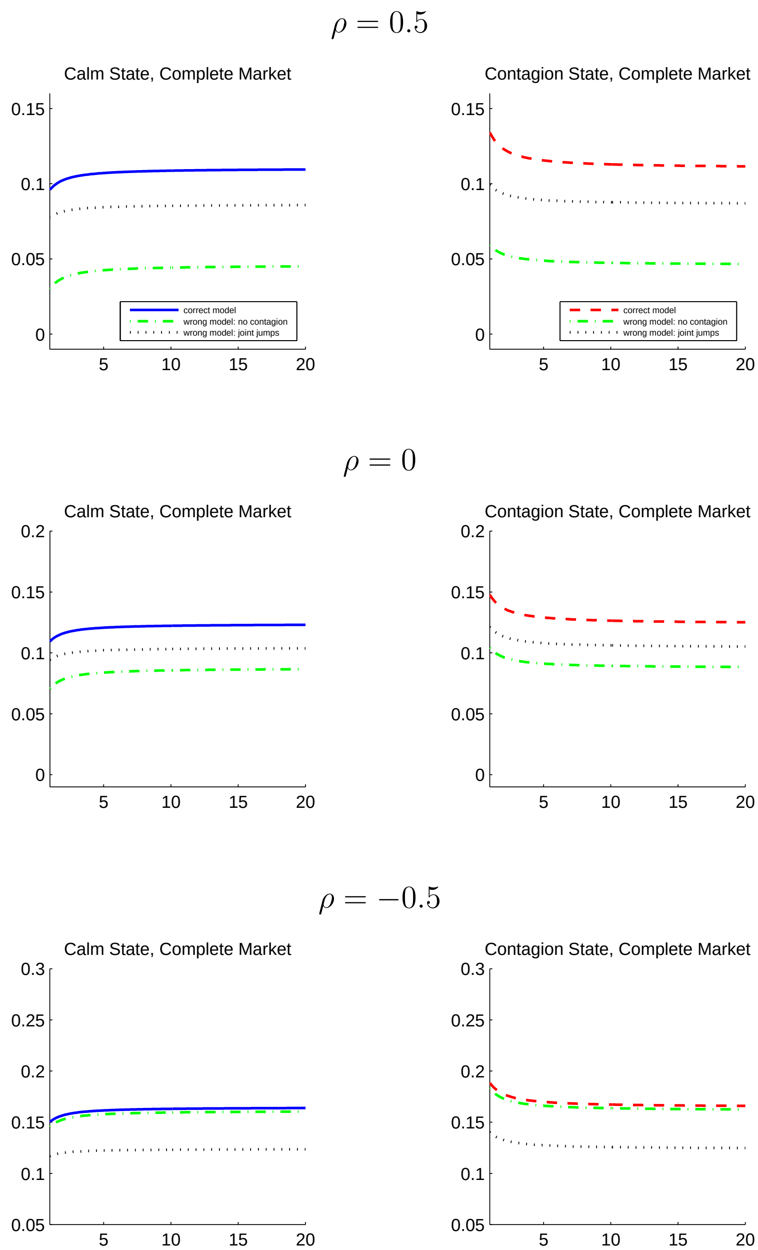


Figure 6: Model Mis-Specification: certainty equivalent returns for different values of ρ

The figures show the certainty equivalent returns for the complete market in case of model mis-specification as a function of the planning horizon for different values of ρ. The solid blue lines and the dashed red lines give the certainty equivalent returns in the calm and contagion state, respectively, if the correct model is used. The dash-dotted green lines indicate the CERs if a model with no contagion is used, the dotted black lines are the CERs if a model with joint jumps is used. The results are based on parametrization 1 (equal market prices of risk) from Table 1 where we have changed the diffusion correlation to $\rho = -0.5, 0, 0.5$.