# STABLE MODELING OF VALUE AT RISK

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**Abstract**

The Value-at-Risk (VAR) measurements are widely applied to estimate exposure to market risks. The traditional approaches to VAR computations - the variance-covariance method, historical simulation, Monte Carlo simulation, and stress-testing - do not provide satisfactory evaluation of possible losses. In this paper we analyze the use of stable Paretian distributions in VAR modeling.

*Key words and phrases.* Market risks, Value-at-Risk, VAR computations, stable Paretian distributions.

# 1 Introduction

One of the most important tasks of financial institutions is evaluating the exposure to market risks, which arise from variations in prices of equities, commodities, exchange rates, and interest rates. The dependence on market risks can be measured by changes in the portfolio value, or profits and losses. A commonly used methodology for estimation of market risks is the *Value at Risk* (VAR).

A *VAR measure* is the highest possible loss over a certain period of time at a given confidence level. For example, if the daily VAR for a given portfolio of assets is reported to be \$2 million at the 95 percent confidence level, it means that, without abrupt changes in the market conditions, one-day losses will exceed \$2 million 5 percent of the time.

Formally, a $\mathrm{VAR} = \mathrm{VAR}_{t,\tau}$ is defined as the upper bound of the one-sided confidence interval:

$$Pr[\Delta P(\tau) < -\mathrm{VAR}] = 1 - c, \tag{1}$$

where $c$ is the confidence level and $\Delta P(\tau) = \Delta P_t(\tau)$ is the *relative change (return)* in the portfolio value over the time horizon $\tau$.

$$\Delta P_t(\tau) = P(t+\tau) - P(t),$$

where $P(t) = \log S(t)$, $S(t)$ is the portfolio value at $t$, the time period is $[t, T]$, with $T - t = \tau$, and $t$ is the current time.

The time horizon, or the holding period, should be determined from the liquidity of the assets and the trading activity. The confidence level should be chosen to provide a comfortable level of downside risk [1].

The essence of the VAR computations is estimation of low quantiles in the portfolio return distributions. The VAR techniques suggest different ways of constructing the portfolio return distributions. The common methods are the delta method, historical simulation, Monte Carlo simulation, and stress-testing. The delta methods are based on the normal assumption for the distribution of financial returns. However, financial data often violate the normality assumption. The empirical observations exhibit "fat" tails and excess kurtosis. The historical method does not impose distributional assumptions but it is not reliable in estimating low quantiles of $\Delta P$ with a

[1] In practice, the time horizon varies from one day to two weeks (10 trading days) and the confidence level - from 95% to 99%. The regulators recommend to calculate VAR at the 10-day holding period and the 99% confidence level.

small number of observations in the tails. The performance of the Monte Carlo method depends on the quality of distributional assumptions on the underlying risk factors.

The existing methods do not provide satisfactory evaluation of VAR. The main drawback is the lack of a convincing unified model for VAR capturing the following phenomena generally observed in financial data, such as asset returns, interest rates, exchange rates, equities:

- heavy tails of the marginal distributions of the process of financial returns,
- time-varying volatility,
- short- and long-range dependence.

In this article we propose using stable distributions for constructing models that encompass these empirical features and develop more precise VAR-estimation techniques. Adequate approximation of distributional forms of portfolio returns is a key condition for accurate VAR derivation. Given the leptokurtic nature (thick tails and excess kurtosis) of empirical financial data, the stable Paretian distributions seem to be the most appropriate distributional models[2]. The conditional heteroskedastic models based on the $\alpha$-stable hypothesis can be applied to describe both thick tails and time-varying volatility. The fractional-stable GARCH models can explain all observed phenomena: heavy-tails, time-varying volatility, and temporal dependence.

The remainder of the paper is organized as follows. In Section 2 we discuss traditional approaches to VAR computations. Section 3 provides a finance-oriented description of stable distributions. In Section 4 we estimate the VAR measurements for financial returns following a stable law [3]. Section 5 states conclusions and outlines future research on VAR modeling with stable processes.

[2]Cheng and Rachev (1995); Chobanov, Mateev, Mittnik, and Rachev (1996); Fama (1965); Gamrowski and Rachev (1994, 1995a, and 1995b); Mandelbrot (1962, 1963a, 1963b, and 1967); McCulloch (1996); Mittnik and Rachev (1991,1993a and 1993b); Mittnik, Rachev, and Chenyao (1996); Mittnik, Rachev, and Paolella (1998).

[3]See also Gamrowski and Rachev (1996).

## 2 Computation of VAR

From the definition of VAR = $\mathrm{VAR}_{t,\tau}$ in equation (1), the VAR values are obtained from the probability distribution of portfolio value returns:

$$1 - c = F_{\Delta P}(-\mathrm{VAR}) = \int_{-\infty}^{-\mathrm{VAR}} f_{\Delta P}(x)dx,$$

where $F_{\Delta P}(x) = Pr(\Delta P \leq x)$ is the cumulative distribution function (cdf) of portfolio returns in one period, and $f_{\Delta P}(x)$ is the probability density function (pdf) of $\Delta P$ [4]. The VAR methodologies mainly differ in the way of constructing $f_{\Delta P}(x)$.

The traditional techniques of approximating the distribution of $\Delta P$ are:

- the *parametric method* (analytic or models-based),
- *historical simulation* (nonparametric or empirical-based),
- *Monte Carlo simulation* (stochastic simulation), and
- *the stress-testing* (scenario analysis). [5]

### 2.1 Parametric Method

If the changes in the portfolio value are characterized by a parametric distribution, VAR can be computed using the distribution parameters. In this section we briefly review: VAR for a single asset, portfolio VAR, a parametric method based on the normal distribution, and linear approximation to price movements.

#### 2.1.1 VAR for a Single Asset

Assume that a portfolio consists of a single asset, which depends only on one risk factor. Traditionally, in this setting, the distribution of asset returns is assumed to be the *univariate normal distribution*, identified by two parameters: the mean $\mu$, and the standard deviation, $\sigma$. The problem of calculating VAR is then reduced to finding the $(1-c)$th percentile of the standard normal distribution $z_{1-c}$ :

[4] If $f_{\Delta P}(x)$ does not exist, then VAR can be obtained from (cdf) $F_{\Delta P}$.

[5] Dave and Stahl (1997); Duffie and Pan (1997); Fallon (1996); Gamrowski and Rachev (1996); Hopper (1996/1997); Jorion (1996); JP Morgan (1995); Linsmeier and Pearson (1996); Mahoney (1996); Phelan (1995); Pritsker (1996/1997); Simons (1996).

$$1 - c = \int_{-\infty}^{X^*} g(x)dx = \int_{-\infty}^{z_{1-c}} \phi(z)dz = N(z_{1-c}), \qquad \text{with} \quad X^* = z_{1-c}\sigma + \mu,$$

where $\phi(z)$ is the standard normal density function, $N(z)$ is the cumulative normal distribution function, $X$ is the portfolio return, $g(x)$ is the normal distribution function for returns with mean $\mu$ and standard deviation $\sigma$, and X* is the lowest return at a given confidence level $c$.

In many applications investors assume that the expected return $\mu$ equals 0. This assumption is based on the conjecture that the magnitude of $\mu$ is substantially smaller than the magnitude of the standard deviation $\sigma$ and, therefore, can be ignored. Then we have

$$X^* = z_{1-c}\,\sigma \ .$$

and, therefore,

$$\mathrm{VAR} = -Y_0 X^* = -Y_0 z_{1-c}\,\sigma \ .$$

where $Y_0$ is the initial portfolio value.

### 2.1.2 Portfolio VAR

If a portfolio consists of many assets, the computation of VAR is performed in several steps. Portfolio assets are decomposed into "building blocks", which depend on a finite number of risk factors. Exposures of the portfolio securities are combined into risk categories. Then, the total portfolio risk is obtained by aggregating risk factors and their correlations. We denote:

- $X_p$ is the portfolio return in one period,
- $N$ is the number of assets in the portfolio,
- $X_i$ is the $i$-th asset return in one period $(\tau = 1), X_i = \Delta P(1) = P_i(1) - P_i(0)$, where $P_i$ is the log-spot price of asset $i$, $i = 1, \ldots, N$. More generally, $X_i$ can be the risk factor that enters linearly[6] in the portfolio return.
- $w_i$ is the $i$-th asset's weight in the portfolio, $i = 1, \ldots, N$.

[6] If the risk factor does not enter linearly (as in a case of an option), then a linear approximation is used.

The portfolio return is

$$X_P = \sum_{i=1}^{N} w_i X_i \,.$$

In matrix notation,

$$X_P = w^T X \,,$$

where $w = (w_1, w_2, \ldots, w_N)^T$, $X = (X_1, X_2, \ldots, X_N)^T$.

Then the portfolio variance is

$$V(X_P) = w^T \underline{\Sigma} w = \sum_{i=1}^{N} w_i^2 \sigma_{ii} + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} w_i w_j \rho_{ij} \sigma_i \sigma_j \,,$$

where $\sigma_{ii}$ is the variance of returns on the $i$-th asset, $\sigma_i$ is the standard deviation of returns on the $i$-th asset, $\rho_{ij}$ is the correlation between the returns on the $i$-th and the $j$-th assets, $\underline{\Sigma}$ is the covariance matrix, $\underline{\Sigma} = [\sigma_{ij}]$, $1 \leq i \leq N$, $1 \leq j \leq N$.

If all portfolio returns are *jointly normally distributed*, the portfolio return, as a linear combination of normal variables, is also *normally distributed*. The portfolio VAR based on the normal distribution assumption is

$$\mathrm{VAR} = -Y_0 z_{1-c}\, \sigma(X_P) \,,$$

where $\sigma(X_P)$ is the portfolio standard deviation (the *portfolio volatility*),

$$\sigma(X_P) = \sqrt{V(X_P)} \,.$$

Thus, risk can be represented by a combination of linear exposures to normally distributed factors.

In this class of parametric models, to estimate risk, it is sufficient to evaluate the covariance matrix of portfolio risk factors (in the simplest case, individual asset returns).

The estimation of the covariance matrix is based on the *historical data* or on *implied data* from securities pricing models.

If portfolios contain zero-coupon bonds, stocks, commodities, and currencies, VAR can be computed from correlations of these basic risk factors and the asset weights. If portfolios include more complex securities, then the securities are decomposed into building blocks.

The portfolio returns are often assumed to be normally distributed[7]. One of methods employing the normality assumption for returns is the *delta* method (the delta-normal or the variance-covariance method).

### 2.1.3 Delta Method

The *delta* method estimates changes in prices of securities using their "deltas" with respect to basic risk factors. The method involves a *linear* (also named as *delta* or *local*) *approximation* to (log) price movements:

$$P(X+U) \approx P(X) + P'(X)\,U\,,$$

or

$$\Delta P(X) = P(X+U) - P(X) \approx P'(X)\,U\,,$$

where $X$ is the level of the basic risk factor (i.e., an equity, an exchange rate), $U$ is the change in $X$, $P(X+U) = P(t+\tau, X+U)$, $P(X) = P(t, X)$[8], $P(X)$ is the (log) price of the asset at the $X$ level of the underlying risk factor, $P'(X) = \partial P/\partial X$ is the first derivative of $P(X)$, it is commonly called the *delta* ($\underline{\Delta} = \underline{\Delta}(X)$) of the asset.

Thus, the price movements of the securities are approximately

$$\Delta P(X) \approx P'(X) U = \underline{\Delta} U\,.$$

The *delta-normal* (the *variance-covariance*) method computes the portfolio VAR as

$$\mathrm{VAR} = -Y_0 z_{1-c} \sqrt{d^T \underline{\Sigma} d}\,,$$

where $d = d(X) = (\underline{\Delta}_1(X), \underline{\Delta}_2(X), \ldots, \underline{\Delta}_n(X))^T$ is a vector of the delta-positions, $\underline{\Delta}_j(X)$ is the security's delta with respect to the $j$-th risk factor, $\underline{\Delta}_j = \partial P/\partial X_j$.

## 2.2 Historical Simulation

The historical simulation approach constructs the distribution of the portfolio value changes $\Delta P$ from historical data without imposing distribution assumptions and estimating parameters. Hence, sometimes the historical simulation method is called a *nonparametric* method. The method assumes

[7] JP Morgan (1995); Phelan (1995).

[8] Because the time horizon ($\tau$) is fixed and $t$ is the present time, we shall omit the time argument and shall write $P(X+U)$ instead of underlying $P(t+\tau,\, X+U)$ and $P(X)$ instead of $P(t,\, X)$. We shall consider the dependency of $P$ on the risk factor $X$ only.

that trends of past price changes will continue in the future. Hypothetical future prices for time $t+s$ are obtained by applying historical price movements to the current (log) prices:

$$P^*_{i,t+s} = P^*_{i,t+s-1} + \Delta P_{i,t+s-\kappa}\,,$$

where $t$ is the current time, $s = 1, 2, \ldots, \kappa$, $\kappa$ is the horizon length of going back in time, $P^*_{i,t+s}$ is the hypothetical (log) price of the $i$-th asset at time $t+s$, $P^*_{i,t} = P_{i,t}$, $\Delta P_{i,t+s-\kappa} = P_{i,t+s-\kappa} - P_{i,t+s-1-\kappa}\,, P_{i,t}$ is the historical (log) price of the $i$-th asset at time $t$. Here we assumed that the time horizon $\tau = 1$.

A portfolio value $P^*_{p,t+s}$ is computed using the hypothetical (log) prices $P^*_{i,t+s}$ and the current portfolio composition. The portfolio return at time $t+s$ is defined as

$$R^*_{p,t+s} = P^*_{p,t+s} - P_{p,t},$$

where $P_{p,t}$ is the current portfolio (log) price.

The portfolio VAR is obtained from the density function of the computed hypothetical returns. Formally, VAR $=$ VAR$_{t,\tau}$ is estimated by the negative of the $(1-c)th$ quantile, VAR$^*$; namely, $F_{\kappa,\Delta P}(-\text{VAR}) = F_{\kappa,\Delta P}(\text{VAR}^*) = 1 - c$, where $F_{\kappa,\Delta P}(x)$ is the empirical cumulative distribution function $F_{\kappa,\Delta P}(x) = \frac{1}{\kappa}\sum_{s=1}^{\kappa} \mathbf{1}\left\{R^*_{p,t+s} \leq x\right\}, \qquad x \in \mathbf{R}$.

## 2.3 Monte Carlo Simulation

The *Monte Carlo* approach requires specification of statistical models for the basic risk factors and the underlying assets. The method simulates the behavior of risk factors and asset prices by generating random price paths. Monte Carlo simulations provide possible portfolio values on a given date $T$ after the present time $t, T > t$. The VAR(VAR$_T$) value can be determined from the distribution of simulated portfolio values. The Monte Carlo method is performed according to the following algorithm:

1. Specify stochastic processes and process parameters for financial variables and correlations.

2. Simulate the hypothetical price trajectories for all variables of interest. Hypothetical price changes are obtained by random draws from the specified distribution.

3. Obtain asset prices at time $T, P_{i,T}$, from the simulated price trajectories. Compute the portfolio value $P_{p,T} = \sum w_{i,T} P_{i,T}$.

4. Repeat steps 2 and 3 many times to form the distribution of the portfolio value $P_{p,T}$.

5. Measure $VAR_T$ as the negative of the $(1-c)th$ percentile of the simulated distribution for $P_{p,T}$.

### 2.4 Stress testing

The parametric, historical simulation, and Monte Carlo methods estimate the VAR (expected losses) depending on risk factors. The *stress testing* method examines the effects of large movements in key financial variables on the portfolio value. The price movements are simulated in line with the certain scenarios[9]. Portfolio assets are reevaluated under each scenario. The portfolio return is derived as

$$R_{p,s} = \sum w_{i,s} R_{i,s},$$

where $R_{i,s}$ $(w_{i,s})$ is the hypothetical return (weight) on the i-th security under the new scenario $s$. Estimating a probability for each scenario $s$ allows to construct a distribution of portfolio returns, from which VAR can be derived.

### 2.5 Weaknesses of Traditional VAR Methods

The traditional VAR methods do not provide accurate estimation of VAR. The *delta methods* are based on the normal assumption for the distribution of financial returns. However, financial data violate the normality assumption. The empirical observations exhibit "fat" tails and excess kurtosis. Thus, the delta-normal technique does not fit well data with heavy tails. The *historical simulation* does not impose distributional assumptions. Models based on historical data assume that the past trends will continue in the future. However, the future might encounter extreme events. The historical simulation technique is limited in forecasting the range of portfolio value changes and is not reliable in estimating low quantiles with a small number of observations in the tails. One weakness of *stress-testing* is that it is

[9]Scenarios include possible movements of the yield curve, changes in exchange rates, etc. together with estimates of the underlying probabilities.

subjective. The performance of the *Monte Carlo* method depends on the quality of distributional assumptions on the underlying risk factors.

We propose the use of stable processes in VAR modeling. In the next section we first provide a finance-oriented description of stable laws. Then, we describe modeling VAR with stable distributions and compare the stable VAR approach with the existing methodologies.

# 3 A Finance-oriented Description of Stable Distributions

In this part we describe parameters and some finance-oriented properties of stable distributions. We also examine methods of estimating parameters of stable laws.

## 3.1 Parameters and Properties of Stable Distributions

A random variable $R$ is said to be *stable*[10] if for any $a > 0$ and $b > 0$ there exist constants $c > 0$ and $d \in \mathbf{R}$ such that

$$aR_1 + aR_2 \stackrel{\mathrm{d}}{=} cR + d$$

where $R_1$ and $R_2$ are independent copies of $R$ and $\stackrel{d}{=}$ denotes the equality in distribution.

In general, the stable distributions do not have closed form expressions for density and distribution functions. Stable random variables (R) are commonly described by their characterestic functions:

$$\begin{aligned}
\Phi_R(\theta) &= E(\exp(iR\theta)) \\
&= \exp\left\{-\sigma^\alpha|\theta|^\alpha\left(1 - i\beta\,\mathrm{sign}(\theta)\tan\frac{\pi\alpha}{2}\right) + i\mu\theta\right\}, \text{if } \alpha \neq 1, \\
\Phi_R(\theta) &= E(\exp\,(iR\theta)) \\
&= \exp\left\{-\sigma|\theta|\left(1 + i\beta\frac{2}{\pi}\mathrm{sign}(\theta)\ln\theta\right) + i\mu\theta\right\}, \text{if } \alpha = 1,
\end{aligned}$$

where $\alpha$ is the *index of stability*, $0 < \alpha \leq 2$, $\beta$ is the *skewness parameter*, $-1 \leq \beta \leq 1$, $\sigma$ is the *scale parameter*, $\sigma \geq 0$, and $\mu$ is the *location parameter*,

[10] Often $R$ is called *$\alpha$-stable* or *Pareto stable* or *Pareto-Lévy-stable* (for $\alpha < 2$).

$\mu \in \mathbf{R}$. To indicate the dependence of a stable random variable $R$ on its parameters, we write $R \sim S_\alpha(\beta, \sigma, \mu)$. If the *index of stability* $\alpha = 2$, then the stable distribution reduces to the Gaussian distribution. In empirical studies the modeling of financial return data is done typically with stable distributions having $1 < \alpha < 2$.[11] Stable distributions are unimodal and the smaller $\alpha$ is, the stronger the leptokurtic feature of the distribution (the peak of the density becomes higher and the tails are heavier). Thus, the index of stability can be interpreted as a *measure of kurtosis*. When $\alpha > 1$, the *location parameter* $\mu$ measures the mean of the distribution. If the *skewness parameter* $\beta = 0$, the distribution of $R$ is symmetric and the characteristic function is

$$\Phi_R(\theta) = E(\exp{(iR\theta)}) = exp\left\{-\sigma^\alpha|\theta|^\alpha + i\mu\theta\right\} .$$

If $\beta > 0$, the distribution is skewed to the right. If $\beta < 0$, the distribution is skewed to the left. Larger magnitudes of $\beta$ indicate stronger skewness. If $\beta = 0$ and $\mu = 0$, then the stable random variable $R$ is called *symmetric* $\alpha$*-stable (s$\alpha$s)*. The *scale parameter* (the volatility) $\sigma$ allows any stable random variable $R$ to be expressed as $R = \sigma R_o$, where $R_o$ has a unit scale parameter, and the same index of stability $\alpha$ and skewness parameter $\beta$ as $R$. The scale parameter generalizes the definition of standard deviation. The stable analog of variance is the *variation*: $\nu_\alpha = \sigma^\alpha$.

In VAR estimations we are interested in investigating the behavior of the distributions in the tails. The *tails* of the stable (non-Gaussian) distributions have a power decay and are characterized by the following properties:

$$\lim_{\lambda \to +\infty} \lambda^\alpha P(R > \lambda) = k_\alpha \frac{1+\beta}{2}\sigma^\alpha$$

and

$$\lim_{\lambda \to +\infty} \lambda^\alpha P(R < -\lambda) = k_\alpha \frac{1-\beta}{2}\sigma^\alpha \ ,$$

where

$$k_\alpha = \frac{1-\alpha}{,\ (2-\alpha)\cos(\frac{\pi\alpha}{2})}, \text{ if } \alpha \neq 1,\ k_\alpha = \frac{2}{\pi}, \text{ if } \alpha = 1^{12} .$$

[11] The financial returns modeled with $\alpha$-stable laws exhibit finite means but infinite variances.

The $p$-th absolute moment, $E|R|^p = \int_0^\infty P(|R|^p > x)dx$, is

- *finite* if $p < \alpha$ or $\alpha = 2$, and
- *infinite* otherwise.

Thus, the second moment of any non-Gaussian stable distribution is infinite.

Stable distributions possess the *additivity property*: a linear combination of independent stable random variables with stability index $\alpha$ is again a stable random variable with the same $\alpha$.[13]

Example: If $R_1, R_2, \ldots, R_n$ are independent stable random variables with stability index $\alpha, R_i \sim S_\alpha(\beta_i, \sigma_i, \mu_i)$, then $R = \sum_{i=1}^n w_i R_i$ is a stable random variable with the same $\alpha$ and parameters:

(a) if $\alpha \neq 1$,

$$\begin{aligned}
\sigma &= ((|w_1|\sigma_1)^\alpha + \cdots + (|w_n|\sigma_n)^\alpha)^{\frac{1}{\alpha}} \ , \\
\beta &= \frac{\operatorname{sign}(w_1)\beta_1(|w_1|\sigma_1)^\alpha + \cdots + \operatorname{sign}(w_n)\beta_n(|w_n|\sigma_n)^\alpha}{(|w_1|\sigma_1)^\alpha + \cdots + (|w_n|\sigma_n)^\alpha} \, , \\
\mu &= w_1\mu_1 + \cdots + w_n\mu_n \ ,
\end{aligned}$$

(b) if $\alpha = 1$,

$$\begin{aligned}
\sigma &= |w_1|\sigma_1 + \cdots + |w_n|\sigma_n \ , \\
\beta &= \frac{\operatorname{sign}(w_1)\beta_1|w_1|\sigma_1 + \cdots + \operatorname{sign}(w_n)\beta_n|w_n|\sigma_n}{|w_1|\sigma_1 + \cdots + |w_n|\sigma_n} \, , \\
\mu &= w_1\mu_1 + \cdots + w_n\mu_n - \frac{2}{\pi}(w_1 \ln|w_1|\sigma_1\beta_1 + \cdots + w_n \ln|w_n|\sigma_n\beta_n).
\end{aligned}$$

Since the Pareto-stable distributions have infinite variances, one cannot estimate risk by variance and dependence by correlations. We shall introduce

[12] Note that, in contrast to the normal case, the tails of the non-Gaussian (Pareto) stable distributions are much fatter, which will be an important issue in estimating VAR.

[13] This property is shared only by normal and stable laws, and is the main advantage of the use of stable laws for portfolio returns.

variance- and covariance-similar notions for stable laws. These notions are based on the multivariate assumptions of stable distributions.

A random vector $R$ of dimension $d$ is *stable* if for any $a > 0$ and $b > 0$ there exist $c > 0$ and a $d$-dimensional vector $D$ such that

$$aR_1 + bR_2 \stackrel{d}{=} cR + D,$$

where $R_1$ and $R_2$ are independent copies of $R$.

If a random vector is stable with $\alpha > 1$, then it means that all components of the vector are stable with the same index of stability and any linear combination (for example, portfolio returns) is again stable[14] .

The characteristic function of a $d$-dimensional vector is given by:

(a) if $\alpha \neq 1$,

$$\begin{aligned}\Phi_R(\theta) &= \Phi_R(\theta_1, \theta_2, \dots, \theta_d) \\ &= E\exp(i\theta^T R) = \exp\left\{-\int_{S_d} |\theta^T s| \left(1 - i\mathrm{sign}(\theta^T s) \tan\frac{\pi\alpha}{2}\right) , (ds) + i\theta^T \mu\right\},\end{aligned}$$

(b) if $\alpha = 1$,

$$\Phi_R(\theta) = \exp\left\{-\int_{S_d} |\theta^T s| \left(1 + i\frac{2}{\pi}\mathrm{sign}\,(\theta^T s) \ln|\theta^T s|\right) , (ds) + i\theta^T \mu\right\},$$

where , is a bounded nonnegative measure on the unit sphere $S_d$, $s$ is the integrand unit vector ($s \in S_d$) and $\mu$ is the shift vector. The measure , is named a *spectral measure*. Let $H$ be the distribution function of , . Then, the characteristic function in polar coordinates is as follows

(a) if $\alpha \neq 1$,

$$\begin{aligned}\Phi_R(\theta) = \exp\Big\{-\rho^\alpha \int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi} &|\cos(\theta,\psi)|^\alpha (1 - i\mathrm{sign}\,(\cos(\theta,\psi)) \\ &\tan\frac{\pi\alpha}{2}) dH(\psi) + i\theta^T\mu\Big\},\end{aligned}$$

(b) if $\alpha = 1$,

$$\begin{aligned}\Phi_R(\theta) = \exp\Big\{-\rho \int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi} &|\cos(\theta,\psi)|(1 + i\mathrm{sign}\,(\cos(\theta,\psi)) \\ &\frac{2}{\pi}\ln(\rho|\cos(\theta,\psi)|)) dH(\psi) + i\theta^T\mu\Big\},\end{aligned}$$

[14]We shall model the dependence structure of the vector of returns $(R_1, \dots, R_d)$ of a portfolio by assuming that $(R_1, \dots, R_d)$ is an $\alpha$-stable vector.

where $\theta = (\rho\cos\phi_1, \rho\sin\phi_1\cos\phi_2, \ldots, \rho\sin\phi_1, \ldots, \sin\phi_{n-2}\cos\phi_{n-1}, \rho\sin\phi_1, \ldots, \sin\phi_{n-1})^T$, $\rho = |\theta|, \psi = (\psi_1, \ldots, \psi_{n-1})^T$, and

$$
\begin{aligned}
\cos(\theta, \psi) &= \left(\prod_{i=1}^{d-1} \sin\phi_i \sin\psi_i\right) \\
&+ \left(\prod_{i=1}^{d-2} \sin\phi_i \sin\psi_i\right) \cos\phi_{d-1}\cos\psi_{d-1} + \ldots + \cos\phi_1\cos\psi_1.
\end{aligned}
$$

If $a > 1$, then $\mu$ is the mean vector, $\mu = ER$. The scale parameter of a linear combination of the components of a stable vector $R$ satisfies the relation:

$$\sigma^\alpha(w^T R) = \sigma^\alpha(w_1 R_1 + \cdots + w_d R_d) = \int_{S_d} |w^T s|^\alpha, (ds).$$

Viewing $R = (R_1, \ldots, R_d)$ as the vector of individual returns in a portfolio with weights $w_1, \ldots, w_d, \sigma^\alpha(w^T R)$ will be the portfolio risk-measure. As we defined above, $\nu_\alpha = \sigma^\alpha$ is the *variation*, the stable equivalent of variance. Similarly to the traditional interpretation of covariance as an indicator of *dependence*, one can use the *covariation* to estimate the dependence between two $s\alpha s$ distributions:

$$[R_1; R_2]_\alpha = \frac{1}{\alpha} \frac{\partial\sigma^\alpha(w_1 R_1 + w_2 R_2)}{\partial w_1}\bigg|_{w_1=0; w_2=1} = \int_{S_d} s_1 s_2^{<\alpha-1>}, (ds),$$

where $(R_1, R_2)$ is a $s\alpha s$ vector $(1 < \alpha < 2)$ and $x^{<k>} = |x|^k \text{sign}(x)$ (signed power). The matrix of covariations $[R_i; R_j]_\alpha, 1 \leq i \leq d, 1 \leq j \leq d$, determines the dependence structure among the individual returns in the portfolio.

### 3.2 Estimation of Parameters of Stable Distributions[15]

We shall examine the methods of estimating the stable parameters and their applicability in VAR computations, where the primary concern is the tail behavior of distributions. It has been proposed that it is more useful to evaluate directly the tail index (the index of stability) instead of fitting the whole distribution. The latter method is claimed to negatively affect the estimation of the tail behavior by its use of "center" observations. We shall describe both approaches: tail estimation and entire-distribution modeling. We suggest a method, which combines the two techniques: it is designed for fitting the overall distribution with greater emphasis on the tails.

#### 3.2.1 Tail Estimation

Tail estimators for the index of stability $\alpha$ are based on the asymptotic Pareto tail behavior of stable distributions[16]. We shall consider the following estimators of tail thickness: the Hill, the Pickands, and the modified unconditional Pickands[17]. The Hill estimator[18] is described by

$$\hat{\alpha}_{\text{Hill}} = \frac{1}{\frac{1}{k}\sum_{j=1}^{k} \ln(X_{n+1-j:n}) - \ln X_{n-k:n}},$$

where $X_{j:n}$ denotes the $j$-th order statistic of sample $X_1, \ldots, X_n$[19]; the integer $k$ points where the tail area "starts". The selection of $k$ is complicated by a tradeoff: it must be adequately small so that $X_{n-k:n}$ is in the tail of the distribution; but if it is too small, the estimator is not accurate. The disadvantage of the estimator is the condition to explicitly determine the order statistic $X_{n-k:n}$. It is proved that, for stable Paretian distributions, the Hill

[15]For additional references on estimation of four parameters of stable univariate laws, see Chobanov, Mateev, Mittnik, and Rachev (1996), Gamrowski and Rachev (1994, 1995a, and 1995b), Klebanov, Melamed, and Rachev (1994), Kozubowski and Rachev (1994), McCulloch (1996), Mittnik and Rachev (1991), Rachev and SenGupta (1993). For the multivariate case estimation of: the spectral measure, the index of stability, the covariation and tests for dependence of stable distributed returns, see Cheng and Rachev (1995), Gamrowski and Rachev (1994, 1995a, 1995b, and 1996), Heathcote, Cheng, and Rachev (1995), Mittnik and Rachev (1993b), Rachev and Xin (1993).

[16]See section 3.1.

[17]For details on the Hill, Pickands, and the modified unconditional Pickands estimators, see Mittnik, Paolella, Rachev (1998c) and references therein.

[18]Hill (1975).

[19]Given a sample of observations $X_1, \ldots, X_n$, we rearrange the sample in increasing order $X_{1:n} \leq \cdots \leq X_{n:n}$, then the $j$-th order statistic is equal to $X_{j:n}$.

Hill estimator with 95% confidence bounds

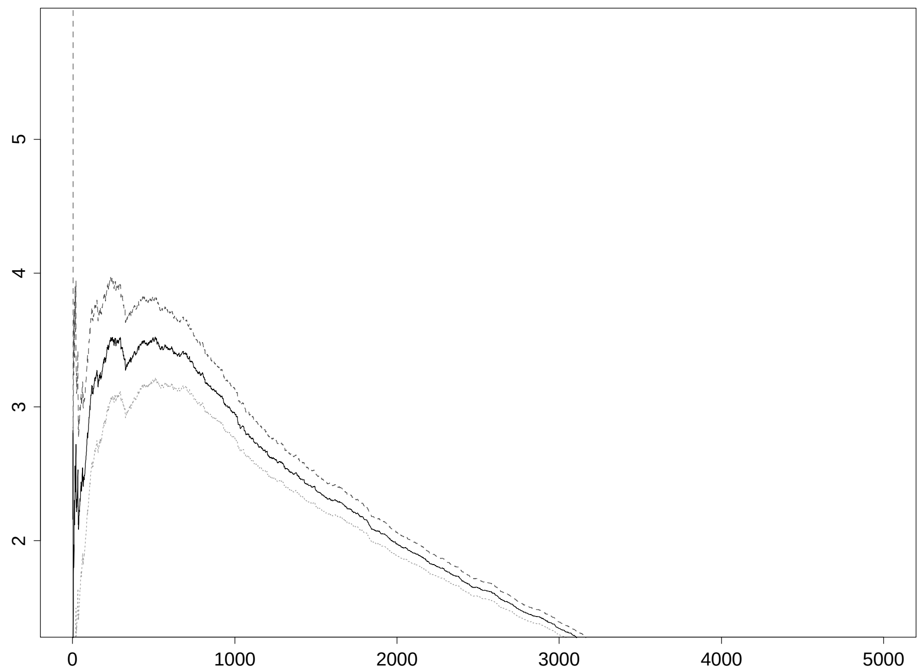


Figure 1: Hill estimator for 10,000 standard stable observations with index alpha=1.9

estimator is consistent and asymptotically normal. Mittnik, Paolella, and Rachev (1998c) found that, the small sample performance of $\hat{\alpha}_{\text{Hill}}$ does not resemble its asymptotic behavior, even for $n > 10,000$ (see Figure 1[20]).

It is necessary to have enormous data series in order to obtain unbiased estimates of $\alpha$, for example, with $\alpha = 1.9$, reasonable estimates are produced only for $n > 100,000$ (see Figure 2[21]). Alternatives to the Hill estimator

[20]In Figure 1, the true value of $\alpha$ is 1.9, the sample size is $n$=10,000; the x-axis shows values of $k$ from 1 to $n/2 = 5000$. Notice that the estimator for $\hat{\alpha} = \hat{\alpha}(k(n), n)$ is unbiased when $\lim_{n\to\infty}(k(n)/n) \to 0$. So, unbiasedness of the estimator requires very small values of $k$. However, for a small value of $k$, the variance of the estimator is large. A close look at the estimator $\hat{\alpha}(k, n)$ suggests a value of $\hat{\alpha}$ around 2.2, whereas $\alpha$=1.9.

[21]In Figure 2, the true $\alpha$ is again 1.9, the sample size is $n = 500,000$, $k = 1, \ldots, n/2 = 250,000$. One can see that, for very small values of $k$, $\alpha \approx 1.9$.

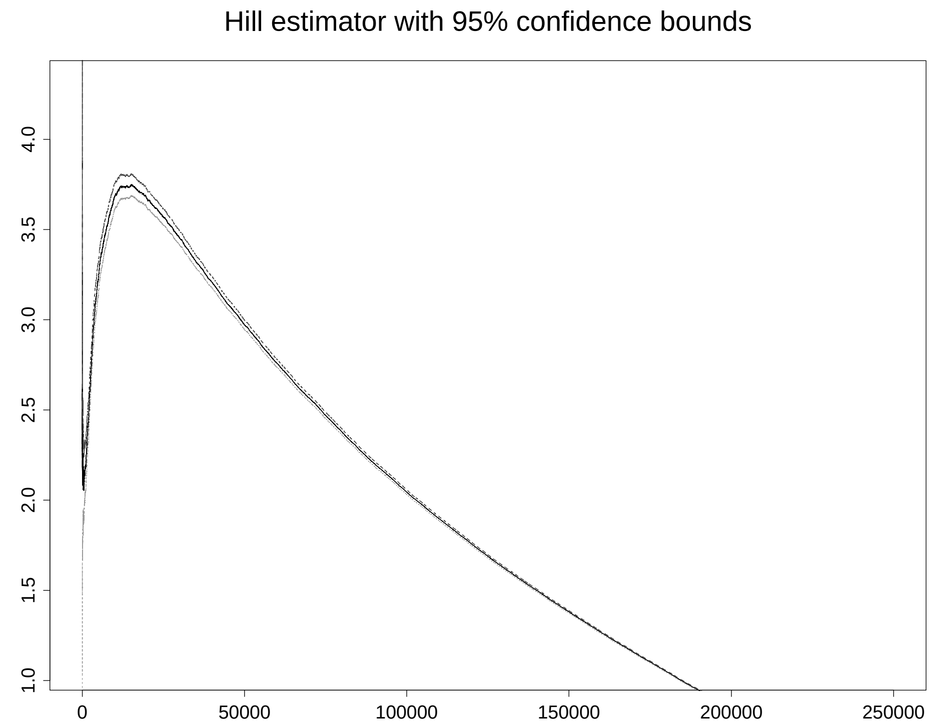


Figure 2: Hill estimator for 500,000 standard stable observations with index alpha=1.9

are the Pickands and the modified unconditional Pickands estimators. The "original" Pickands estimator[22] takes the form

$$\hat{\alpha}_{Pick} = \frac{\ln 2}{\ln(X_{n-k+1:n} - X_{n-2k+1:n}) - \ln(X_{n-2k+1:n} - X_{n-4k+1:n})}, 4k < n\,.$$

The Pickands estimator requires choice of the optimal $k$, which depends on the true unknown $\alpha$. Mittnik and Rachev (1996) proposed a new tail estimator named "the modified unconditional Pickands (MUP) estimator", $\hat{\alpha}_{MUP}$. An estimate of $\alpha$ is obtained by applying the nonlinear least squares method to the following system:

[22] Pickands (1975).

$$k_2 \approx X_2 X_1^{-1} k_1 + \varepsilon,$$

where

$$X_1 = \begin{bmatrix} X_{n-k+1:n}^{-\alpha} & X_{n-k+1:n}^{-2\alpha} \\ X_{n-2k+1:n}^{-\alpha} & X_{n-2k+1:n}^{-2\alpha} \end{bmatrix}, X_2 = \begin{bmatrix} X_{n-3k+1:n}^{-\alpha} & X_{n-3k+1:n}^{-2\alpha} \\ X_{n-4k+1:n}^{-\alpha} & X_{n-4k+1:n}^{-2\alpha} \end{bmatrix},$$

$$k_1 = \begin{bmatrix} k-1 \\ 2k-1 \end{bmatrix}, \text{and } k_2 = \begin{bmatrix} 3k-1 \\ 4k-1 \end{bmatrix}.$$

Mittnik, Paolella, and Rachev (1998c) found that the optimal $k$ for $\hat{\alpha}_{MUP}$ is far less dependent on $\alpha$ than in the case of either the Hill or Pickands estimators. Studies demonstrated that $\hat{\alpha}_{MUP}$ is approximately unbiased for $\alpha \in [1.00, 1.95)$ and nearly normally distributed for large sample sizes. The MUP estimator appears to be useful in empirical analysis.

### 3.2.2 Entire-Distribution Modeling

We shall describe the following methods of estimating stable parameters with fitting the entire distribution: quantile approaches, characteristic function (CF) techniques, and maximum likelihood (ML) methods.
Fama and Roll (1971) suggested the first quantile approach based on observed properties of stable quantiles. Their method was designed for evaluating parameters of symmetric stable distributions with index of stability $\alpha > 1$. The estimators exhibited a small asymptotic bias. McCulloch (1986) offered a modified quantile technique, which provided consistent and asymptotically normal estimators of all four stable parameters, for $\alpha \in [0.6, 2.0]$ and $\beta \in [-1, 1]$. The estimators are derived using functions of five sample quantiles: the 5%, 25%, 50%, 75%, and 95% quantiles. Since the estimators do not consider observations in the tails (below the 5% quantile and above the 95% quantile), the McCulloch method does not appear to be suitable for estimating parameters in VAR modeling.

Characteristic function techniques are built on fitting the sample CF to the theoretical CF. Press (1972a and 1972b) proposed several CF methods: the minimum distance, the minimum r-th mean distance, and the method of moments. Koutrouvelis (1980, 1981) developed the iterative regression procedure. Kogon and Williams (1998) modified the Koutrouvelis method by eliminating iterations and limiting the estimation to a common frequency interval[23]. CF estimators are consistent and under certain conditions are asymptotically normal[24].

Maximum likelihood methods for estimating stable parameters differ in a way of computing the stable density. DuMouchel (1971) evaluated the density by grouping data and applying the fast Fourier transform to "center" values and asymptotic expansions - in the tails. Mittnik, Rachev, and Paolella (1998) calculated the density at equally spaced grid points via an fast Fourier transform of the characteristic function and at intermediate points - by linear interpolation. Nolan (1998a) computed the density using numerical approximation of integrals in the Zolotarev integral formulas for the stable density[25]. DuMouchel (1973) proved that the ML estimator is consistent and asymptotically normal. In Section 4 we analyze applicability of the ML method in VAR estimations.

### 3.2.3 Tail Estimation: Fast Fourier Transform Method

Tail estimation using the Fourier Transform (FT) method is based on fitting the characteristic function in a neighborhood of the origin t=0. Here we use the classical tail estimate:

$$P(X \leq -\frac{1}{a}) \leq P(|X| \geq \frac{1}{a}) \leq \frac{K}{a}\int_0^a (1 - f_x(t))dt^{26}, \text{for all}\, a > 0\,,$$

where $f_x(t)$ is the characteristic function of a random variable $X$. Precise estimation of the characteristic function guarantees accurate tail estimation, which leads to an adequate evaluation of VAR.

Suppose that the distribution of returns $r$ is symmetric-$\alpha$-stable[27] , that is: the *characteristic function* of $r$ is given by $f_r(t) = Ee^{irt} = e^{it\mu - |ct|^\alpha}$. If $\alpha > 1$[28], then, given observations $r_1, \ldots, r_n$, we estimate $\mu$ by the sample mean $\bar{\mu} = \bar{r} = \frac{1}{n}\sum_{i=1}^n r_i$. For large values of $n$ , the characteristic function of observations $R_i = r_i - \bar{r}$ approaches $f_R(t) = e^{-|ct|^\alpha}$. Consider the *empirical characteristic function* of the centered observations: $\hat{f}_{R,n}(t) = \frac{1}{n}\sum_{i=1}^n e^{iR_k t}$.

[23] For additional references, see Arad (1980); Feuerverger and McDunnough (1981); Mittnik, Rachev, and Paolella (1998); Paulson, Holcomb, and Leitch (1975).

[24] Heathcote, Cheng, and Rachev (1995).

[25] For additional references, see Mittnik, Rachev, Dogagnoglu, and Chenyao (1997).

[26] The last inequality is by Lemma 3 on p.321 in Shiryayev, 1984.

[27] Empirical evidence suggests that $\beta$ does not play a significant role for VAR estimation.

[28] As we have already observed, in all financial return data, fitting an $\alpha$-stable model results in $\alpha > 1$, which implies existence of the first moment.

Because the theoretical characteristic function, $f_R(t)$, is real and positive, we have that $\hat{f}_{R,n}(t) = Re\left(\frac{1}{n}\sum_{i=1}^{n} e^{iR_k t}\right) = \frac{1}{n}\sum_{i=1}^{n}\cos(R_k t)$.

Now the problem of estimating $\alpha$ and $c$ is reduced to determining $\hat{\alpha}$ and $\hat{c}$ such that $\int_0^M |\frac{1}{n}\sum_{k=1}^{n}\cos(R_k t) - e^{-(\hat{c}t)^{\hat{\alpha}}}|dt$ is minimal, where $M$ is a sufficiently large value.

The realization of the FT method is performed in the following steps:

Step 1. Given the asset returns $r_1, \ldots, r_n$, compute the centered returns $R_i = r_i - \bar{r}, i = 1, \ldots, n$, where $\bar{r} = \frac{1}{n}\sum_{k=1}^{n} r_i$.

Step 2. Construct the sample characteristic function

$$\hat{f}(t_j) = \frac{1}{n}\sum_{k=1}^{n}\cos(R_k t_j),$$

where $t_j = j\frac{\kappa\pi}{\tau}, j = 1, \ldots, \tau, \kappa\pi$ is the maximal value of $t$, $\tau$ is the number of grid points on $(0, \kappa\pi]$[29].

Step 3. Do the search for best $\hat{\alpha}$ and $\hat{c}$ such that $\sum_{j=1}^{\tau}\left|\frac{1}{n}\sum_{k=1}^{n}\cos(R_k t_j) - e^{-(\hat{c}t_j)^{\hat{\alpha}}}\right|$ is minimal.

## 4 VAR estimates for Stable Distributed Financial Returns

In this section we consider a stable VAR model, which assumes that the portfolio return distribution follows a stable law. We derive "stable" VAR estimates and analyze their properties applying in-sample and forecast evaluations. We use "normal" VAR measurements as benchmarks for investigating characteristics of "stable" VAR measurements. We conduct analysis for various financial data sets:

- the Yen/British Pound (BP) exchange rate,
- the BP/US$ exchange rate,
- the Deutsche Mark (DM)/BP exchange rate,
- the S&P 500 index,

[29] For computation purposes, we have chosen $\kappa = 20$ and $\tau = 10000$. In the realization of the FT method we selected the following grid steps $ht$: if $0 \leq t \leq 1$, $ht = 20\pi/50000$; if $t > 1$, $ht = 20\pi/1000$. In order to emphasize the tail behavior, we refined the mesh near $t = 0$ and named that approach *FT-Tail* (FTT): if $0 \leq t \leq 0.1$, $ht = 20\pi/100000$; if $0.1 \leq t \leq 1.0$, $ht = 20\pi/10000$; if $t > 1$, $ht = 20\pi/1000$. The numerical results are reported in section 4.

- the DAX30 index,
- the CAC40 index,
- the Nikkei 225 index,
- the Dow Jones Commodities Price Index (DJCPI).

A short description of the data is given in Table 1.

Table 1: Financial data series

| Series | Source | Number of observations | Time period | Frequency |
|---|---|---|---|---|
| Yen/BP | Datastream | 6285 | 1.02.74-1.30.98 | Daily (D) |
| BP/US$ | D. Hindanov | 6157 | 1.03.74-1.30.98 | D |
| DM/BP | Datastream | 6285 | 1.02.74-1.30.98 | D |
| S&P 500 | Datastream | 7327 | 1.01.70-1.30.98 | D |
| DAX30 | Datastream | 8630 | 1.04.65-1.30.98 | D |
| CAC40 | Datastream | 2756 | 7.10.87-1.30.98 | D |
| Nikkei 225 | Datastream | 4718 | 1.02.80-1.30.98 | D |
| DJCPI | Datastream | 5761 | 1.02.76-1.30.98 | D |

### 4.1 In-sample evaluation of VAR estimates

In this part we evaluate stable and normal VAR models by examining distances between the VAR estimates and the empirical VAR measures.

By a formal definition of VAR in equation (1), VAR estimates, $\overline{VAR}_{t,\tau}$, are such that

$$Pr\left[\Delta P_t(\tau) < -\overline{VAR}_{t,\tau}\right] \approx 1 - c, \tag{2}$$

where $c$ is the confidence level, $\Delta P_t(\tau)$ is the relative change in the portfolio value over the time horizon $\tau$, i.e., $\Delta P_t(\tau) = R_{t,\tau}$, is the portfolio return at moment $t$ over the time horizon $\tau$, and $t$ is the current time.

For the purpose of testing VAR models financial regulators advise to choose a time horizon of one day, so we take $\tau = 1$. In the text below, if the time horizon is not stated explicitly, it is assumed to equal one day. At each time $t$, an estimate $\overline{\mathrm{VAR}}_t$ is obtained using $lw$ recent observations of portfolio returns $R_{t-1}, R_{t-2}, \ldots, R_{t-lw}$:

$$\overline{\mathrm{VAR}}_t = \mathrm{VAR}(R_{t-1}, R_{t-2}, \ldots, R_{t-lw}). \tag{3}$$

The $lw$ parameter is called the *window length*. In this subsection, VAR is estimated employing the entire sample of observations, i.e., $lw = N$, where $N$ is the sample size. Hence, we do not point out the present time $t$.

We obtain "stable" ("normal") VAR measurements at the confidence level $c$ in two steps:

(i) fitting empirical data by a stable (normal) distribution,

(ii) calculating a VAR as the negative of the (1-c)th quantile of a fitted stable (normal) distribution.

"Stable" fitting is implemented using three methods: maximum likelihood (ML), Fourier Transform (FT), and Fourier Transform-Tail (FTT)[30]. Estimated parameters of densities and corresponding confidence intervals are presented in Table 2. In the FT and FTT fitting we assume that distributions of returns are symmetric, i.e., the skewness parameter $\beta$ is equal to zero. Since the index of stability $\alpha > 1$ for our data series, the location parameter $\mu$ is approximated by the sample mean. The ML estimates were computed applying the STABLE program by J.P. Nolan[31]. The confidence intervals (CI) for the FT and FTT parameter estimates were derived using a bootstrap method with 1000 replications[32]. Empirical analysis showed that a set of 1000 replications is: (i) satisfactory for constructing 95% CI; (ii) insufficient for obtaining reliable 99% CI. In our experiments, sets of 1000 replications generated: (i) 95% CI for $\alpha$ and $\sigma$ whose bounds coincided up to two decimal points; 95% CI for $\mu$ with slightly varying bounds; (ii) varying 99% CI, with insignificant variation of left limits.

[30]Evaluation of parameters of stable distributions is provided in Section 3.2

[31]The STABLE program is described in Nolan (1997).

[32]For references on bootstrapping, see Heathcote, Cheng, and Rachev (1995); for discussion on CI based on ML parameter estimates, see Nolan (1998a).

Table 2: Parameters of stable and normal densities*

| Series | Normal | | Stable | | | | |
|---|---|---|---|---|---|---|---|
| | Mean | Standard deviation | Method | $\alpha$ | $\beta$ | $\mu$ | $\sigma$ |
| Yen/BP | -0.012 | 0.649 | ML | 1.647 | -0.170 | -0.023 | 0.361 |
| | | | FT | 1.61 | | -0.018 | 0.34 |
| | | | | [1.57, 1.66] | | [-0.095, 0.015] | [0.33, 0.36] |
| | | | | [1.55, 1.68] | | [-0.178, 0.025] | [0.33, 0.37] |
| | | | FTT | 1.50 | | -0.018 | 0.32 |
| | | | | [1.46, 1.55] | | [-0.131, 0.034] | [0.31, 0.34] |
| | | | | [1.44, 1.64] | | [-0.261, 0.070] | [0.31, 0.39] |
| BP/US | 0.006 | 0.658 | ML | 1.582 | 0.038 | 0.007 | 0.349 |
| | | | FT | 1.57 | | 0.006 | 0.33 |
| | | | | [1.53, 1.65] | | [-0.096, 0.045] | [0.32, 0.36] |
| | | | | [1.51,1.75] | | [-0.393, 0.065] | [0.32, 0.47] |
| | | | FTT | 1.45 | | 0.006 | 0.31 |
| | | | | [1.41, 1.51] | | [-0.134, 0.070] | [0.30, 0.33] |
| | | | | [1.40, 1.62] | | [-0.388, 0.097] | [0.30, 0.47] |
| DM/BP | -0.012 | 0.489 | ML | 1.590 | -0.195 | 0.018 | 0.256 |
| | | | FT | 1.60 | | -0.012 | 0.24 |
| | | | | [1.54, 1.75] | | [-0.064, 0.013] | [0.23, 0.26] |
| | | | | [1.53, 1.75] | | [-0.165, 0.022] | [0.23, 0.27] |
| | | | FTT | 1.45 | | -0.012 | 0.23 |
| | | | | [1.41, 1.55] | | [-0.114, 0.038] | [0.22, 0.26] |
| | | | | [1.40, 1.77] | | [-0.402, 0.061] | [0.22, 0.40] |
| S&P 500 | 0.032 | 0.930 | ML | 1.708 | 0.004 | 0.036 | 0.512 |
| | | | FT | 1.82 | | 0.032 | 0.54 |
| | | | | [1.78, 1.84] | | [-0.013, 0.057] | [0.53, 0.54] |
| | | | | [1.77, 1.84] | | [-0.062, 0.067] | [0.53, 0.55] |
| | | | FTT | 1.60 | | 0.032 | 0.48 |
| | | | | [1.56, 1.65] | | [-0.066, 0.078] | [0.47, 0.49] |
| | | | | [1.54, 1.66] | | [-0.120, 0.095] | [0.46, 0.50] |

| Series | Normal | | Stable | | | | |
|---|---|---|---|---|---|---|---|
| | Mean | Standard deviation | Me-thod | $\alpha$ | $\beta$ | $\mu$ | $\sigma$ |
| DAX30 | 0.026 | 1.002 | ML | 1.823 | -0.084 | 0.027 | 0.592 |
| | | | FT | 1.84 | | 0.026 | 0.60 |
| | | | | [1.81, 1.88] | | [-0.015, 0.050] | [0.59, 0.60] |
| | | | | [1.80, 1.89] | | [-0.050, 0.057] | [0.58, 0.62] |
| | | | FTT | 1.73 | | 0.026 | 0.57 |
| | | | | [1.69, 1.77] | | [-0.031, 0.061] | [0.56, 0.58] |
| | | | | [1.68, 1.79] | | [-0.124, 0.073] | [0.56, 0.59] |
| CAC40 | 0.028 | 1.198 | ML | 1.784 | -0.153 | 0.027 | 0.698 |
| | | | FT | 1.79 | | 0.028 | 0.70 |
| | | | | [1.73, 1.85] | | [-0.050, 0.088] | [0.68, 0.73] |
| | | | | [1.71, 1.87] | | [-0.174, 0.103] | [0.67, 0.74] |
| | | | FTT | 1.76 | | 0.028 | 0.69 |
| | | | | [1.71, 1.84] | | [-0.053, 0.091] | [0.67, 0.72] |
| | | | | [1.69, 1.87] | | [-0.394, 0.101] | [0.66, 0.77] |
| Nikkei 225 | 0.020 | 1.185 | ML | 1.444 | -0.093 | -0.002 | 0.524 |
| | | | FT | 1.58 | | 0.02 | 0.59 |
| | | | | [1.53, 1.64] | | [-0.127, 0.102] | [0.57, 0.62] |
| | | | | [1.52, 1.67] | | [-0.421, 0.130] | [0.57, 0.69] |
| | | | FTT | 1.30 | | 0.02 | 0.49 |
| | | | | [1.26, 1.47] | | [-0.451, 0.316] | [0.47, 0.69] |
| | | | | [1.05, 1.67] | | [-1.448, 0.860] | [0.47, 1.10] |
| DJCPI | 0.006 | 0.778 | ML | 1.569 | -0.060 | 0.003 | 0.355 |
| | | | FT | 1.58 | | 0.006 | 0.35 |
| | | | | [1.53, 1.66] | | [-0.026, 0.100] | [0.34, 0.37] |
| | | | | [1.52, 1.67] | | [-0.140, 0.120] | [0.33, 0.39] |
| | | | FTT | 1.49 | | 0.006 | 0.33 |
| | | | | [1.44, 1.55] | | [-0.160, 0.062] | [0.32, 0.36] |
| | | | | [1.44, 1.69] | | [-0.396, 0.100] | [0.32, 0.46] |

*The CIs right below the estimates are the 95% CIs, the next CIs are the 99% CIs.

VAR measurements were calculated at confidence levels $c = 99\%$ and $c = 95\%$. The 99% (95%) VAR was determined as the negative of the 1% (5%) quantile. For calculating stable quantiles we used our program, built on the Zolotarev integral representation form of the cumulative distribution function. The 99% and 95% VAR estimates are reported in Tables 3 and 4, respectively. Biases of stable and normal VAR measurements are provided in Table 5[33].

[33] Biases are computed by subtracting the empirical VAR from the model VAR estimates.

Table 3: Empirical, Normal, and Stable 99% VAR estimates*

| Series | 99% VAR | | | | |
|---|---|---|---|---|---|
| | Empirical | Normal | Stable | | |
| | | | ML | FT | FTT |
| Yen/BP | 1.979 | 1.528 | 2.247 | 2.212 | 2.494 |
| | | | | [1.968, 2.252] | [2.276, 2.736] |
| | | | | [1.919, 2.415] | [2.230, 2.836] |
| BP/US$ | 1.774 | 1.526 | 2.221 | 2.200 | 2.668 |
| | | | | [2.014, 2.412] | [2.436, 2.925] |
| | | | | [1.956, 2.593] | [2.358, 3.029] |
| DM/BP | 1.489 | 1.149 | 1.819 | 1.520 | 1.996 |
| | | | | [1.190, 1.712] | [1.792, 2.211] |
| | | | | [1.179, 1.742] | [1.700, 2.329] |
| S&P 500 | 2.293 | 2.131 | 2.559 | 2.200 | 2.984 |
| | | | | [2.117, 2.258] | [2.757, 3.243] |
| | | | | [2.106, 2.470] | [2.700, 3.336] |
| DAX 30 | 2.564 | 2.306 | 2.464 | 2.375 | 2.746 |
| | | | | [2.260, 2.502] | [2.557, 2.949] |
| | | | | [2.240, 2.569] | [2.523, 2.997] |
| CAC 40 | 3.068 | 2.760 | 3.195 | 3.019 | 3.144 |
| | | | | [2.753, 3.364] | [2.788, 3.504] |
| | | | | [2.682, 3.520] | [2.700, 3.841] |
| Nikkei 225 | 3.428 | 2.737 | 4.836 | 3.842 | 6.013 |
| | | | | [3.477, 4.254] | [5.190, 6.701] |
| | | | | [3.367, 4.453] | [4.658, 19.950] |
| DJCPI | 2.053 | 1.804 | 2.446 | 2.285 | 2.603 |
| | | | | [1.955, 2.423] | [2.382, 2.870] |
| | | | | [1.916, 2.474] | [2.288, 3.035] |

*The CIs right below the estimates are the 95% CIs, the next CIs are the 99% CIs.

Table 4: Empirical, Normal, and Stable 95% VAR estimates*

| Series | 95% VAR | | | | |
|---|---|---|---|---|---|
| | Empirical | Normal | Stable | | |
| | | | ML | FT | FTT |
| Yen/BP | 1.103 | 1.086 | 1.033 | 0.968<br>[0.926, 1.047]<br>[0.911, 1.186] | 0.995<br>[0.937, 1.132]<br>[0.911, 1.329] |
| BP/US$ | 1.038 | 1.077 | 0.981 | 0.944<br>[0.898, 1.072]<br>[0.876, 1.599] | 0.986<br>[0.917, 1.158]<br>[0.895, 1.588] |
| DM/BP | 0.806 | 0.816 | 0.772 | 0.687<br>[0.652, 0.749]<br>[0.641, 0.894] | 0.748<br>[0.695, 0.894]<br>[0.678, 1.418] |
| S&P 500 | 1.384 | 1.497 | 1.309 | 1.308<br>[1.275, 1.361]<br>[1.265, 1.411] | 1.319<br>[1.265, 1.423]<br>[1.246, 1.503] |
| DAX 30 | 1.508 | 1.623 | 1.449 | 1.451<br>[1.415, 1.500]<br>[1.402, 1.533] | 1.452<br>[1.405, 1.521]<br>[1.395, 1.650] |
| CAC 40 | 1.819 | 1.943 | 1.756 | 1.734<br>[1.653, 1.837]<br>[1.621, 1.944] | 1.734<br>[1.647, 1.845]<br>[1.616, 2.288] |
| Nikkei 225 | 1.856 | 1.929 | 1.731 | 1.666<br>[1.570, 1.839]<br>[1.558, 2.280] | 1.840<br>[1.582, 2.512]<br>[1.500, 5.022] |
| DJCPI | 1.066 | 1.274 | 1.031 | 0.994<br>[0.888, 1.047]<br>[0.870, 1.200] | 1.011<br>[0.944, 1.188]<br>[0.915, 1.615] |

*The CIs right below the estimates are the 95% CIs, the next CIs are the 99% CIs.

We accompany our computations with plots of:

- daily price levels,
- daily returns,
- fitted empirical, normal, and stable densities with the ML, FT, and FTT estimated parameters,

- daily empirical, normal, and stable VAR* estimates at the 99% and 95% confidence levels[34].

Combined plots of price levels, returns, densities, and VAR estimation are displayed in Figures 3-10. In order to illustrate that confidence intervals for the FT parameter estimates are sufficiently narrow, we show stable densities and VAR measures at boundary values of confidence intervals for $\hat{\alpha}_{Yen,FT}$ and $\hat{\sigma}_{Yen,FT}$ in Figures 11-14.

As Figures 3-10 demonstrate, the VAR estimates obtained at confidence level $c$=95% seem to belong to the area between the "tail" and the "center". The VAR at level $c$=99% is really in the tail area. Hence, we compare performance of stable and normal models separately for the cases $c = 95\%$ and $c = 99\%$.

In general, the stable modeling (ML, FT, and FTT) provided evaluations of the 99% VAR greater than the empirical 99% VAR (see Figures 3-10, Tables 3 and 5). It underestimated the sample 99% VAR in the applications of two methods: FT - for the CAC40, S&P 500, and DAX30 indices, and ML - for the DAX30 index. Biased downwards stable VAR estimates were closer to the true VAR than the normal estimates (see Table 5). Among the methods of stable approximation, the FT method provided more accurate VAR estimates for 7 data sets (see Table 5). For all analyzed data sets, the normal modeling underestimated the empirical 99% VAR. Stable modeling provided more accurate 99% VAR estimates: mean absolute bias [35] under the stable (FT) method is 42% smaller than under the normal method.

At 95% confidence level, the stable VAR estimates were lower than the empirical VAR for all data sets. The normal VAR measurements exceeded the true VAR, except the Yen/BP exchange rate series (see Table 6). For the exchange rate series (Yen/BP, BP/US$, and DM/BP), the normal method resulted in more exact VAR estimates. For the S&P 500, DAX30, CAC40, and DJCPI indices, stable methods underestimated VAR, though the estimates were closer to the true VAR than the normal estimates. Mean absolute biases under stable and normal modeling are of comparable magnitudes.

---

[34] The VAR* numbers are the negative values of the VAR estimates, VAR* = -VAR.

[35] Let $b_{m,s}$ be a bias of a VAR estimate: $b_{m,s} = \text{VAR}_{m,s} - \text{VAR}_{Empirical,s}$. The mean absolute bias equals $MAB_m = (\sum_{s=1}^{8} |b_{m,s}|)/8$, where $m$ denotes normal, stable-ML, stable-FT, and stable-FTT methods, and $s - a$ series.

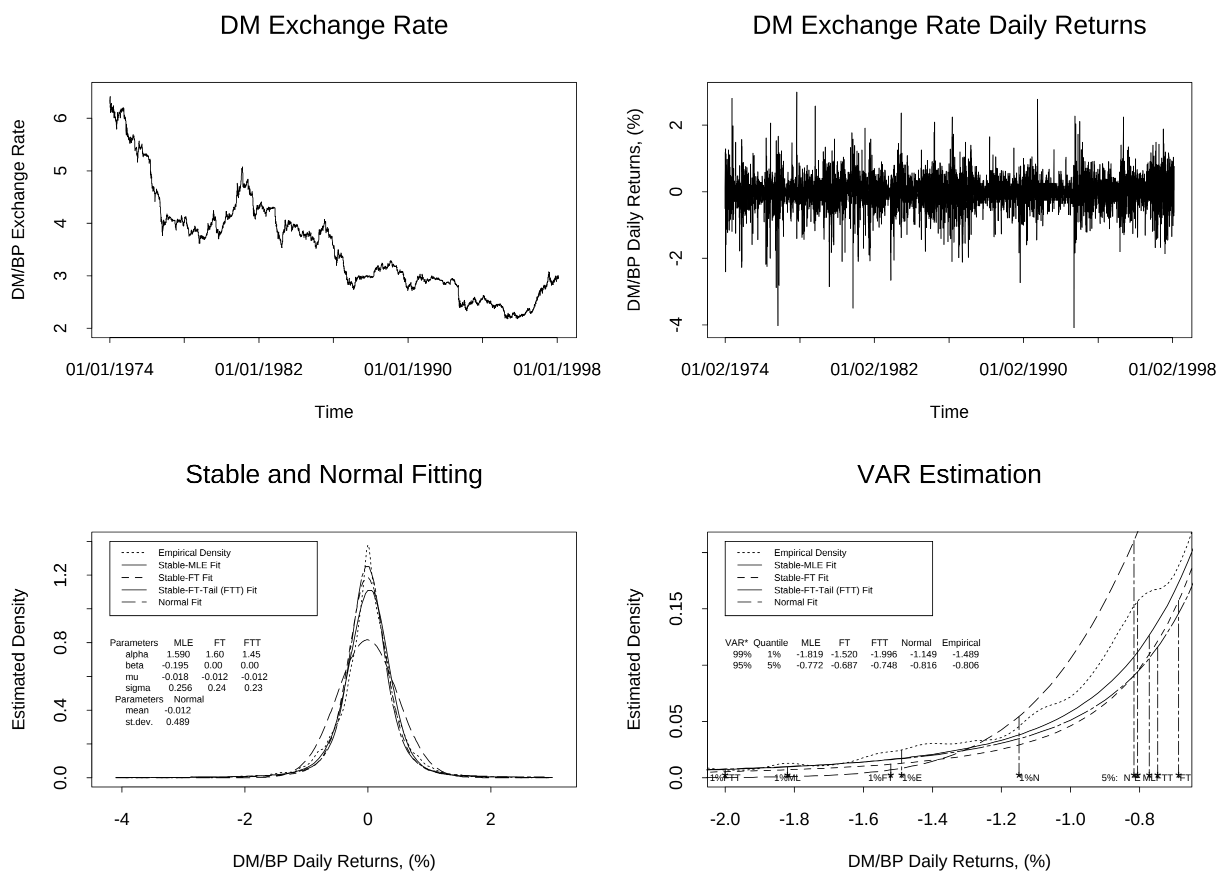


Figure 3. VAR estimation for the DM/BP exchange rate.

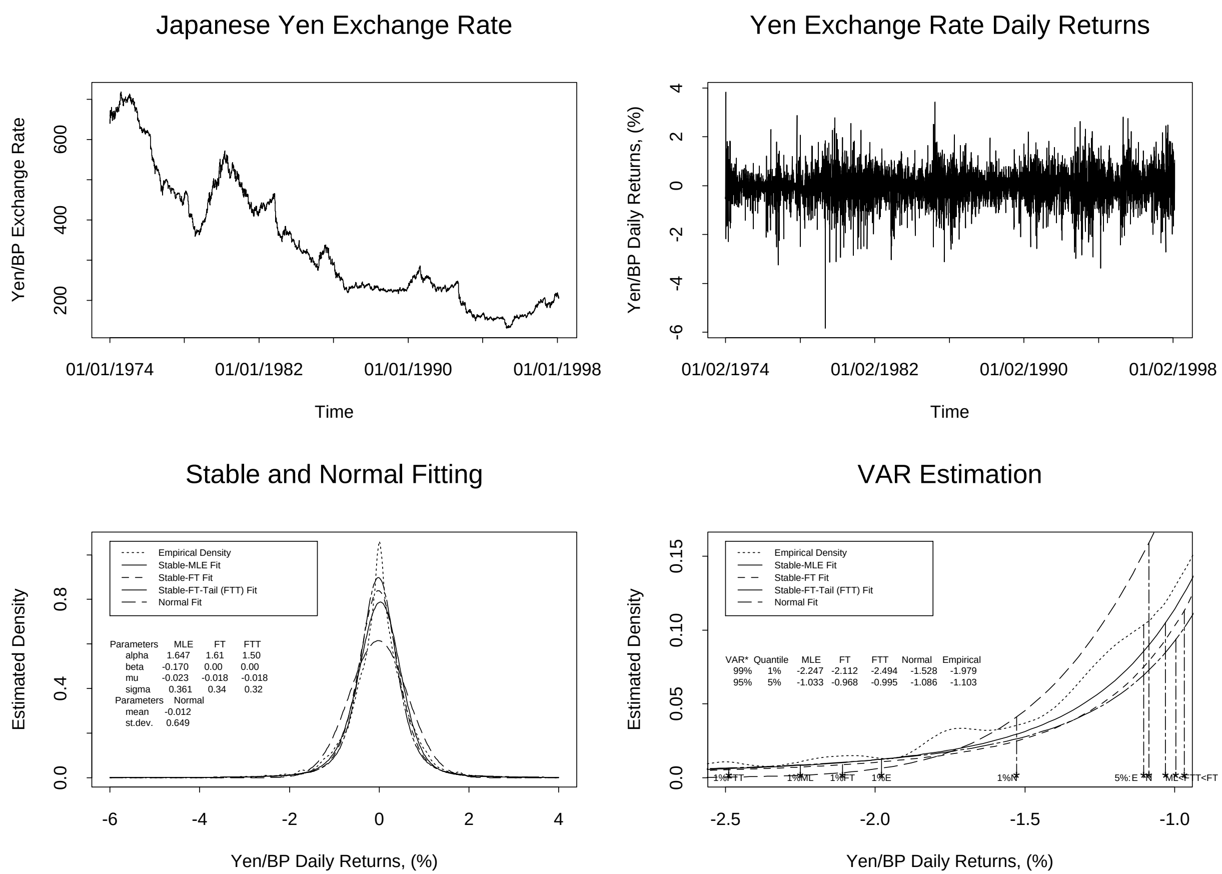


Figure 4. VAR estimation for the Yen/BP exchange rate.

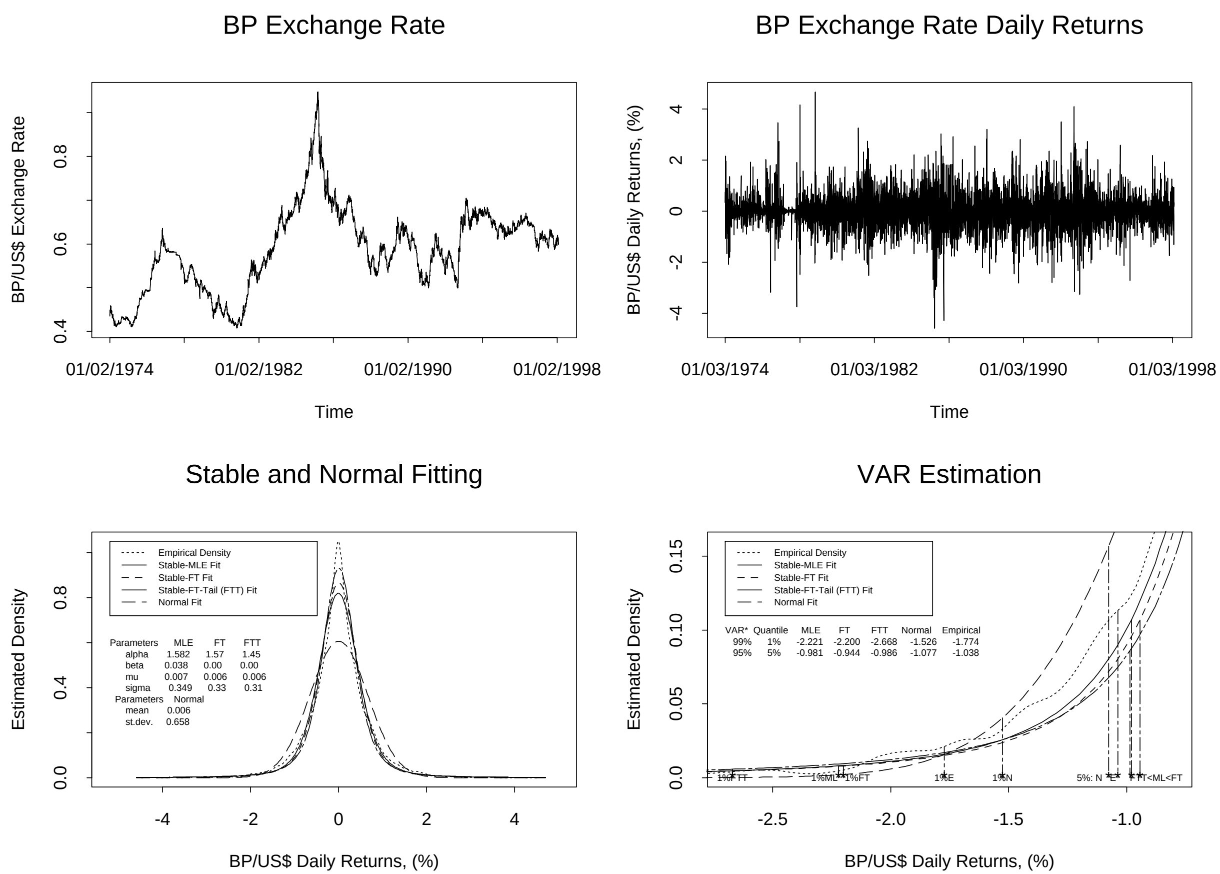


Figure 5. VAR estimation for the BP/US$ exchange rate.

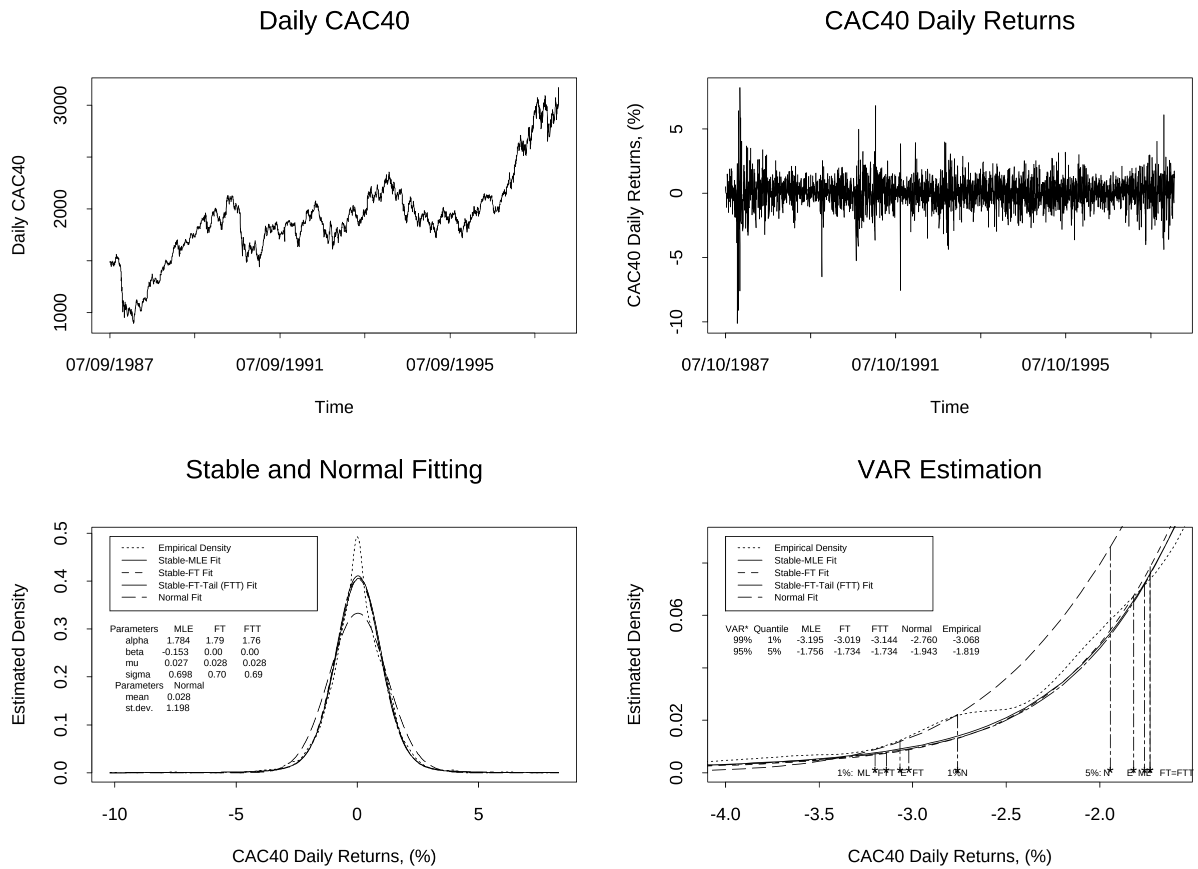


Figure 6. VAR estimation for the CAC40 index.

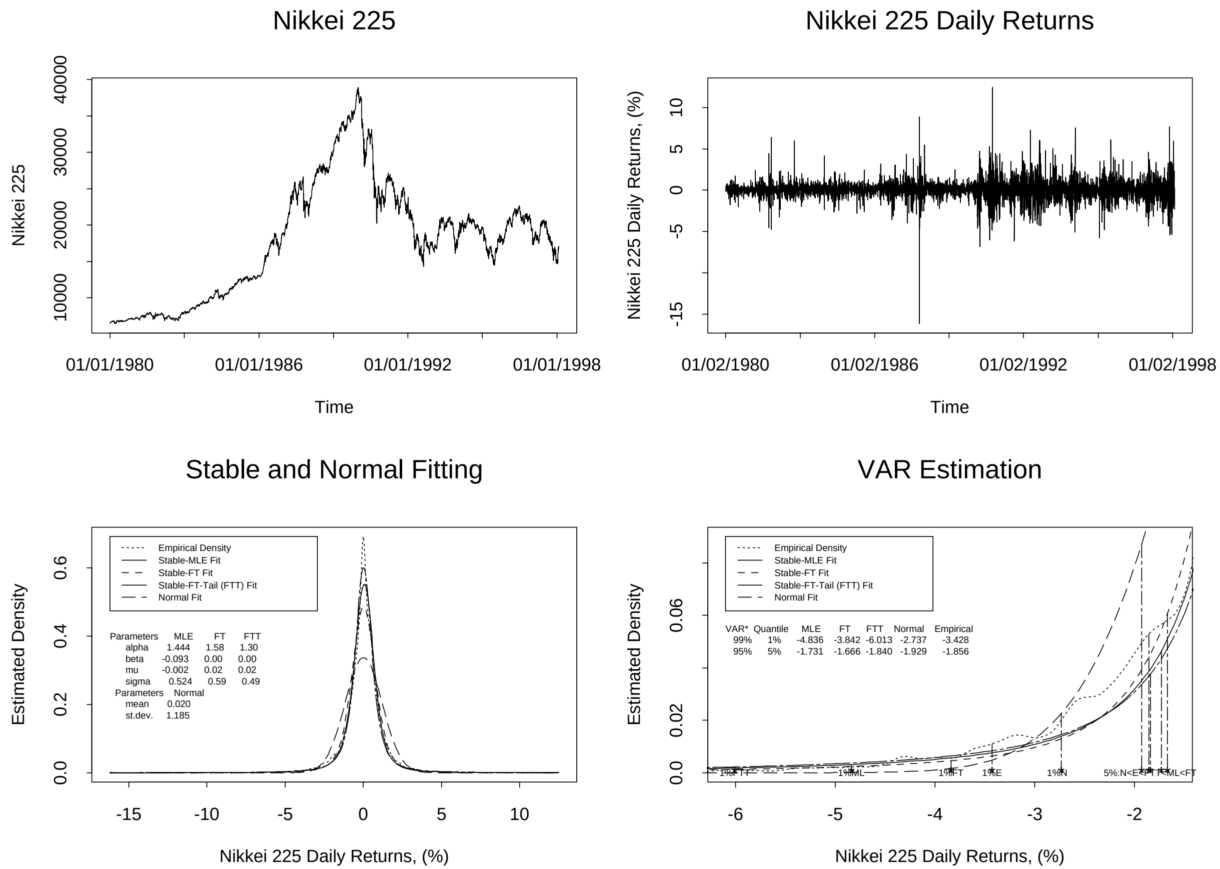


Figure 7. VAR estimation for the Nikkei 225 index.

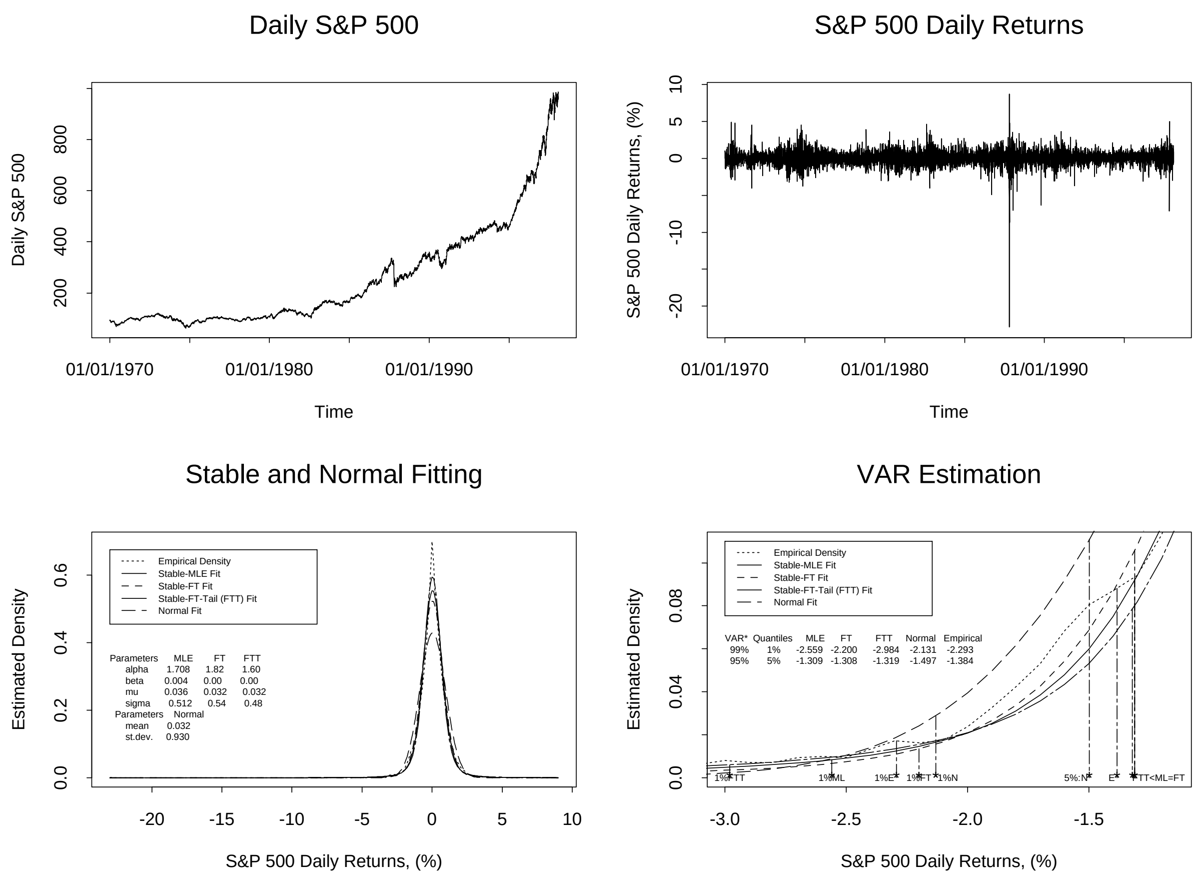


Figure 8. VAR estimation for the S&P 500 index.

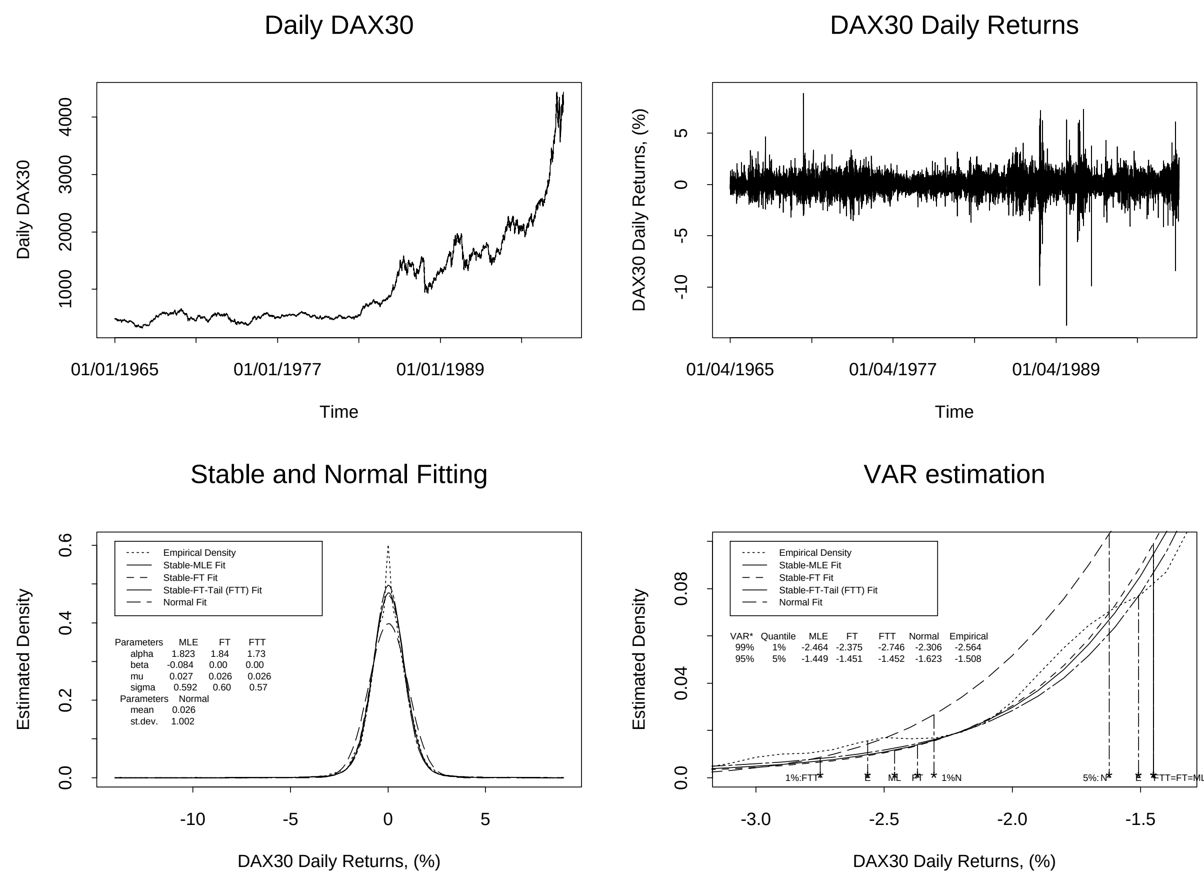


Figure 9. VAR estimation for the DAX30 index.

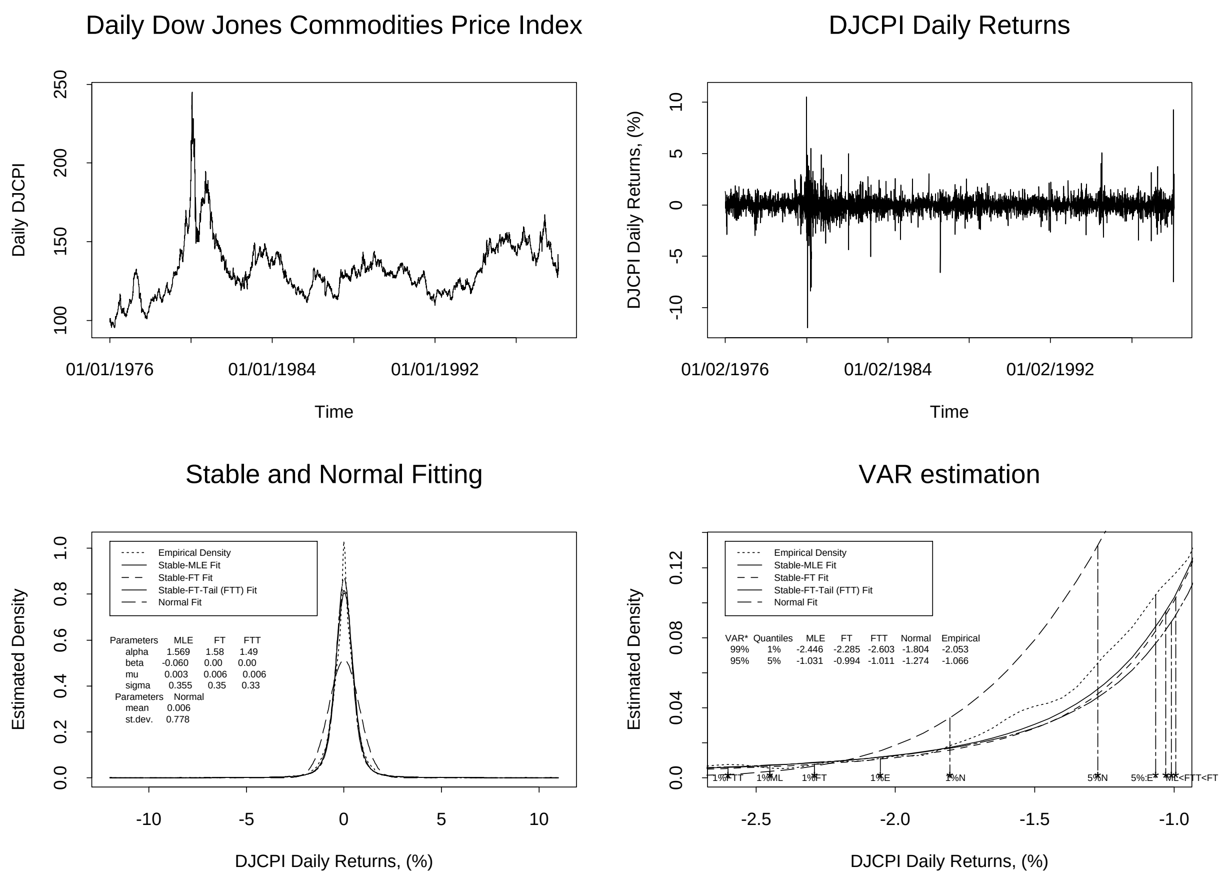


Figure 10. VAR estimation for the DJCPI index

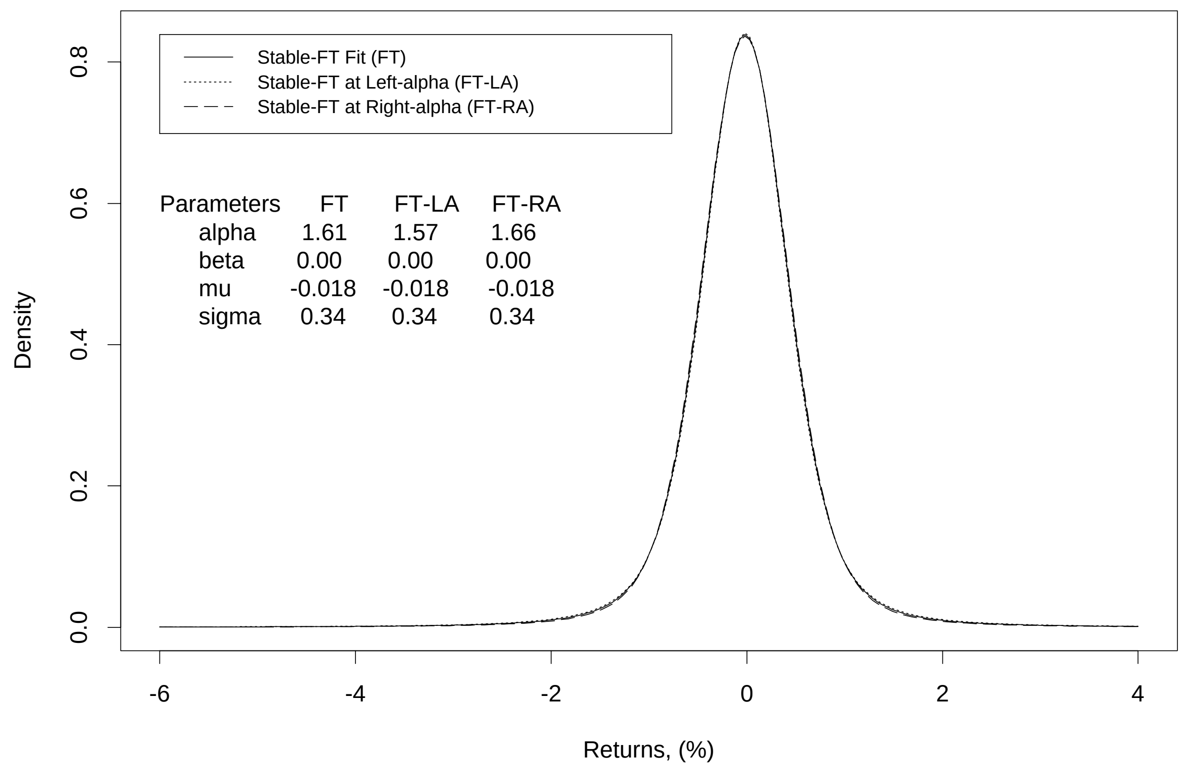


Figure 11. Stable fitting at limiting values of a confidence interval for alpha

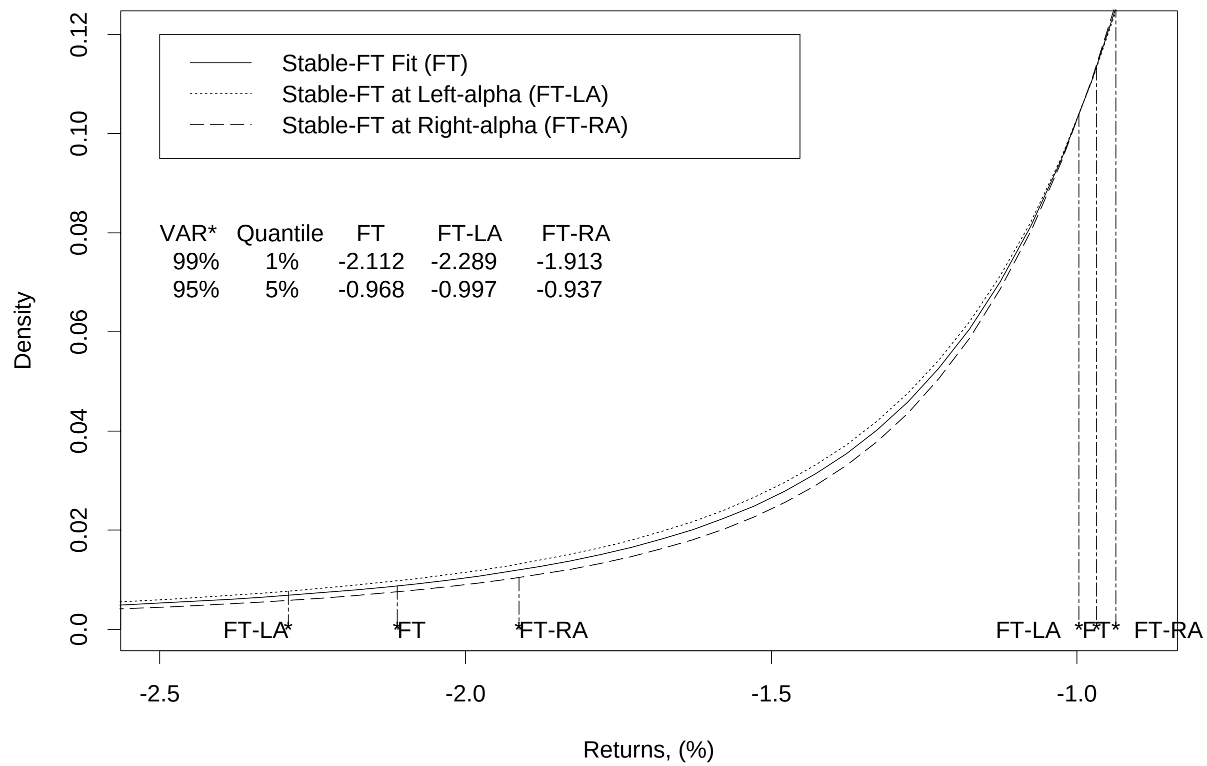


Figure 12. VAR estimation at limiting values of a confidence interval for alpha

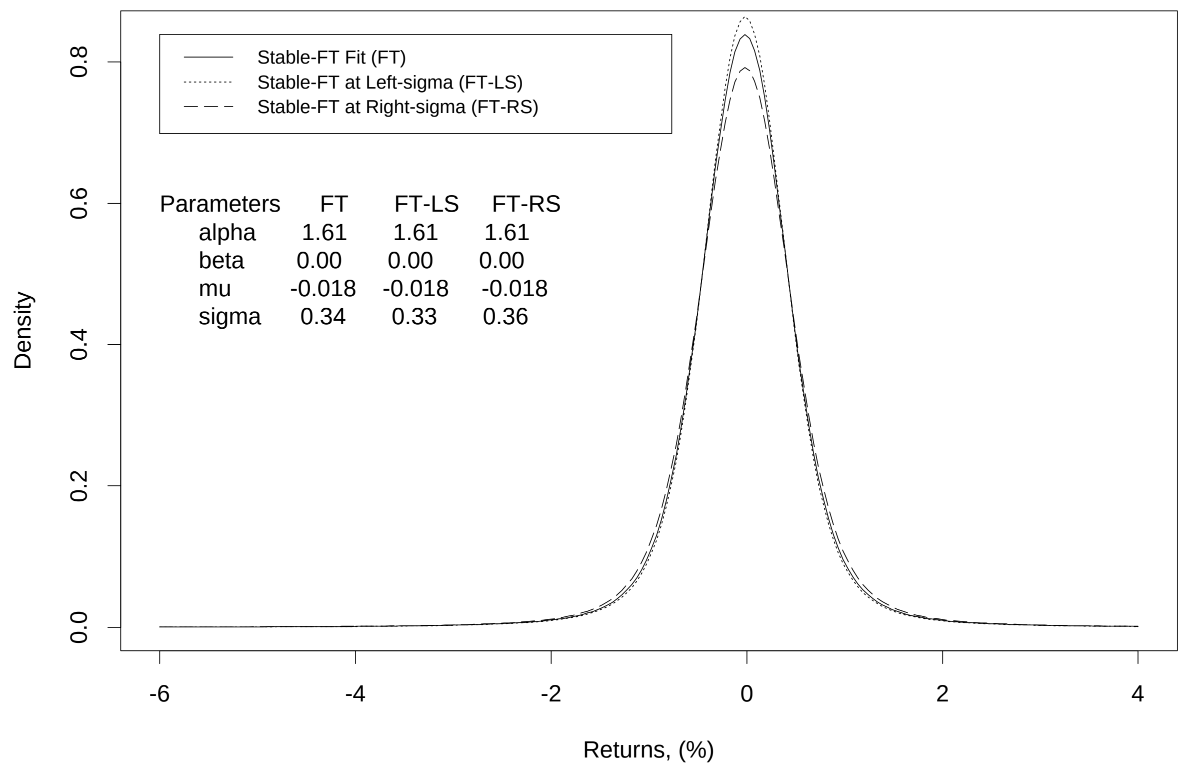


Figure 13. Stable fitting at limiting values of a confidence interval for sigma

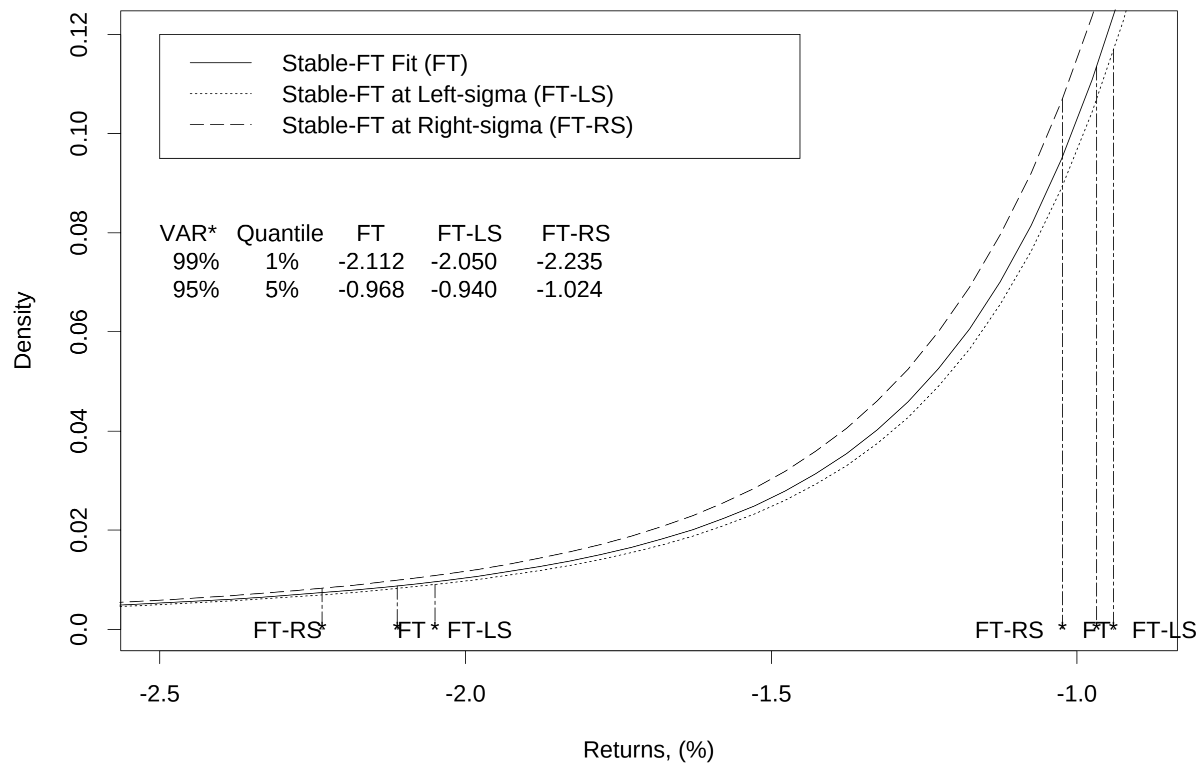


Figure 14. VAR estimation at limiting values of a confidence interval for sigma

Table 5: Biases of Normal and Stable 99% VAR estimates

| Series | 99% VAR$_m$-99% VAR$^*_{Emprirical}$ | | | |
|---|---|---|---|---|
| | | Stable | | |
| | Normal | ML | FT | FTT |
| Yen/BP | -0.451 | 0.268 | 0.133 | 0.515 |
| BP/US$ | -0.248 | 0.447 | 0.426 | 0.894 |
| DM/BP | -0.340 | 0.330 | 0.031 | 0.507 |
| S&P 500 | -0.162 | 0.266 | -0.093 | 0.691 |
| DAX30 | -0.258 | -0.100 | -0.189 | 0.182 |
| CAC40 | -0.308 | 0.127 | -0.049 | 0.076 |
| Nikkei 225 | -0.691 | 1.408 | 0.414 | 2.585 |
| DJCPI | -0.249 | 0.393 | 0.232 | 0.550 |
| Mean absolute bias | 0.338 | 0.416 | 0.196 | 0.750 |

$^*m$ denotes normal, stable-ML, stable-FT, and stable-FTT methods.

Table 6: Biases of Normal and Stable 95% VAR estimates

| Series | 95% VAR$_m$-95% VAR$^*_{Emprirical}$ | | | |
|---|---|---|---|---|
| | | Stable | | |
| | Normal | ML | FT | FTT |
| Yen/BP | -0.017 | -0.070 | -0.135 | -0.108 |
| BP/US | 0.039 | -0.057 | -0.094 | -0.052 |
| DM/BP | 0.010 | -0.034 | -0.119 | -0.058 |
| S&P 500 | 0.113 | -0.075 | -0.076 | -0.065 |
| DAX30 | 0.115 | -0.059 | -0.057 | -0.056 |
| CAC40 | 0.124 | -0.063 | -0.085 | -0.085 |
| Nikkei 225 | 0.073 | -0.125 | -0.190 | -0.016 |
| DJCPI | 0.208 | -0.035 | -0.072 | -0.055 |
| Mean absolute bias | 0.087 | 0.065 | 0.104 | 0.070 |

$^*m$ denotes normal, stable-ML, stable-FT, and stable-FTT methods.

In-sample examination of VAR models showed:

- the stable modeling generally results in conservative and accurate 99% VAR estimates, which is preferred by financial institutions and regulators[36],
- the normal approach leads to overly optimistic forecasts of losses in the 99% VAR estimation,
- from a conservative point of view, the normal modeling is acceptable for the 95% VAR estimation,
- the stable models underestimate the 95% VAR. In fact, the stable 95% VAR measurements are closer to the empirical VAR than the normal 95% VAR measurements.

The next step in evaluating VAR models is analysis of their forecasting characteristics.

### 4.2 Forecast-evaluation of VAR estimates

In this section we investigate the forecasting properties of stable and normal VAR modeling by comparing predicted VAR with observed returns.

We test the null hypothesis that equation (1) for a time horizon of 1 day ($\tau$=1) holds at any time $t$:

$$Pr[\Delta P_t < -\mathrm{VAR}_t] = 1 - c, \tag{4}$$

where $\Delta P_t$ is the relative change (return) in the portfolio value, i.e. $\Delta P_t = R_t$ is the portfolio return at moment $t$, $\mathrm{VAR}_t$ is the VAR measure at time $t$, $c$ is the VAR confidence level, $t$ is the current time, $t \in [1, T]$, and $T$ is the length of the testing interval. The test is performed by checking whether $Pr[R_t < -\overline{\mathrm{VAR}}_t]$ is reasonably close to 1 - $c$, where $\overline{\mathrm{VAR}}_t$ is the estimate of $\mathrm{VAR}_t$. Recall that $\overline{\mathrm{VAR}}_t$ is computed using the last $lw$ observations[37].

Let $b_t$ be the indicator function $\mathbf{1}\{R_t < -\mathrm{VAR}_t\}, 1 \leq t \leq T$. If the equation (4) holds, then

$$b_t = \mathbf{1}\{R_t < -\overline{\mathrm{VAR}}_t\} = \begin{cases} 1, & \text{probability} = 1 - c \\ 0, & \text{probaility} = c. \end{cases}$$

[36] In the 99% VAR estimation for data series from Table 1, mean absolute bias under the stable modeling was 42% smaller than under the normal modeling.

[37] See equation 3

Let us denote by $E$ the number of exceedings $(R_t < -\overline{\text{VAR}}_t)$[38] over the testing interval $[1, T]$. If equation (4) is valid, then the variable $E = \sum_{t=1}^{T} b_t$ has a binomial distribution. We can formulate a testing rule: reject the null hypothesis at level of significance $x$ if

$$\sum_{t=0}^{E} \binom{T}{t} (1-c)^t c^{T-t} \leq \frac{x}{2} \text{ or } \sum_{t=0}^{E} \binom{T}{t} (1-c)^t c^{T-t} \geq 1 - \frac{x}{2}.$$

For large $T$ and sufficiently high VAR confidence levels, the binomial distribution can be approximated by the normal distribution. Hence, the testing rule for large $T$ is: reject the null hypothesis at level of significance $x$ if

$$E < T(1-c) - z_{1-x/2}\sqrt{T(1-c)c} \text{ or } E > T(1-c) + z_{1-x/2}\sqrt{T(1-c)c},$$

where $z_p$ is the $p$% standard normal quantile. The bounds of admissible VAR exceedings $E$ and exceedings frequencies, $\frac{E}{T}$ , for testing at level of sigificance 5% and 1% are provided in Table 7.

Table 7: Admissible VAR exceedings and exceeding frequencies

| VAR confidence level, c | Length of a testing interval, $T$ | Admissible VAR exceedings, $E$ | | Admissible VAR frequencies, $E/T$ | |
|---|---|---|---|---|---|
| | | Significance level, x | | Significance level, $x$ | |
| | | 5% | 1% | 5% | 1% |
| 95% | 500 | [17,33] | [14,36] | [3.40%, 6.60%] | [2.80%, 7.20%] |
| | 1500 | [61,89] | [56,94] | [4.07%, 5.93%] | [3.73%, 6.27%] |
| 99% | 500 | [2,8] | [0,10] | [0.40%, 1.60%] | [0.00%, 2.00%] |
| | 1500 | [9,21] | [6,23] | [0.60%, 1.40%] | [0.40%, 1.53%] |

We examined forecasting properties of stable and VAR models for data series described in Table 1. In testing procedures we considered the following parameters:

[38]In nominal levels, an exceeding implies a case when actual losses exceeded the predicted losses.

- window lengths $lw = 260$ observations (data over 1year) and $lw = 1560$ observations (data over 6 years),
- lengths of testing intervals T= 500 days and T=1500 days.

Evaluation results are reported in Tables 8 and 9. We indicate by the bold font the numbers, which are outside of acceptable ranges.

Table 8: 99% VAR exceedings

| Series | Length of a testing interval, $T$ | 99% VAR exceedings | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | Window length = 260 obs. | | | | Window length = 1560 obs. | | | |
| | | Normal | | FT | | Normal | | FT | |
| | | $E$ | $E/T$ | $E$ | $E/T$ | $E$ | $E/T$ | $E$ | $E/T$ |
| Yen/BP | 500 | **15** | **3.00%** | **13** | **2.60%** | 10 | 2.00% | 2 | 0.40% |
| | 1500 | **40** | **1.67** | **34** | **2.27** | **45** | **3.00** | 21 | 1.40 |
| BP/US$ | 500 | 10 | 2.00 | 5 | 1.00 | 1 | 0.20 | 0 | 0.00 |
| | 1.500 | **26** | **1.73** | 13 | 0.86 | 17 | 1.33 | **5** | **0.33** |
| DM/BP | 500 | **18** | **3.60** | **14** | **2.80** | **17** | **3.40** | 8 | 1.60 |
| | 1500 | **45** | **3.00** | **33** | **2.20** | **50** | **3.33** | 19 | 1.27 |
| S&P 500 | 500 | **17** | **3.40** | **13** | **2.60** | **25** | **5.00** | **13** | **2.60** |
| | 1500 | **35** | **2.33** | **27** | **1.80** | **28** | **1.87** | 14 | 0.93 |
| DAX30 | 500 | **21** | **4.20** | **14** | **2.80** | **19** | **3.80** | **18** | **3.60** |
| | 1500 | **41** | **2.73** | **29** | **1.93** | **25** | **1.67** | 20 | 1.33 |
| CAC40 | 500 | **16** | **3.20** | **14** | **2.80** | **14** | **2.80** | **13** | **2.60** |
| | 1500 | **34** | **2.27** | **29** | **1.93** | 17 | 1.63 | 19 | 1.27 |
| Nikkei 225 | 500 | **15** | **3.00** | **14** | **2.80** | **13** | **2.60** | 7 | 1.40 |
| | 1500 | **31** | **2.07** | 23 | 1.53 | **26** | **1.73** | 10 | 0.67 |
| DJCPI | 500 | **12** | **2.40** | 7 | 1.40 | **15** | **3.00** | 10 | 2.00 |
| | 1500 | **29** | **1.93** | 15 | 1.00 | **28** | **1.87** | 17 | 1.13 |

From Table 8 we can see that normal models for the 99% VAR computations commonly produce numbers of exceedings above the acceptable range, which implies that normal modeling significantly underestimates VAR (losses). At window length of 260 observations, stable modeling is not satisfactory. It provided permissible number of exceptions only for the BP/US$ and DJCPI series. At sample size of 1560 and testing interval of 500 observations, exceedings by the stable-FT method are outside of the admissible interval for the S&P 500, DAX30, and CAC40 indices. Testing on the longer interval with T=1500 showed that numbers of "stable" exceptions are within

Table 9: 95% VAR exceedings

| Series | Length of a testing interval, $T$ | 95% VAR exceedings | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | Window length = 260 obs. | | | | Window length = 1560 obs. | | | |
| | | Normal | | FT | | Normal | | FT | |
| | | $E$ | $E/T$ | $E$ | $E/T$ | $E$ | $E/T$ | $E$ | $E/T$ |
| Yen/BP | 500 | 35 | 7.00% | **38** | **7.60%** | 27 | 5.40% | 31 | 6.2% |
| | 1500 | 94 | 6.27 | **104** | **6.93** | **109** | **7.27** | **122** | **8.13** |
| BP/US$ | 500 | 33 | 6.60 | **45** | **9.00** | **10** | **2.00** | 17 | 3.40 |
| | 1.500 | 73 | 4.87 | **96** | **6.40** | **46** | **3.07** | 57 | 3.80 |
| DM/BP | 500 | 32 | 6.40 | **38** | **7.60** | 29 | 5.80 | **37** | **7.40** |
| | 1500 | 89 | 5.93 | **114** | **7.60** | **105** | **7.00** | **139** | **9.27** |
| S&P 500 | 500 | 34 | 6.80 | **39** | **7.80** | **43** | **8.60** | **47** | **9.40** |
| | 1500 | 79 | 5.27 | **98** | **6.53** | 62 | 4.13 | 69 | 4.60 |
| DAX30 | 500 | **47** | **9.40** | **50** | **10** | **42** | **8.40** | **45** | **9.00** |
| | 1500 | **98** | **6.53** | **109** | **7.27** | 62 | 4.13 | 79 | 5.27 |
| CAC40 | 500 | 32 | 6.40 | 34 | 6.80 | 31 | 6.20 | 32 | 6.40 |
| | 1500 | 81 | 5.40 | 87 | 5.80 | **51** | **4.90** | 82 | 5.47 |
| Nikkei 225 | 500 | **37** | **7.40** | **40** | **8.00** | 28 | 5.60 | 33 | 6.60 |
| | 1500 | 85 | 5.67 | 90 | 6.00 | 68 | 5.43 | 87 | 5.80 |
| DJCPI | 500 | 29 | 5.80 | 35 | 7.00 | **37** | **7.40** | **46** | **9.20** |
| | 1500 | 70 | 4.67 | 93 | 6.20 | 77 | 5.13 | **108** | **7.20** |

permissible range. Table 8 demonstrates that increasing the window length from 260 observations to 1560 observations reduces the number of stable-FT exceedings. In contrast, extending the window length for normal models does not decrease E, in some cases, even elevates it. Results illustrate that stable modeling outperforms normal modeling in the 99% VAR estimations.

The 95% VAR normal estimates (except the DAX30 series), obtained using 260 observations, are within the permissible range. Increasing the window length generally worsens the normal VAR measurements. The stable-FT method provided sufficient 95% VAR estimates for the Yen/BP and BP/US$ exchange rates and the CAC40 and Nikkei 225 indices. A study of the predictive power of VAR models suggests that:

- the normal modeling significantly underestimates 99% VAR,
- the stable method results in reasonable 99% VAR estimates,

- 95% normal measurements are in the admissible range for the window length of 260 observations. Increasing $lw$ to 1560 observations might deteriorate the precision of the estimates.

# 5 Conclusions

The Value-at-Risk (VAR) measurements are widely applied to estimate the exposure to market risks. The traditional approaches to VAR computations - the delta method, historical simulation, Monte Carlo simulation, and stress-testing - do not provide satisfactory evaluation of possible losses. The delta-normal methods do not describe well financial data with heavy tails. Hence, they underestimate VAR measurements in the tails. The historical simulation does not produce robust VAR estimates since it is not reliable in approximating low quantiles with a small number of observations in the tails. The stress-testing VAR estimates are subjective. The Monte Carlo VAR numbers might be affected by model misspecification.

We suggest to apply stable processes in VAR estimation. The in-sample- and forecast-evaluation shows that stable VAR modeling outperforms the normal modeling for high values of the VAR confidence level:

- the stable modeling generally produces conservative and accurate 99% VAR estimates, which is preferred by financial institutions and regulators,
- the normal method leads to overly optimistic forecasts of losses in the 99% VAR estimation,
- the normal modeling is acceptable for the 95% VAR estimation.

The stable Paretian model, while sharing the main properties of the normal distribution leading to the CLT (Central Limit Theorem), provides at the same time superior fit in modeling VAR. However, additional research is needed. Future work is this direction will be construction of models that capture the features of financial empirical data such as heavy tails, time-varying volatility, and short and long range dependence[39]. In order to describe thick tails, one can employ the conditional homoskedastic models based on the stable hypothesis[40]. ARMA-stable-GARCH models can incorporate both heavy

[39]For some preliminary results see Liu and Brorsen (1995), Mittnik, Rachev, and Paolella (1998), Mittnik, Paolella, and Rachev (1997, 1998a, and 1998b), Panorska, Mittnik, and Rachev (1995).

[40]These models are named as ARMA-$\alpha$-stable models.

tails and time-varying volatility[41]. The fractional-stable GARCH model can capture all observed phenomena in financial data: heavy tails, time-varying volatility, and short- and long-range dependence. An analysis of VAR estimation with ARMA-$\alpha$-stable, ARMA-stable-GARCH, and fractional-stable GARCH models will be provided in a subsequent paper.

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