The Integration of Rare Events into "Classical" Diversification Considerations

by

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Abstract: Stock price movements are characterized by occasional extreme vibrations (rare events), resulting in dramatic losses during stock market crises. This can be demonstrated by the October crises in 1987 and 1989 as well as the Southeast Asia, Russia and Brazil crises in 1998/99.

Despite its obvious importance, literature has neglected jumps in the context of optimum portfolio strategies. For that reason, the objective of our paper is the integration of price jumps into portfolio selection. To that end, we are looking to find in a first step a well-founded model of the jump/diffusion process: Reasonable representations of jumps distinguish between firm-specific, cluster-specific and market jumps (scope of jumps) as well as crashes and explosions (direction of jumps). In addition, jump probabilities and amplitudes have to be bounded from both above and below. Only now, with this economically founded jump model can we be ready to establish portfolio selection in a next step: Through the calculation of optimum portfolio weights, we are able to prove the second best nature of the following portfolio strategy: One ignores jumps completely due to their rare occurrence or one simply adjusts stocks' means and variances/covariances to jumps and uses these new stock characteristics as input of the old diffusion based portfolio rules. Instead, price jumps demand a totally unique portfolio strategy: Optimum portfolio weights are a linear combination of the μ-σ-efficient portfolio of the diffusion component and jump-induced correction terms. Moreover, both parts of the optimum portfolio can be combined to an extended Tobin separation.

Keywords: dynamic optimization, jump/diffusion processes, portfolio selection, rare events, stock market crises

1 Preliminaries

1.1 Introduction to the problem

Stock price movements are characterized by occasional extreme vibrations (rare events), resulting in dramatic losses during stock market crises. This can be demonstrated by the October crises in 1987 and 1989 as well as the Southeast Asia, Russia and Brazil crises in 1998/99[1]. The problem is gaining additional importance because the standard strategy of protecting a minimum portfolio wealth ("floor"), the Portfolio Insurance, can only handle small price movements, but no price jumps[2]. Therefore, we intend to raise the question of how crashes could be neutralized by improved portfolio planning.

Specifying the last point, we will discover the following underlying theoretical problem: The exact form of portfolio strategies critically depends on the stock price process and consequently the return distribution assumed. For that reason, careful selection of the stock price process is of decisive importance: The model has to deal with empirical observations on the one hand and special information of the decision maker on the other hand. This last point also includes identification of significant loss potential, e.g. stock market crashes.

The first method pursued in literature to integrate the above requirement into a stock price process models not only the stock price itself, but in addition mean and variance/covariance matrices as diffusion processes. This results in stochastically varying means and variance/covariance matrices (stochastic opportunity set[3]) and thus a better fit to empirical data. Yet, diffusion processes only encompass "normal" - mathematically speaking still smooth - price movements and fail to recognize an additional source of short term price movements: price jumps. Hence, diffusion processes do have difficulties in portraying the decision makers' information except for the fact that small[4] variations are unable to represent a significant loss potential.

Due to their technical features, combined jump/diffusion processes are a natural choice to depict price jumps and to further improve stock price models. Moreover, empirical results also support the jump/diffusion theory: Merton's[5] review of the empirical work on stock price distributions in the 60s and 70s documents that combined jump/diffusions processes really fit to empirically observable return characteristics. In addition, more recent data underpins the theory of the jump/diffusion process: Ball/Torous (1985) observe jumps in daily

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1 The following percentages of indexes' declines stress the importance of losses: Dow Jones: on 19/10/1987 -20.39% (largest percentage decline in US history, see Schwert (1990): 80), on 10/27/1987 -7.2% (at the same time largest point loss in the history of this index); DAX: on 10/19/1987 -9.86% (see Grünewald/Trautmann (1997): 44), on 10/28/1987 -8%, on 08/19/1991 -9.87%, on 10/16/1996 -13.71%, on 10/01/1998 -7.59%, on 01/13/199 -5.16% and on 03/15/2000 –2.72%.

2 See Geman (1992): 186.

3 See Merton (1973a) and Breeden (1979).

4 We use the word "small" to characterize price movements caused by pure diffusion processes.

5 See Merton (1976): 127, fn. 4 and Merton (1982): 21-22.

US stock returns and Jorion (1988) confirms their findings even in the case of the heteroscedasticity of the diffusion process (stochastic opportunity set). Both studies only differ with regard to the time span of their observations: Ball/Torous are unable to identify jumps in weekly or monthly stock returns, whereas Jorion finds his perceptions to be true in both cases (weekly: significant, monthly: weak). For German stocks, Trautmann/Beinert (1995) and Beinert (1997) find statistically significant jump components of daily and weekly DAX-returns. Additional evidence of jumps - though without statistical tests - can be found in Schwert (1990), Stehle/Hartmond (1991) and Turner/Weigel (1992).

Besides these empirical facts, jumps do coincide with the notion of special information of the decision maker: Jumps denote an explicit description of short-run vibrations caused by extraordinary events. Thus, jumps can take into consideration information not integrable into (pure) diffusion processes. For example, information about a large order and a resulting profit jump of the company. Moreover, it is possible to model extreme losses, Spremann (1997)[6] calls them stress case of portfolio planning, directly via stock price jumps.

To sum up, combined jump/diffusion processes are a founded stock price model, as Merton (1975)[7] stated

> *"...since virtually any reasonable stochastic process arising in an economics context can be adequately approximated by some mixture of these two types (author's comment: combined jump and diffusion processes), I would expect any serious disagreement with the stochastic process assumption would be on the use of a special form of the processes, rather than with the processes themselves".*

Hence, the first objective of our work is the development of an economically motivated stock price process, an important task according to Merton (1975) and nevertheless neglected in literature. To that end, we characterize jumps in a first step by distinguishing between firm-specific, cluster-specific and market jumps depending on their scope and between crashes and explosions with respect to their direction. In a second step, we use this information to adequately model jump probabilities and amplitudes.

Interestingly, many authors have described crashes[8] and tried to explain their existence[9]. On the other hand, jump processes are common in option pricing theory[10], but their integration into portfolio selection has been overlooked.

Alone Aase (1984) as well as, with some restrictions, Eastham/Hastings (1988), Hastings (1992) and Spremann (1997) have attempted such a model. However, the first three authors

6 See Spremann (1997): 867.

7 Merton (1975): 661.

8 See for example descriptions of the 1987 Crash in Gammill/Marsh (1988), Grossman (1988b), Lockwood/Linn (1990) and Kleidon/Whaley (1992).

9 See for example Black (1988), Grossman (1988a), Leland/Rubinstein (1988), Fama (1989), Roll (1989), Gennotte/Leland (1990) and Malliaris/Urrutia (1992).

10 See for example Cox/Ross (1976) and Merton (1976) or, more recently, the survey article Chang (1995).

focus on mere technical aspects of jump/diffusion processes. They neither describe the jump component economically nor do they compute optimum portfolio strategies. Spremann (1997), to an extent, constitutes the counterpole to them. He exclusively uses non-quantitative arguments in a static framework and illustrates by means of examples his main thesis: "Classical" portfolio considerations and the Tobin separation fail in a jump environment. But, he is also unable to establish a complete portfolio theory under price jumps.
For that reason, the second objective of our article is the integration of price jumps into portfolio selection. By calculating optimum portfolio weights, we can prove the second best nature of the following portfolio strategy: One ignores jumps completely due to their rare occurrence or one simply adjusts stocks' means and variances/covariances to jumps and uses these new stock characteristics as input of the old diffusion-based portfolio rules. Instead, price jumps demand a totally unique portfolio strategy: The optimum portfolio weights are a linear combination of the μ-σ-efficient portfolio of the diffusion component and jump-induced correction terms. So far, we can confirm Spremann's conjecture[11] that jumps call for a new portfolio theory. On the other hand, we refute his statement[12] with respect to the collapse of the Tobin separation because it is possible to decompose the correction terms into parts depending on investors' preferences and others that do not. These parts, however, can be combined to an extended Tobin separation.

The twofold objective of our paper - to develop an economically founded characterization of stock price jumps (rare events) and to establish a portfolio selection on that basis - demands the following structure of our article: In *chapter 2* we repeat important results of the diffusion based portfolio selection to attain a reference model. Next, we offer a first introduction to jump/diffusion portfolio selection by means of a naive modeling of jumps (*chapter 3*). However, this naive jump model leaves economically important questions unanswered and calls for a more thorough modeling of jumps (*chapter 4*). Based on the now established economically motivated jumps, we develop a whole portfolio theory under jumps (*chapter 5*). A summary of the most important results ends the paper (*chapter 6*).

1.2 Description of the decision problem

To be able to illustrate the effects of jumps, we will use the simplest possible notion of an economic environment. Therefore, we assume a perfect capital market, a special initial endowment of the individual investor and certain preferences. To be more precise, we rely on the framework developed by Merton (1969) and Merton (1971):

– Markets are free of arbitrage, frictionless and investors act as price takers.

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11 See Spremann (1997): 880.

12 See Spremann (1997): 866.

- All revenues from short sales can be used for investment purposes and there are no additional restrictions on investment policies.
- All investors have free and equal access to information important to pricing.
- There is no exogenous income i.e. solely stock transactions and consumption cause wealth changes.

The investor seeks to maximize his expected utility from consumption and bequest[13] by continuously optimizing his consumption and portfolio decisions (choice between a riskless asset and n stocks characterized by an invertible variance/covariance matrix). The utility functions are

- additively separable (consumption utility only), continuous, strictly monotone increasing and strictly concave referring to their arguments consumption as well as end of period wealth. Additionally, we must take into account the time dependency of each utility function.

Introducing a constant time preference rate as special model of this time dependency,

(1) $$U\left[C(t),t\right] = e^{-\rho t}\, U\left[C(t)\right]$$

or rather,

(2) $$B\left[W(T),T\right] = e^{-\rho T}\, B\left[W(T)\right]$$

U[C(t),t]	consumption utility function in t
C(t)	consumption per unit time in t ("instantaneous consumption"); consumption can only be measured as a flow (we get $C(t)\,dt$ between t and $t + dt$ [14]).
B[W(T),T]	bequest utility function in T
W(t)	wealth in t
ρ	investor's time preference rate

we can formalize the investor's decision problem as follows:

(3) $$\underset{C,\mathbf{w}}{\mathrm{Max}}\; E_0 \left\{ \underbrace{\int_0^T e^{-\rho s}\, U\left[C(s)\right] ds}_{\text{consumption utility}} + \underbrace{e^{-\rho T}\, B\left[W(T)\right]}_{\text{bequest utility}} \right\}$$

w(t)	$n \times 1$ vector of portfolio weights
$E_t\{\}$	expectation conditional on information in t
T	planning horizon

[13] The existence of a bequest function implies a planning horizon smaller than infinity. This does not constitute any problem because we argue with individuals not corporations.

[14] See Ingersoll (1987): 271 or Cox/Huang (1991): 468.

2 The basis of dynamic portfolio selection: optimization under diffusion processes[15]

Every continuous time model needs a further assumption specifying the price process, i.e. the return distribution. In general, literature uses price processes without jumps (diffusion processes) to model "normal" price movements "due to temporary imbalance between supply and demand, changes in capitalization rates, changes in the economic outlook, or other information, that causes marginal changes in the stock's value" etc.[16]

The first approach, the basis model, uses the geometric Brownian motion to capture "normal" price movements. Therefore, the following additional assumption results:

– Stock prices are governed by a geometric Brownian motion.

Hence, the price process of stock j reads

(4) $$d\,P_j(t) = \alpha_j P_j(t)\,d\,t + \sigma_j P_j(t)\,d\,z_j(t)$$

α_j	constant instantaneous mean ex dividend return of stock j
$dP_j(t)$	change of the ex dividend price of stock j between t and t + dt
dt	infinitesimal change of time
σ_j	constant instantaneous standard deviation of the ex dividend return of stock j
$d\,z_j(t)$	increment of stock j's Wiener process

Price changes are composed of a drift, i.e. expected (1st term on the right side of equation (4)), and a superimposed stochastic, i.e. unexpected, component (2nd term). Such a model of stock prices coincides with economic intuition: Stock prices following a geometric Brownian motion have a logarithmic normal distribution. Thus, they are restricted to nonnegative values, a fact reflecting the nature of assets with limited liability (returns are normally distributed).

The price of the riskless asset moves according to the following diffusion process[17]:

(5) $$d\,P_0(t) = r\,P_0(t)\,d\,t$$

r	riskless instantaneous return

Transactions in those assets characterized by (4) and (5) as well as the instantaneous consumption determine the wealth change:[18]

[15] See Merton (1969), Merton (1971) and Merton (1973b).

[16] Merton (1976): 127.

[17] In an economy without free lunches, all riskless assets must have the same instantaneous return. Therefore, it suffices to examine only one riskless asset.

[18] For a proof, see Merton (1971): 379.

(6) $$d\,W(t) = \mathbf{w}^T(t)\left(a - \mathbf{1}r\right)W(t)\,d\,t + \left(r\,W(t) - C(t)\right)d\,t + W(t)\mathbf{w}^T(t)\;s\;d\mathbf{z}(t)$$

^{T}	transposition of a vector or matrix
a	$n \times 1$ vector of constant instantaneous mean ex dividend returns
s	$n \times n$ diagonal matrix of constant instantaneous standard deviations of the ex dividend returns
1	$n \times 1$ vector that solely has the number one as components
d **z**(t)	$n \times 1$ vector of the increments of correlated Wiener processes holding: $d\,z_i\;d\,z_j = \eta_{ij}\,d\,t$ (η_{ij}: correlation coefficient of stock i's and j's price changes)

In order to prepare for calculation of optimum consumption and portfolio programs, we define a function J[Y,t] denoting the expected utility of the optimum strategy from t to the planning horizon T if the investor has initial wealth Y in t:

(7) $$J[Y,t] \equiv \underset{C,\mathbf{w}}{\mathrm{Max}}\; E_t\left\{\int_t^T e^{-\rho s}\,U[C(s)]\,d\,s + e^{-\rho T}B[W(T)]\right\}$$

We can characterize optimum consumption portfolio decisions by means of J[.] in two respects: Firstly, further portfolio modification cannot increase expected utility, the change of J[.] with respect to a marginal change of consumption or portfolio weights is zero. If the expected utility could still be increased, we would - by definition - not have reached the optimum strategy. Secondly, the expected utility of the optimum strategy must equal the bequest utility in the planning horizon. The so-called Hamilton/Jacobi/Bellman equation inclusive of boundary condition gives us a formal treatment of this problem as follows[19]:

(8) $$0 = \underset{C,\mathbf{w}}{\mathrm{Max}}\left\{e^{-\rho t}\,U[C(t)] + J_t + J_W\,(r\,W(t) - C(t)) + J_W\,W(t)\,\mathbf{w}^T(t)(a - \mathbf{1}r) + \frac{1}{2}J_{WW}\,W^2(t)\,\mathbf{w}^T(t)\,W\,\mathbf{w}(t)\right\}$$

with boundary condition: $J[W(T),T] = e^{-\rho T}B[W(T)]$

W	$n \times n$ matrix of constant instantaneous variances/covariances of the ex dividend returns (instantaneous variance/covariance matrix)

To find the solely interesting optimum portfolio weights, we differentiate and solve the Hamilton/Jacobi/Bellman equation (8) with respect to **w**(t)[20]

(9) $$\mathbf{w}(t) = -\frac{J_W}{J_{WW}\,W(t)}\,W^{-1}(a - \mathbf{1}r)$$

[19] The derivation of this formula is well-documented in literature, e.g. Kamien/Schwartz (1981): 246 p. Moreover, for the ease of exposition, we evaluate the theoretically exact formulation J[Y,t] at Y = W(t) and use J[W(t),t] in every future calculation.

[20] The sufficient conditions for a maximum will be met if J_{WW} is negative (see Merton (1971): 382).

From (9) we conclude that two elements determine the optimum portfolio weights: The structural component $W^{-1}(a - \mathbf{1}r)$ (which suggests how the resources intended for risky investment should be divided between stocks) is independent of investors' preferences, is exclusively governed by the trade-off between excess return and risk and therefore represents a μ-σ-efficient portfolio[21]. The weight $-\frac{J_W}{J_{WW} W(t)}$, i.e. the relative risk tolerance based on the derived utility function J[.], depends on investors' preferences and determines the volume of funds desired at all for risky investment.
Normalizing the portfolio weights (9) by using $\mathbf{1}^T W^{-1}(a - \mathbf{1}r)$, we can achieve a more detailed interpretation:

(10) $$\mathbf{w}(t) = w_{\tan g}(t)\mathbf{T}$$

$\mathbf{T} \equiv \frac{1}{\mathbf{1}^T W^{-1}(a - \mathbf{1}r)} \cdot W^{-1}(a - \mathbf{1}r)$ efficient portfolio of risky assets if there is also a riskless asset (tangency portfolio)

$w_{\text{tang}}(t) \equiv -\frac{J_W}{J_{WW} W(t)} \mathbf{1}^T W^{-1}(a - \mathbf{1}r)$ weight invested in the tangency portfolio

Pursuant to equation (10), a decision maker is indifferent between holding optimum portfolio weights **w**(t) according to (9) or investing the weight $w_{\text{tang}}(t)$ in the tangency portfolio **T**. Therefore, investors' needs to allocate risk will be satisfied if they buy the fund **T**. Since the composition of **T** is, in addition, independent of the indirect utility function J[.] and thus investors' preferences, this statement does not only hold for one specific, but all decision makers (Tobin separation)[22].
This fundamental statement of "classical" continuous-time portfolio selection, the construction of portfolio weights (9) and the Tobin separation (10), must be analyzed in a jump environment, indeed for that reason, because Spremann (1997)[23] postulates a collapse of both statements under jumps.

3 A first introduction to price jumps - the case of two stocks

3.1 General characteristics of price jumps (rare events)

Chapter 2 is based on diffusion processes with small changes and for that reason, continuous, although not differentiable sample paths. This chapter's price jumps differ significantly. On the one hand, they do not vary permanently, but only at certain times. On the other

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[21] For an exact proof, see Ingersoll (1987): 283.

[22] To be precise, investors have to have homogeneous expectations, that is identical opinions with respect to α and Ω.

[23] See Spremann (1997): 866, 880.

hand, their movement is to a large extent[24]. Hence, discontinuous sample paths characterize jump/diffusion processes[25].

Apart from this formal representation of jump, we are, of course, interested in their economic description: The diffusion process portrays "normal" movements. Therefore, jumps denote an explicit description of "short-run" vibrations caused by extraordinary events and can take into consideration forecasts and hence information not integrable into diffusion processes.

3.2 Mathematical modeling of large variations

Compression of the last section's jump information (variation to a large extent and occurrence at only finite points in time) to a stock price model yields the following result: At every time t the stock price follows the geometric Brownian motion[26] (4). At time τ_i a stock price jump occurs in addition. This large price change is firstly independent of the "normal" risk to deal with the extraordinary nature of jumps. Secondly, we capture it by a random variable, the so-called jump amplitude, expressing the jump as percentage of the current stock price[27]. Formally, we get a price change of

stock 1:

(11a)
$$\begin{aligned} d P_1(t) &= \alpha_1 P_1(t)\, dt + \alpha_1 P_1(t)\, d z_1(t) && t \in [\tau_{i-1}, \tau_i) \\ P_1(\tau_i) &= (1+\tilde{\varphi}_1(\tau_i)) P_1(\tau_i^-) && t = \tau_i \end{aligned}$$

$\tilde{\varphi}_1(\tau_i)$	random variable specifying the jump amplitude of stock j in τ_i
τ_i	$\tau_0 = 0 \le \tau_1 \le \tau_2 \le \ldots \tau_i \le \ldots$ time at which jumps occur
τ_i^-	time immediately prior to the jump time τ_i; to be more precise, we have to formulate: $P_j(\tau_i^-) = \lim_{s \to \tau_i} P_j(s)$ (s < τ_i; see Neftci (1996): 214)

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24 "Large" means subject to extraordinary price vibrations and should be understood as counter concept of "small" changes caused by pure diffusion processes.

25 See Merton (1982): 43.

26 Basically, it would be no problem to use a more general model of the diffusion component. However, the actually interesting jump component would not be affected by this change (see Nietert (1996): 118 pp.) so that we continue to utilize the simplest diffusion model.

27 The use of a percentage $\tilde{\varphi}(t)$ for the jump size ensures W(t) to be positive for all t. Hence, there is no free lunch even in a continuous time environment (see Dybvig/Huang (1988): 390, theorem 2). For, free lunches exclude a solution to the portfolio problem from the outset.

or rather,

stock 2:

(11b) $$d\,P_2(t) = \alpha_2\, P_2(t)\, d\,t + \alpha_2\, P_2(t)\, d\,z_2(t) \qquad \text{for all t}$$

Based on the previous price model, we can formalize the wealth change in a jump diffusion environment - every single stock enters wealth only with its portfolio weight - as[28]:

(12) $$d\,W(t) = \left[w_1(t)\left(\alpha_1 - r\right) + w_2(t)\left(\alpha_2 - r\right)\right] W(t)\, d\,t + \left(r\,W(t) - C(t)\right) d\,t$$
$$+\left[w_1(t)\, d\,z_1(t) + w_2(t)\, d\,z_2(t)\right] W(t)$$
for $t \in \left[\tau_{i-1}, \tau_i\right)$ (see (6) adapted to the case of two stocks)
$$W(\tau_i) - W(\tau_i^-) = w_1(\tau_i^-)\, \tilde{\varphi}_1(\tau_i)\, W(\tau_i^-)$$
for $t = \tau_i$

Of course, ex ante we do not know at which time τ_i a jump occurs. Therefore, the equation (12) corresponds more to the illustration of jump effects than their realistic modeling. We can, however, capture the occurrence at just finite points of times by means of so-called jump probabilities. For stock 1, based on jump probabilities calculated from Poisson processes[29], we can write:

(13) probability{one jump occurs in the time interval (t,t + dt)}
$= \lambda_1\, d\,t$

probability{more than one jump occurs in the time interval (t,t + dt)}
= o(dt)

probability{no jumps occur in the time interval (t,t + dt)}
$= 1 - \lambda_1\, d\,t$

λ_1	intensity, that is average number of jumps per unit time, of the Poisson process of stock 1
o(dt)	function of greater order than one; such terms will be negligible if dt is infinitesimal small

Therefore, the second characteristic of jumps, occurrence at a solely finite number of times, can be handled by means of a simple switch: "one jump occurs/no jumps occur". Consequently, we specify the wealth change as:

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[28] Since consumption is measured over a time period, there can be no consumption at time τ_i. Consequently, the wealth's jump component in (12) does not contain consumption.

[29] For a derivation of these probabilities, see e.g. Nietert (1996): 253 p.

$$\text{(14)} \qquad dW(t) = \left[w_1(t)(\alpha_1 - r) + w_2(t)(\alpha_2 - r)\right]W(t)\,dt + \left(rW(t) - C(t)\right)dt$$
$$+\left[w_1(t)\,dz_1(t) + w_2(t)\,dz_2(t)\right]W(t)$$

with probability $1-\lambda_1 dt$ (diffusion case)

$$\Delta_S W(t) = \left[w_1(t)(\alpha_1 - r) + w_2(t)(\alpha_2 - r)\right]W(t)\,dt + \left(rW(t) - C(t)\right)dt$$
$$+ w_1(t)\tilde{\varphi}_1(t)W(t)$$

with probability $\lambda_1 dt$ (jump case)[30]

ΔW_S wealth change due to jumps of stock 1

To sum up, we can conclude from equation (12) in connection with equation (14) that both parameters, jump amplitude and jump probability, completely describe the jump risk.

3.3 Optimization in the case of two stocks

From chapter 2, we know that a portfolio strategy will be optimum if the investor cannot increase his expected utility by further portfolio adjustments ($dJ[.] = 0$). We formalize this notion by means of the jump-adjusted Hamilton/Jacobi/Bellman equation[31] inclusive of boundary condition. Differentiating and solving the Hamilton/Jacobi/Bellman equation with respect to the portfolio weights, yields the following optimum portfolio strategy[32]:

$$\text{(15a)} \qquad w_1(t) = \underbrace{-\frac{J_W}{J_{WW}W(t)}\frac{\sigma_2^2(\alpha_1 - r) - \sigma_1\sigma_2\eta_{12}(\alpha_2 - r)}{\sigma_1^2\sigma_2^2 - (\sigma_1\sigma_2\eta_{12})^2}}_{\text{1st term}}$$
$$\underbrace{-\frac{E_t\left\{\tilde{\varphi}_1 J\left[(1+w_1(t)\tilde{\varphi}_1(t))W(t),t\right]\right\}}{J_{WW}W(t)}\frac{\sigma_2^2}{\sigma_1^2\sigma_2^2 - (\sigma_1\sigma_2\eta_{12})^2}\cdot\lambda_1}_{\text{2nd term}}$$

and

30 To keep our symbols readable, we do not distinguish - in accordance with literature - between W(t) and $W(t^-)$ from now own.

31 The derivation of this equation can be found in appendix 1.

32 The sufficient conditions will be fulfilled if $J_{WW} < 0$ holds (see Nietert (1996): 262).

(15b) $$w_2(t) = \underbrace{-\frac{J_W}{J_{WW}W(t)}\frac{\sigma_1^2(\alpha_2 - r) - \sigma_1\sigma_2\eta_{12}(\alpha_1 - r)}{\sigma_1^2\sigma_2^2 - (\sigma_1\sigma_2\eta_{12})^2}}_{\text{1st term}}$$

$$\underbrace{-\frac{E_t\{\tilde{\varphi}_1 J[(1+w_1(t)\tilde{\varphi}_1(t))W(t),t]\}}{J_{WW}W(t)}\frac{-\sigma_1\sigma_2\eta_{12}}{\sigma_1^2\sigma_2^2 - (\sigma_1\sigma_2\eta_{12})^2}\cdot\lambda_1}_{\text{2nd term}}$$

The optimum portfolio weights consist of two basic parts: the well-known μ-σ-efficient portfolio of "normal" price movements (1st term) and a correction term (2nd term) taking stock 1's jumps into consideration. In analogy to the interpretation of the 1st term (see the statement following equation (9)), we can identify two components of the correction term: a structural component $\frac{\sigma_2^2}{\sigma_1^2\sigma_2^2 - (\sigma_1\sigma_2\eta_{12})^2}\cdot\lambda_1$ or rather $\frac{-\sigma_1\sigma_2\eta_{12}}{\sigma_1^2\sigma_2^2 - (\sigma_1\sigma_2\eta_{12})^2}\cdot\lambda_1$, which is independent of investors' preferences and weights the average number of jumps per unit of time against the risk from "normal" price movements. Moreover, we have a volume factor $-\frac{J_W}{J_{WW}W(t)}\cdot\frac{E_t\{\tilde{\varphi}_1 J[(1+w_1(t)\tilde{\varphi}_1)W(t),t]\}}{J_W}$, which is dependent on investors' preferences and judges the importance of jumps by means risk tolerance and expected utility comparisons before and after jumps. These different volume factors $-\frac{E_t\{\tilde{\varphi}_1 J[(1+w_1(t)\tilde{\varphi}_1)W(t),t]\}}{J_{WW}W(t)}$ and $-\frac{J_W}{J_{WW}W(t)}$ ensure furthermore that it is not possible to combine the μ-σ-efficient portfolio and the correction term to one single term. Therefore, (15a) and (15b) call for a totally new portfolio strategy under price jumps.

Our thesis up to now illustrates the tight connection between the jump model chosen and the optimum portfolio weight under jumps. Of course, the naive jump model of equation (11a) and (11b) does not represent an economically founded jump model because it leaves unanswered the following important questions:

- Is it economically permissible to make only one stock jump, or rather, what jump model are we thereby implying?
- What will happen if there are more stocks subject to jumps? How does wealth change under these circumstances?
- What values can the jump amplitude reach without violating arbitrage or institutional conditions?

Generalizing the above arguments, we have to scrutinize the proliferation of jump models existing in literature with respect to their economic foundation: Eastham/ Hastings (1988)[33]

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[33] See Eastham/Hastings (1988): 589, formula (1.1).

and Aase (1993)[34] postulate a simultaneous jump of all stocks and a wealth change as follows: "wealth jumps/wealth does not jump". Hastings (1992)[35] studies only one risky asset and reaches similar results. On the contrary, Aase (1984)[36] allows every stock to jump with its own probability and in this way creates a more sophisticated wealth path.
Since, as just seen, the jump description exerts an enormous influence on the model's results, we have to analyze the economic characterization of jumps before we can use them. For, without such reasoning, equations (15a) and (15b) as well as the deduced demand for a new portfolio strategy under jumps would only be conjectures.

4 More precise inspection of price jumps

4.1 Economic modeling

In chapter 3 we stated that jumps were the explicit description of short-term price movements due to extraordinary events. We also developed a naive jump model for the case of two stocks. Now we have to lay the basis of an economically founded portfolio theory under price jumps by using a reasonably motivated jump model.
To that end, we narrow down in the first step the scope of jumps through the analysis of the causes of jumps: Literature cites new fundamental information about a company, industry or the market[37]. In addition, literature offers non-fundamental explanations for market jumps[38]. Hence, we can identify:

– firm-specific jumps: These affect solely the stocks of one company and leave all other stocks untouched. A good example is the BMW stock, whose price rose by 11% on 03/15/2000 due to rumors that they will restructure their Rover investment.

– cluster-specific jumps: These affect every stock of a cluster (e.g. industry) and cause the jump sizes of the stocks to move together whilst leaving the stocks of other clusters untouched. A typical case would be the price rise of bank stocks in Germany on 03/07/2000 after the annoucement of the Deutsche Bank/Dresdner Bank merger.

– market jumps: These affect every stock of the market and cause the jump sizes of all stocks to move together. The price decrease of the DAX by 2.72% on 03/15/2000 due to interest rate fears.

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[34] See Aase (1993): 75, formula (3.6).

[35] See Hastings (1992): 61, formula (2.2).

[36] See Aase (1984): 82, formula (1).

[37] See Merton (1976): 127 for firm- and industry-specific jumps respectively. For market jumps, see Black (1988), Roll (1989) and Fama (1989).

[38] For instance, overreaction of the market (French (1988)) or information asymmetries in connection with Portfolio Insurance (Gennotte/Leland (1990) and Jacklin/Kleidon/Pfleiderer (1992)).

In a second step, we specify the range of possible jump outcomes. Stock price movements will only be regarded as jumps if they exceed that minimum price change *a*, that is no longer "normal". Two factors influence *a*: Firstly, the trading span. Daily transactions demand higher "normal" vibrations than hourly portfolio revisions. Secondly, the market conditions. Highly volatile environments (e.g. in the third quarter of 1999 DAX-movements by 1.5 % a day occurred often) need other boundaries than low volatility phases. However, every investor has a different notion of "normality" - we can call the lower boundary *a* subjective boundary on that account[39]. - A similar reasoning yields an upper boundary *b* for jumps: Let us call the downward jump "crash" and the upward one "explosion". Due to arbitrage arguments, there must be an upper boundary for crashes. A stock is an asset with limited liability, does not require any payment beyond its purchasing price and therefore offers a cash flow of zero at worst. Thus, the crash amplitude must be ≤ 100%. In addition, upper limits can be motivated from an institutional point of view. In the case of explicit price limits[40] the upper boundary is obvious. In Germany however, there are no such limits. The Börsengesetz (German stock market law) requires merely orderly price formation[41]. So far, according to § 43 I BörsG, there are only implicit price limits within the scope of a trading stop. Unfortunately, § 43 I BörsG stresses that a trading stop needs circumstances resting on the behavior of the issuing company or third party action. It explicitly excludes circumstances owing to technical market situations, such as significant supply or demand surplus. Consequently, in Germany there are neither explicitly nor implicitly legally binding price limits. On the other hand, we can read in § 8 IV "Börsenordnung" of the Frankfurt Stock Exchange from 02/25/1997 (rules developed by the stock exchanges themselves) that price formation under heavy price fluctuations can only take place after an appropriate period of time. In addition, a trading stop can, according to § 43 I BörsG, occur in the case of publication of insider information by the issuing firm. Putting both facts together, there will practically be an upper boundary for price changes even in Germany. - At any rate, *b* is an exogenous upper limit from the investor's point of view.

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[39] Of course, price jumps smaller than 1% a day can no longer be given the attribute of a jump beyond the subjective character of the minimum jump size. For, simulations of the diffusion process (e.g. instantaneous mean: 0.15 and instantaneous standard deviation: 0.2 for a geometric Brownian motion as well as setting dt equal to one day, that is $1/360$) show that price changes between + 1,3% and -1,3% can be modeled by an diffusion process without any problem. - In so far, using a lower limit for jumps does not forgo any information of the price process.

[40] See Roll (1989): 54 for a list of stock exchanges using that device.

[41] See § 11 II BörsG, whereas we have to interpret "orderly" in the sense of equal opportunities and transparency (see Kümpel (1996): 77). For further details with respect to price formation § 11 II 6 BörsG is referring to the "Börsenordnungen" (rules developed by the stock exchanges themselves). But, also in the "Börsenordnung" of the Frankfurt Stock Exchange from 02/25/1997 there are no explicit price limits (see §8, provisions in the cause of price fluctuations).

To sum up the above paragraph:

- A crash is a downward jump with
  $0 < a_C \leq |\text{jump size}| \leq b_C \leq 100\%$
- An explosion is an upward jump with
  $0 < a_E \leq \text{jump size} \leq b_E \leq \infty$

4.2 Consequences of economically reasonable jump description for modeling

4.2.1 Consequences for the jump probability

After the findings of chapter 4.1, we know jumps to have different scopes. We deal with this fact by using improved models of the jump probability:

Our naive jump model of equation (12) implies that solely stock 1 is subject to jumps. Therefore, we assumed implicitly firm-specific jumps. Due to this, we can generalize: Under firm-specific jumps every stock has a jump possibility and thus probability of its own, that is $\lambda_1 \mathrm{d}t$, $\lambda_2 \mathrm{d}t$, etc. Consequently, the probability of a simultaneous occurrence of two jumps under firm-specific jumps is negligible[42]. Hence, the probability of a wealth jump under firm-specific jumps equals the probability that stock 1 jumps or stock 2 or ..., i.e. we simply sum over the individual probabilities.

Eventually, we have to take into consideration the two manifestations (crash and explosion) of jumps. For that reason, the above described jump probabilities need an adaptation to this specification: In the period between t and $t + \mathrm{d}t$ there occurs with the probability $1 - \lambda_{Cj} \mathrm{d}t - \lambda_{Ej} \mathrm{d}t$ no jump (stock j's price is governed by the diffusion component), with probability $\lambda_{Ej} \mathrm{d}t$ an explosion and with probability $\lambda_{Cj} \mathrm{d}t$ a crash. Such an approach can be justified from a practical point of view besides the mere theoretical construction: A highly volatile environment restricts investors to forecasts only stating large movement, but does not allow a reliable estimation of the vibrations' direction[43]. Hence, even in this environment it is desirable to integrate both forms of jumps into one formula in order to gain optimum portfolio strategies.

As opposed to firm-specific jumps, market jumps call for simultaneous jumps of all stocks of the market; the probability that only one stock does not jump equals zero. Therefore, we can no longer rely on the individual jump probability $\lambda_j \mathrm{d}t$. We have to focus on the probability of a jump of the whole market $\lambda_M \mathrm{d}t$, instead, that is: $\lambda_1 \mathrm{d}t = \ldots = \lambda_k \mathrm{d}t = \ldots$

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[42] For, the probability that a simultaneous jumps of stock 1 and 2 occurs is $\lambda_1 \mathrm{d}t \cdot \lambda_2 \mathrm{d}t = o(\mathrm{d}t)$.

[43] We are talking about situations in which buying e.g. a straddle would be the optimum strategy. The most prominent example is presented by Schwert (1990): 80: On October 28th and 29th 1929 stocks of a portfolio representing the US stock market dropped by more than 10%, on October 30th stock prices soared by more than 10%. Both on 29th and on 30th the market movement's direction was unclear.

$= \lambda_n \, dt = \lambda_M \, dt$. Of course, we also have to distinguish between crashes and explosions under market jumps.

4.2.2 Consequences for the jump amplitude

The jump amplitude - the second parameter characterizing jumps - is intended to describe the extent of the price change due to occurrence of an exceptional event. Since extraordinary events at different dates can lead to different price movements, we do not use a constant to capture jump amplitudes, but a random variable. In general, every probability distribution can be used to represent the jump size $\tilde{\varphi}_j(t)$, as long as it does not violate the range of crashes and explosions prescribed, that is: $a_j \leq \tilde{\varphi}_j(t) \leq b_j$. For example, possible candidates are: uniform distribution, when an investor has no idea about the jump size, triangular distribution to model crudely and Beta distribution to thoroughly analyze distributions skewed to the right and to the left respectively[44]. The logarithmic normal distribution of $1+\tilde{\varphi}_j(t)$ widely used in literature (e.g. Merton (1976) and Chang (1995)) due to its computational advantages is not a permissible model of the jump size from a strictly theoretical point of view because its input ranges from zero to infinity and lies beyond upper and, especially, lower limits of the amplitude.

Eventually, we have to differentiate between crashes and explosions and to distinguish between mutually uncorrelated (firm-specific jumps) and jointly distributed (market jumps) jump amplitudes[45].

4.2.3 Consequences for the wealth change

Putting together our results with respect to jump probabilities and amplitudes, we obtain the following economically founded[46] wealth change equation:

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44 For an explicit calculation of optimum portfolio weights under consideration of these distributions, see Nietert (1996): 98 p. and 101.

45 Cluster-specific jumps are, from a theoretical point of view, a mixture between firm-specific and market jumps: Within the cluster, stocks behave similarly to a market jump since every stock has to jump. Between different clusters, stocks behave like under firm-specific jumps because different clusters can jump independently of each other. To sum up, cluster-specific jumps do not contain new information and will be skipped for the rest of the article.

46 Strictly speaking, stochastically changing jump probabilities would be suitable to capture the occurrence of surprising information and thus to consider economic needs of a jump model. However, we would have to rely on a far more complex model of jumps without observing consequences on portfolio strategy. For that reason, we skip further details and refer to Nietert (1998) and Nietert (1999).

- firm-specific jumps:

(16a) $$dW(t) = \mathbf{w}^T(t)(a - \mathbf{1}r)W(t)\,dt + (r\,W(t) - C(t))\,dt + W(t)\mathbf{w}^T(t)\,s\,d\mathbf{z}(t)$$

with probability $1 - \left(\sum_{j=1}^{n}\lambda_{Ej} + \sum_{j=1}^{n}\lambda_{Cj}\right)dt$ (diffusion case)

$$\Delta_S W(t) = \mathbf{w}^T(t)(a - \mathbf{1}r)W(t)\,dt + (r\,W(t) - C(t))\,dt + W(t)\mathbf{w}^T(t)\,s\,d\mathbf{z}(t) + \sum_{j=1}^{n} w_j(t)\tilde{\varphi}_j(t)W(t)$$

with probability $\left(\sum_{j=1}^{n}\lambda_{Ej} + \sum_{j=1}^{n}\lambda_{Cj}\right)dt$ (jump case)

- market jumps:

(16b) $$dW(t) = \mathbf{w}^T(t)(a - \mathbf{1}r)W(t)\,dt + (r\,W(t) - C(t))\,dt + W(t)\mathbf{w}^T(t)\,s\,d\mathbf{z}(t)$$

with probability $1 - (\lambda_{EM} + \lambda_{CM})\,dt$ (diffusion case)

$$\Delta_S W(t) = \mathbf{w}^T(t)(a - \mathbf{1}r)W(t)\,dt + (r\,W(t) - C(t))\,dt + W(t)\mathbf{w}^T(t)\,s\,d\mathbf{z}(t) + \sum_{j=1}^{n} w_j(t)\tilde{\varphi}_{Mj}(t)W(t)$$

with probability $(\lambda_{EM} + \lambda_{CM})\,dt$ (jump case)

$\tilde{\varphi}_{Mj}(t)$ jump amplitude of stock j under market jumps

Hence, we do not postulate large wealth changes, but derive them from the changes of single parts of wealth - the individual stocks. We therefore avoid those problems that appeared during the discussion of literature's jump models at the end of section 3.3.

In addition, a comparison of equation (16a) and (16b) shows that every stock has a jump probability as well as amplitude under both firm-specific and market jumps. Although we are considering scenarios completely different at the first glance, they are theoretically very similar to handle. For that reason, it suffices to analyze one version, the basic results also apply to the other case.

Due to this, we will focus on firm-specific jumps; they are able to illustrate the jump component in the best way: For, under firm-specific jumps every stock has its own jump possibility. Moreover, this is a useful feature even in the case of long lasting stock market crises, e.g. the Southeast Asia crisis, because even under those circumstances stock market crashes do not happen every day. We observe special movements of selected stocks instead.

5 Optimization under (firm-specific) price jumps

5.1 Theoretical analysis

Only with the economically founded jump model developed in chapter 4 do we have the basis for a portfolio theory under price jumps.
To obtain optimum portfolio weights, we use the same technique as in chapter 3 during the optimization under naive jump models. That is, we differentiate the Hamilton/Jacobi/Bellman equation[47] with respect to the portfolio weights and rearrange:

$$\text{(17)}\qquad \mathbf{w}(t) = \underbrace{-\frac{J_W}{J_{WW}W(t)}W^{-1}(a-\mathbf{1}r)}_{\text{1st term}}$$

$$\underbrace{-\sum_{j=1}^{n}\frac{E_t\left\{\tilde{\varphi}_{Ej}(t)J_W\left[(1+w_j(t)\tilde{\varphi}_{Ej}(t))W(t),t\right]\right\}}{J_{WW}W}W^{-1}\begin{pmatrix}\mathbf{0}\\ \lambda_{Ej}\\ \mathbf{0}\end{pmatrix}}_{\text{2nd term}}$$

$$\underbrace{+\sum_{j=1}^{n}\frac{E_t\left\{\tilde{\varphi}_{Cj}(t)J_W\left[(1-w_j(t)\tilde{\varphi}_{Cj}(t))W(t),t\right]\right\}}{J_{WW}W}W^{-1}\begin{pmatrix}\mathbf{0}\\ \lambda_{Cj}\\ \mathbf{0}\end{pmatrix}}_{\text{3rd term}}$$

$E_t\left\{\tilde{\varphi}_{Ej}(t)J_W[j_E]\right\}$	short for $E_t\left\{\tilde{\varphi}_{Ej}(t)J_W\left[(1+w_j(t)\tilde{\varphi}_{Ej}(t))W(t),t\right]\right\}$
$\begin{pmatrix}\mathbf{0}\\ \lambda_{Ej}\\ \mathbf{0}\end{pmatrix}$	$n \times 1$ vector with λ_{Ej} as jth component and zero as other elements

Thus, optimum portfolio weights (17) reproduce both results of the naive jump model (see equations (15a) and (15b)): the optimum portfolio consists of two basic parts, the μ-σ-efficient portfolio of "normal" price fluctuations (1st term) and correction terms (2nd term: explosion and 3rd term: crash). The correction terms have two components: a structural component $W^{-1}\begin{pmatrix}\mathbf{0}\\ \lambda_{kj}\\ \mathbf{0}\end{pmatrix}$ independent of investors' preferences and volume factors $\frac{E_t\{J_W[j]\}}{J_{WW}W(t)}$ ($j = j_{E1},\dots,j_{En}, j_{C1},\dots,j_{Cn}$) dependent on investors' preferences. Again, the different volume factors of terms 1 ($-\frac{J_W}{J_{WW}W(t)}$) to 3 ($\frac{E_t\{J_W[j]\}}{J_{WW}W(t)}$) ensure that we are unable to combine μ-σ-efficient portfolio and correction terms to one single term.

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47 The derivation of this equation can be found in appendix 1.
The sufficient conditions will be fulfilled if $J_{WW} < 0$ holds (see Nietert (1996): 262).

Moreover, we can now confirm the conjecture "jumps demand explicit correction terms" formulated in equations (15a) and (15b) by means of equation (17) and the economically founded jump model. To be more precise, there is exactly one correction term j for the jump of each stock j[48]. Therefore, (17) calls for a totally new portfolio strategy: It does not work to simply ignore jumps due to their rare occurrence or to simply adjust stocks' characteristics to jumps and continue to use the old portfolio rules of the "normal" risk case. So far, we have confirmed Spremann's guess[49] that jump risks demand a new portfolio selection strategy. We are also able to justify a related statement via (17): good diversification under "normal" risk does not necessarily mean good diversification under jumps; for, correlation between stocks' "normal" risk cannot take into consideration jumps because they are unable to generate the 2nd and 3rd term in equation (17).

In order to analyze Spremann's second statement, the collapse of the Tobin separation under jumps, and to be simultaneously able to scrutinize the correction terms, we transform the optimum portfolio weights under jumps (17) into the funds representation - by analogy to equation (10) of the "normal" portfolio selection. That is, normalizing the correction terms in (17) by $\mathbf{1}^T W^{-1}\begin{pmatrix}\mathbf{0}\\ \lambda_{ki}\\ \mathbf{0}\end{pmatrix}$ the optimum portfolio weights under jumps read

(18) $$\mathbf{w}(t) = w_{tang}(t)\mathbf{T} + \sum_{i=1}^{n} w_{HEi}(t)\,\mathbf{H}_{Ei} - \sum_{i=1}^{n} w_{HCi}(t)\,\mathbf{H}_{Ci}$$

$$\mathbf{H}_{ki} \equiv \frac{1}{\mathbf{1}^T W^{-1}\begin{pmatrix}\mathbf{0}\\ \lambda_{ki}\\ \mathbf{0}\end{pmatrix}} \cdot W^{-1}\begin{pmatrix}\mathbf{0}\\ \lambda_{ki}\\ \mathbf{0}\end{pmatrix}$$ jump correction portfolio i (k = E, C)

$$w_{Hki}(t) \equiv -\frac{E\left\{\tilde{\varphi}_{ki}(t)J_W\left[(1+w_i(t)\tilde{\varphi}_{ki}(t))W(t),t\right]\right\}}{J_{WW}W(t)}\mathbf{1}^T W^{-1}\begin{pmatrix}\mathbf{0}\\ \lambda_{ki}\\ \mathbf{0}\end{pmatrix}$$ weight invested in jump correction portfolio i

Pursuant to equation (18), a decision maker is indifferent between holding optimum portfolio weights $\mathbf{w}(t)$ according to (17) or investing the weights $\mathbf{w}_{tang}(t)$ and $w_{Hki}(t)$ in the funds $\mathbf{T}$ and $\mathbf{H}_{ki}$. Therefore, investors' needs to allocate risk will be satisfied with the funds $\mathbf{T}$ and

48 This is also true in the case of market jumps as we can illustrate in a two-stock-environment. The correction terms (without distinguishing between crashes and explosions) read:

$$\frac{E_t\left\{\tilde{\varphi}_{M1}(t)J_W\left[(1+w_1(t)\tilde{\varphi}_{M1}(t)W(t),t\right]\right\}}{J_{WW}W(t)}W^{-1}\begin{pmatrix}\lambda_M\\ 0\end{pmatrix}$$

or rather,

$$\frac{E_t\left\{\tilde{\varphi}_{M2}(t)J_W\left[(1+w_2(t)\tilde{\varphi}_{M2}(t)W(t),t\right]\right\}}{J_{WW}W(t)}W^{-1}\begin{pmatrix}0\\ \lambda_M\end{pmatrix}$$

49 See Spremann (1997): 866 and 867 p.

$\mathbf{H}_{ki}$ alone. Since the composition of **T** and $\mathbf{H}_{ki}$ is, in addition, independent of the indirect utility function J[.] and thus investors' preferences, this statement does not only hold for one specific, but all decision makers (extended Tobin separation). Consequently, we restate the time-induced portfolio separation discovered by Franke[50] under geometric Brownian motion in a jump environment. For that reason, we contradict Spremann (1997)[51], who claims that investors will no longer hold identically structured portfolios under jumps.

Of course, we cannot combine the funds **T** and $\mathbf{H}_{ki}$, due to their different volume factors $\mathbf{w}_{tang}(t)$ and $w_{H\,ki}(t)$. We need the tangency portfolio, n correction funds against explosions and n correction funds against crashes, that is 2 n + 1 funds, to span the risk-return-characteristics of n stocks. Hence, delegation of portfolio decision, the original concern of portfolio separation, does not fail due to general lack of delegable decisions, but because it is easier to construct the optimum portfolio directly from stocks rather than relying on funds. In addition, we are faced with the unrealistic assumption of homogenous expectations referring to the jump parameters - jump parameters are part of the "short-run" price movement.

Therefore, it is important to identify those circumstances under which we will need less than 2 n + 1 funds to span the risk-return-characteristics of n stocks. Obviously, we will need less than 2 n + 1 funds if the jump probability of stock j equals zero. More interesting however, are the following scenarios of funds' number reduction. Firstly, in the case of identical structural components, secondly, in the case of identical volume factors and thirdly in the case of a combination of the first two points.

The first case - the standard approach of existing jumps models in literature[52] - implies identical crash and explosion probabilities for one stock j and yields in (17):

$$(19) \qquad \left[\frac{E_t\left\{\tilde{\varphi}_{Cj}(t)\,J_W\left[j_C\right]\right\}}{J_{WW}W(t)} - \frac{E_t\left\{\tilde{\varphi}_{Ej}(t)\,J_W\left[j_E\right]\right\}}{J_{WW}W(t)}\right] W^{-1}\begin{pmatrix}\mathbf{0}\\ \lambda_j\\ \mathbf{0}\end{pmatrix}$$

According to (19), we need only one instead of two funds to correct stock j's jump consequences.

The second case focuses on the combination of funds due to identical volume factors. This can happen on the one hand with different stocks by putting together e.g. explosion amplitudes:

$$(20a) \qquad E_t\left\{\tilde{\varphi}_{Ek}(t)\,J_W\left[k_E\right]\right\} = E_t\left\{\tilde{\varphi}_{Ej}(t)\,J_W\left[j_E\right]\right\}$$

and thus in (17)

[50] See Franke (1983): 249.

[51] See Spremann (1997): 866.

[52] See Aase (1984), Eastham/Hastings (1988), Hastings (1992) and Aase (1993).

$$(21a) \qquad -E_t\left\{\tilde{\varphi}_{Ej}(t)\,J_W\left[j_E\right]\right\}\left[W^{-1}\begin{pmatrix}\mathbf{0}\\ \lambda_{Ej}\\ \mathbf{0}\end{pmatrix}+W^{-1}\begin{pmatrix}\mathbf{0}\\ \lambda_{Ek}\\ \mathbf{0}\end{pmatrix}\right]$$

Hence, it does not suffice that the explosion amplitudes of two stocks coincide since their portfolio weights can diverge (due to the "normal" risk). Instead, we have to demand the identity of the risk-neutralized[53] jump amplitudes of stock j and k - according to equation (20a).

On the other hand, identical volume factors will appear with just one stock j if the risk-neutralized crash and explosion sizes coincide:

$$(20b) \qquad E_t\left\{\tilde{\varphi}_{Cj}(t)\,J_W\left[j_C\right]\right\} = E_t\left\{\tilde{\varphi}_{Ej}(t)\,J_W\left[j_E\right]\right\}$$

and therefore in (17)

$$(21b) \qquad E_t\left\{\tilde{\varphi}_{Ej}(t)\,J_W\left[j_E\right]\right\}\left[W^{-1}\begin{pmatrix}\mathbf{0}\\ \lambda_{Cj}\\ \mathbf{0}\end{pmatrix}-W^{-1}\begin{pmatrix}\mathbf{0}\\ \lambda_{Ej}\\ \mathbf{0}\end{pmatrix}\right]$$

The third case combines identical crash and explosion probabilities for stock j with identical risk-neutralized crash and explosion amplitudes. Under these conditions, jumps of stock j do not contribute correction terms to the optimum portfolio weights (17) because the bracket in (19) becomes zero due to (20b).

To sum up, the discussion of the correction terms stresses the importance of jumps. The number of jump terms only diminishes for obvious trivial cases $\lambda_j = 0$ or rather $\lambda_{Ej} = \lambda_{Cj}$. Further reductions can merely occur in the pronounced special cases of equation (20a) and (20b) or in the limiting case $\varphi_{E\,kj} \to \infty$[54] for sufficiently risk avers investors.

In a final step, let us analyze the exact correction behavior of the correction terms. This task can be achieved most easily by using the fund representation: Calculating the return covariance between stock j and correction term i, yields

for i = j

$$(22a) \qquad \begin{aligned} \operatorname{cov}(R_j;R_{Hki}) &= \begin{pmatrix}\mathbf{0} & 1 & \mathbf{0}\end{pmatrix}\Omega\mathbf{H}_{kj}\,dt \\ &= \frac{1}{\begin{pmatrix}\mathbf{0} & \lambda_{kj} & \mathbf{0}\end{pmatrix}\Omega^{-1}\mathbf{1}}\cdot\begin{pmatrix}\mathbf{0} & 1 & \mathbf{0}\end{pmatrix}\Omega\,\Omega^{-1}\begin{pmatrix}\mathbf{0}\\ \lambda_{kj}\\ \mathbf{0}\end{pmatrix}dt \\ &= \frac{1}{\begin{pmatrix}\mathbf{0} & \lambda_{kj} & \mathbf{0}\end{pmatrix}\Omega^{-1}\mathbf{1}}\cdot\lambda_{kj}\,dt \end{aligned}$$

or rather,

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[53] In the spirit of option pricing theory under jumps, see Nietert (1997): 19.

[54] See the ceteris paribus analysis of the following chapter.

for $i \neq j$

(22b) $$\operatorname{cov}(R_j; R_{Hki}) = \frac{1}{(\mathbf{0} \quad \lambda_{ki} \quad \mathbf{0})\Omega^{-1}\mathbf{1}} \cdot (\mathbf{0} \quad \mathbf{0} \quad 1 \quad \mathbf{0})\Omega\Omega^{-1}\begin{pmatrix}\mathbf{0}\\ \lambda_{ki}\\ \mathbf{0}\end{pmatrix} dt$$

$$= 0$$

$(\mathbf{0} \quad 1 \quad \mathbf{0})$	vector that has in its i's column the number one and in every other column the number zero
R_j	stock j's return during period dt
R_{Hki}	return of the ith correction fund against jump k (k = E, C) during period dt

That means the correction portfolio $\mathbf{H}_{kj}$ eliminates, according to (22a), essentially[55] stock j's jump probability and, therefore, an important part of its jump risk. For that reason, the optimum portfolio weights of (17) are on the one hand able to consider huge loss dangers and on the other hand are able to take into account the chances of immense price explosions. Due to this, we establish equation (17) as an alternative to traditional Shortfall[56] and Portfolio Insurance approaches.

Although correction portfolio $\mathbf{H}_{kj}$ corrects class k jumps of stock j, it does not consist solely of stock j's weight changes, that is, $\mathbf{H}_{kj}$ is not identical with stock j: For, calculating the structural component of $\mathbf{H}_{kj}$, $\Omega^{-1}\begin{pmatrix}\mathbf{0}\\ \lambda_{kj}\\ \mathbf{0}\end{pmatrix}$, illustrates that we extract the jth column from the inverted variance/covariance matrix weighted with λ_{kj}. Thus, the correction term influences the portfolio weights of every stock because of complementary and substitutive connections between stocks' "normal" risk and our fund representation (18) proves to be nontrivial.

Moreover, our portfolio adaptation via correction terms reminds us of the portfolio strategy under stochastic opportunity set (Merton (1973a)/Breeden (1979)). - However, the similarity to Merton (1973a) and Breeden (1979) is restricted to the basic idea and diverges in its exact mechanism. There, under stochastic opportunity set, we have a vector $\mathbf{x}(t)$ influencing all stocks. Hence, we structure the optimum portfolio by means of correction terms against changes of $\mathbf{x}(t)$ in the following way: A loss in one stock due to e.g. an unfavorable change of $\mathbf{x}(t)$ should be (partially) offset by a gain of another stock reacting positively on the movement of $\mathbf{x}(t)$. Thus, there is direct compensation and for that reason, the correction terms are called "hedge terms". Here, under jumps, there are no correction terms against changes of $\mathbf{x}(t)$ in the equation of optimum portfolio weights (17) as, due to the firm-specific character of the jumps, there cannot be a common factor $\mathbf{x}(t)$. For that reason, we

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55 The constant $\frac{1}{(\mathbf{0} \quad \lambda_{kj} \quad \mathbf{0})\Omega^{-1}\mathbf{1}}$ and the weight $w_{Hkj}(t)$ investors hold of the correction portfolio eliminate each other, see equation (18).

56 See e.g. Albrecht (1993).

are only able to restructure the portfolio to deal with jumps, but a direct (causal) compensation effect does not exist. We therefore prefer the phrase "correction term".

5.2 Ceteris paribus analysis

To gain further insights into the optimum portfolio weights (17), we conduct a ceteris paribus analysis with respect to the jump parameters amplitude and probability. As a preparatory work, we separate the crash from the explosion case and use constant jump parameters because this will facilitate stronger results:
Since the optimum portfolio weights **w**(t) appear on both sides of equation (17), we use implicit differentiation and set in the crash case

$$\text{(23)}\qquad M(w_{Ck}(t), \varphi_{Ci}, \lambda_{Ci}, \gamma, v_{ki}, \alpha_i, r) = -\frac{J_W}{J_{WW} W(t)} \sum_{j=1}^{n} v_{kj}(\alpha_j - r) + \sum_{j=1}^{n} v_{kj} \lambda_{Cj} \varphi_{Cj} \frac{J_W[j_C]}{J_{WW} W(t)} - w_{Ck}(t) = 0$$

$w_{Ck}(t)$	stock k's portfolio weight if crashes are the only type of jumps
v_{kj}	kjth element of the inverted variance/covariance matrix

Let us start our analysis with a change of the crash amplitude φ_C. Basically, we are interested in two effects on k's portfolio weight: Changes of parameters of stock k itself and of another stock j, that is formally for i = (k, j):

$$\text{(24)}\qquad \frac{\partial w_{Ck}(t)}{\partial \varphi_{Ci}} = -\frac{\partial M / \partial \varphi_{Ci}}{\partial M / \partial w_{Ck}(t)}$$

The common denominator of above fractions reads

$$\text{(25)}\qquad \frac{\partial M}{\partial w_{Ck}(t)} = -v_{kk} \lambda_{Ck} \varphi_{Ck}^2 \frac{J_{WW}[k_C]}{J_{WW}} - 1$$

with negative sign since it generally holds $v_{kk} > 0$[57]. For the numerator we have

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[57] We can prove the positive sign of v_{kk} as follows: According to its definition, the (instantaneous) variance/covariance matrix Ω is positive definite, i.e. we have for every vector $\mathbf{m} \neq \mathbf{0}$ (see e.g. Karlin/Taylor (1975): 542): $\mathbf{m}^T \Omega \mathbf{m} > 0$. Defining $\mathbf{y} \equiv \Omega^{-1}\mathbf{m}$, yields $\mathbf{m} = \Omega \mathbf{y}$. In addition, the existence of Ω^{-1} assures $\Omega\, \mathbf{y} \neq \mathbf{0}$. Hence:
$\mathbf{m}^T \Omega^{-1} \mathbf{m} = \mathbf{y}^T \Omega \Omega^{-1} \Omega \mathbf{y} = \mathbf{y}^T \Omega \mathbf{y} > 0$, i.e. Ω^{-1} is positive definite.
Especially, we have for all identity vectors $\mathbf{e}_k$ (k = 1,...,n) $\mathbf{e}_k \Omega^{-1} \mathbf{e}_k > 0$. As $\mathbf{e}_k \Omega^{-1} \mathbf{e}_k$ (k = 1,...n) portrays the diagonal elements of Ω^{-1}, we have proven the positive sign of v_{kk}.

$$(26a)\quad \frac{\partial M}{\partial \varphi_{Ck}} = \frac{v_{kk}\lambda_{Ck}}{J_{WW}W(t)}\left[J_W\left[k_C\right] - J_{WW}\left[k_C\right]w_{Ck}(t)\varphi_{Ck}W(t)\right]$$

or rather,

$$(26b)\quad \frac{\partial M}{\partial \varphi_{Cj}} = \frac{v_{kj}\lambda_{Cj}}{J_{WW}W(t)}\left[J_W\left[j_C\right] - J_{WW}\left[j_C\right]w_{Cj}(t)\varphi_{Cj}W(t)\right]$$

The sign of (26a) is undetermined because the bracket has an ambiguous sign. The same is true for (26b). In addition, we do not know the sign of v_{kj}.

Hence, stock k's portfolio weight will increase, remain constant or decrease if its crash amplitude or the jump size of another stock rises. Complex complementary and substitutional connections due to the "normal" risk of the stocks disprove the naive statement "an increase in stock k's crash amplitude results in a reduction of its portfolio weight" and "an increase in stock j's crash amplitude results in an increase of stock k's portfolio weights". Thus, the connection to the "normal" risk is also responsible for the fact that some stocks have positive portfolio weights despite a crash and that there is no automatic short selling[58]. This result, economically not intuitive at the first glance, can be explained as follows: The cumulative effect trading off chances/risks of "normal" price movements against chances/risks of jumps consists on the one hand of a direct jump effect on portfolio weights (primary effect). On the other hand, portfolio weights necessary because of the primary effect induce consequences with respect to the "tolerance towards risk" $-\frac{J_W\left[k_C\right]}{J_W W(t)}$ (secondary effect). - An example may help to illustrate this statement: Stock j's crash demands a significant reduction of its portfolio weights (primary effect). But, these low portfolio weights do mean a loss of diversification against "normal" risk and a different evaluation of this weak diversification (secondary effect) compared to the pre-crash situation (risk tolerance is governed by the additional term $\frac{J_W\left[k_C\right]}{J_W}$). Hence, we end with a smaller reduction of the portfolio weights (cumulative effect).

In the special case $\frac{\partial w_{Ck}(t)}{\partial \varphi_{Ck}}$ however, we are able to exactly specify and interpret these conditions under which an increase in stock k's jump amplitude has negative or positive consequences on its portfolio weights. Restructuring the bracket in (26a), yields:

58 See the results of our numerical analysis in appendix 2.
By the same token, in the case of explosions investors do not solely hold the stock subject to jumps.

$$(27a) \qquad \underbrace{-\frac{J_W\left[(1-w_{Ck}(t)\varphi_{Ck})W(t)\right]}{J_{WW}\left[(1-w_{Ck}(t)\varphi_{Ck})W(t)\right]}}_{\substack{\text{absolute risk tolerance}\\ \text{measured}\\ \text{at the wealth after the crash}}} > \underbrace{-w_{Ck}(t)\varphi_{Ck}W(t)}_{\substack{\text{wealth change}\\ \text{due to the crash}}}$$ [59]

or rather, analogously derived, for the explosion:

$$(27b) \qquad \underbrace{-\frac{J_W\left[(1+w_{Ek}(t)\varphi_{Ek})W(t)\right]}{J_{WW}\left[(1+w_{Ek}(t)\varphi_{Ek})W(t)\right]}}_{\substack{\text{absolute risk tolerance}\\ \text{measured}\\ \text{at the wealth after the explosion}}} > \underbrace{w_{Ek}(t)\varphi_{Ek}W(t)}_{\substack{\text{wealth change}\\ \text{due to the explosion}}}$$

Only if the absolute risk tolerance measured at the wealth after the jump is larger than the wealth change caused by stock k's jump, will in the case of an increase of the jump amplitude φ_{Ck} (φ_{Ek}) stock k's portfolio weight decrease (increase), that is, will the primary effect outweigh the secondary effect. Otherwise, the secondary effect dominates. But, the secondary effect is unable to force the optimum portfolio weight of the stock subject to the jump to rise above (fall below) the one without a crash (an explosion). Due to strictly positive marginal utility, the correction term of stock k always decreases (increases) the portfolio weight in the crash case (in the explosion case). At worst, marginal utility of wealth after the jump and the volume factor approach zero. Therefore, despite an explosion investors hold the same portfolio weights as in the (pure) diffusion case when $\varphi_{Ek} \to \infty$ and the risk tolerance is smaller than specified in (27b). Under crashes, we can construct such a behavior only numerically. Economically, the crash amplitude is limited to - 100% and such extreme reactions do not occur[60].

We approach variations of the jump probability by analogy:

$$(28a) \qquad \frac{\partial M}{\partial \lambda_{Ck}} = v_{kk}\varphi_{Ck}\frac{J_W[k_C]}{J_{WW}W(t)}$$

or rather,

$$(28b) \qquad \frac{\partial M}{\partial \lambda_{Cj}} = v_{kj}\,\varphi_{Cj}\frac{J_W[j_C]}{J_{WW}W(t)}$$

According to equations (25) and (28a), the weights of stock k will decrease ($J_{WW} < 0$) if its crash probability increases. Due to the complex complementary and substitutional relation-

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59 Since $w_{Ck}(t)$ can be smaller than zero (see the figures of the numerical analysis in appendix 2), (27a) is no tautology, but a real constraint.

60 See the figures of the numerical analysis in appendix 2 concentrating on 50% jump probability, 1% or rather 50% amplitude and high or rather low relative risk aversion.

ship (the sign of v_{kj} is unequivocal) between "normal" risk, this statement is not valid for variations of stock j's crash probabilities.

Analyzing explosions, yields similar results: If stock k's explosion probability rises, its weight will raise and stock j's weights will react ambivalently.

As we find in the case $\frac{\partial w_k(t)}{\partial \lambda_k}$ that there is only the primary effect and that the "tolerance towards risk" $-\frac{J_W[k_C]}{J_{WW}W(t)}$ is not influenced by a variation of the jump probability, the unequivocal consequences of $\frac{\partial w_k(t)}{\partial \lambda_k}$ compared to $\frac{\partial w_K(t)}{\partial \varphi_k}$ become economically clear.

5.3 Numerical analysis

Finally, we will illustrate our theoretical results by means of numerical examples. Thereby, we want to firstly stress the extent of the jump induced portfolio weight modification, secondly the separation into structural and volume terms, and thirdly, the importance of "normal" risk for the optimum decision under jumps. Since our calculations serve solely for explanatory purposes, we choose a framework as simple as possible: We argue within the case of only two stock which are both subject to "normal" risk, but only stock 1 underlying jumps. In such an environment, we are able to completely control the correlation between the stocks and, moreover, interpret our results economically. In addition, we use, as already introduced in the ceteris paribus analysis, the simplest case of constant jump amplitudes. That way, we can portray extreme scenarios: an amplitude probably representing the lower boundary of a jump (1%) and a fairly fierce change (50%), without fine-tuning the probability distribution. Eventually, we have to specify the utility function to calculate explicitly the volume factors. To that end, we use isoelastic consumption ($\frac{C^{\gamma}(t)}{\gamma}$) and bequest utility functions ($\frac{W^{\gamma}(T)}{\gamma}$) yielding the following portfolio weights of stock 1 and 2[61]:

(29a)
$$
\begin{aligned}
w_1(t) = {} & \frac{1}{1-\gamma}\left[\frac{\sigma_2^2(\alpha_1 - r) - \sigma_1\sigma_2\eta_{12}(\alpha_2 - r)}{\sigma_1^2\sigma_2^2 - (\sigma_1\sigma_2\eta_{12})^2}\right] \\
& + \frac{1}{1-\gamma}\frac{\sigma_2^2}{\sigma_1^2\sigma_2^2 - (\sigma_1\sigma_2\eta_{12})^2}\lambda_{E1}\varphi_{E1}(1 + w_1(t)\varphi_{E1})^{\gamma-1} \\
& - \frac{1}{1-\gamma}\frac{\sigma_2^2}{\sigma_1^2\sigma_2^2 - (\sigma_1\sigma_2\eta_{12})^2}\lambda_{C1}\varphi_{C1}(1 - w_1(t)\varphi_{C1})^{\gamma-1}
\end{aligned}
$$

$1 - \gamma$ relative risk aversion

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[61] See Nietert (1996): S. 89.

or rather,

$$\text{(29b)} \qquad w_2(t) = \frac{1}{1-\gamma}\left[\frac{\sigma_1^2(\alpha_2 - r) - \sigma_1\sigma_2\eta_{12}(\alpha_1 - r)}{\sigma_1^2\sigma_2^2 - (\sigma_1\sigma_2\eta_{12})^2}\right]$$

$$+\frac{1}{1-\gamma}\frac{-\sigma_1\sigma_2\eta_{12}}{\sigma_1^2\sigma_2^2 - (\sigma_1\sigma_2\eta_{12})^2}\lambda_{E1}\varphi_{E1}(1+w_1(t)\varphi_{E1})^{\gamma-1}$$

$$-\frac{1}{1-\gamma}\frac{-\sigma_1\sigma_2\eta_{12}}{\sigma_1^2\sigma_2^2 - (\sigma_1\sigma_2\eta_{12})^2}\lambda_{C1}\varphi_{C1}(1-w_1(t)\varphi_{C1})^{\gamma-1}$$

In all[62], our calculations demonstrate the significant influence of jump terms on optimum portfolio weights: Even for relatively moderate jumps (e.g. 5% probability and 1% amplitude) optimum portfolio weights with and without jumps deviate from each other visibly with the result holding for both low and high risk aversion.

From the rows of the tables in appendix 2 we can moreover see the separation into structural component independent of investor's preferences and into volume factor dependent on investors' preferences. The structural term is responsible for the unique sign of portfolio weights for both low and high (relative) risk aversion. The volume factor is able to influence the weights' levels only and can definitely not change the sign of portfolio weights.

It becomes also clear that low jump probabilities considerably moderate the effect of jumps with a low jump probability (≈ structural term) exerting more influence than a low amplitude (≈ volume term). For instance, we could increase the jump probability from 0.5% to 5% by 10 times and decrease the amplitude by 50 times from 0.5 to 0.01. Nevertheless, the portfolio weights of the combination (0.5%;0.5) are for low risk aversion above those of (5%;0.01). - Thus, we can again illustrate the theoretically derived risk eliminating behavior of the correction terms: According to equation (22a) these care for the jump probability.

Finally, with the help of our numerical examples we want to emphasize the enormous importance of the interdependence between jumps and "normal" risk. "Normal" risk makes a decision optimal that does not immediately short sell the crashing stock or rather invests solely in the exploding stock. On the contrary, we have to take into account complementary and substitutional connections apostrophized in the theoretical part of our article. Moreover, these connections can be handled in our simplified 2-stock-framework: In the case of positive correlation of stock 1's and 2's "normal" risk, a crash of stock 1 causes a rise in the portfolio weight of stock 2 (and the riskless asset) financed by a decrease of stock 1's weight. For, both stocks have a substitutional relationship: Higher correlation weakens diversification of "normal" risk and increases total risk. Since jumps are independent of "normal" risk, the fundamental substitutional relationship from the "normal" risk remains valid in a jump/diffusion environment. - In the case of negative correlation of "normal" risk, both stocks lose significant portfolio weights in favor of the riskless asset and stock 2 behaves similar to a stock that is itself subject to crashes. The reason for this can be found in the

[62] For the figures in detail, please refer to the tables of appendix 2.

complementary relationship between both stocks: Due to their contrary movement regarding "normal" risk, investors combine both stocks to facilitate portfolio variance reductions of "normal" risk below the variance of each single stock. Because of the cooperation of both stocks, just one stock promises a lower "utility" and the observed portfolio weight adaptations follow to maintain the correction effect with respect to the "normal" risk. - If "normal" risk is uncorrelated, a crash-induced decrease in portfolio weights of stock 1 will solely foster the weight of the riskless asset; the weight of stock 2 remains constant because there is no link to stock 1 either from the diffusion or the jumps side of stock 1.

Analogous results hold in the case of price explosions: Substitutional connections lead to a weight increase of the exploding stock 1 at cost of the not jumping stock 2 and the riskless asset, complementary connections to an increase of both stocks' weights at cost of the weight of the riskless asset. In the case of uncorrelated "normal" risk, the weight of stock 2 does not change, the weight of the riskless asset decreases.

In this way, the extreme case of 500% jump probability and 50% amplitude combined with a high risk aversion ($\gamma = -9$) demonstrates the theoretically forecasted convergence behavior against the portfolio weights under "normal" risk. For, portfolio weights are under those circumstances far less extreme than in the case of 50% jump probability and 1% amplitude.

6 Summary

Stock price movements are characterized by occasional extreme vibrations, resulting in dramatic losses during stock market crises. This can be demonstrated by the October crises in 1987 and 1989 as well as the Southeast Asia, Russia and Brazil crises in 1998/99. Therefore, we raised the question of how crashes could be neutralized by improved portfolio planning. Our analysis has produced the following results:

- To be economically reasonable, jump models have to be distinguished between firm-specific, cluster-specific and market jumps (scope of the jump) as well as crashes and explosions (direction of the jump). In addition, jump amplitudes have to be bounded from both below and above.
- Jumps call for a completely new portfolio theory: We have to integrate jump risks explicitly into portfolio planning via correction terms.
- Therefore, the classical Tobin separation is no longer valid, but an extended Tobin separation holds. - We need 2 n + 1 funds in lieu to span the risk characteristics of n shares.

Appendix:

1 Derivation of the modified Hamilton/Jacobi/Bellman equation under combined jump/diffusion processes

The Hamilton/Jacobi/Bellman equation is the prerequisite for determining optimum portfolio weights. Therefore, we have to develop its adaptation to firm-specific jumps. For the ease of exposition, we will commence with the analysis of just two stocks with only stock 1 subject to jumps; the n-stock case will follow by induction:
We start our derivation at the definition of the expected utility of the optimum strategy from t to T:

(7) $$J[W(t),t] \equiv \underset{C,w}{\mathrm{Max}}\; E_t\left\{\int_t^T e^{-\rho s}\, U[C(s)]\, ds + e^{-\rho T} B[W(T)]\right\}$$

Due to the Bellman principle of dynamic programming and the linear approximation of the consumption integral between t and $t+\Delta t$, we can write for (7)

(7') $$J[W,t] = \underset{C,w}{\mathrm{Max}}\left\{e^{-\rho t}\, U[C(t)]\Delta t + E_t\left\{J[W+\Delta W, t+\Delta t]\right\}\right\}$$

Δ change of a quantity

The term $E_t\left\{J[W+\Delta W, t+\Delta t]\right\}$ is the only unknown in (7'). Its development depends on the occurrence of jumps, i.e. consists of the expected changes conditional on the event "a jump occurs /no jump occurs"[63]. Formally:

(A1.1) $$E_t\left\{J[W+\Delta W, t+\Delta t]\right\} = E_t\left\{\underbrace{E_t\left\{J[W+\Delta W_D, t+\Delta t]\right\}}_{\text{mean conditional on the event "no jump occurs"}} + \underbrace{E_t\left\{J[W+\Delta W_S, t+\Delta t]\right\}}_{\text{mean conditional on the event "W jumps"}}\right\}$$

ΔW_D change of wealth due to the diffusion component
ΔW_S change of wealth due to stock 1's price jumps

The probability that a wealth jumps occurs within the period between t and $t+\Delta t$ is identical to the probability of stock 1 jumping and equals $\lambda_1 \Delta t$. Combining the jump probabilities of W with the appropriate values of J[.] and expanding referring to all variables not subject to jumps, we can calculate the "outer" mean in (A1.1) explicitly and get

[63] See Tapiero (1998): 255 p.

(A1.2) $$E_t\left\{J\left[W+\Delta W, t+\Delta t\right]\right\} =$$
$$+\left(1-\lambda_1\,\Delta t\right)\left[J\left[W,t\right]+J_W\,\Delta W_D+J_t\,\Delta t+\frac{1}{2}J_{WW}\,(\Delta W_D)^2+o\left(\Delta t\right)\right]$$
$$+\ \lambda_1\,\Delta t\left[E_t\left\{J\left[\left(1+w_1(t)\tilde{\varphi}_1(t)\right)W(t),t\right]\right\}+J_t\,\Delta t+o\left(\Delta t\right)\right]$$

Substituting (A1.2) into (7'), dividing by Δt and taking limits $\Delta t \to 0$, we gain the following Hamilton/Jacobi/Bellman equation

(A1.3) $$0 = \underset{C,\mathbf{w}}{\mathrm{Max}}\left\{e^{-\rho t}U\left[C(t)\right]+\lambda_1\left[E_t\left\{J\left[(1+w_1(t)\tilde{\varphi}_1(t))W(t),t\right]\right\}-J\left[W,t\right]\right]\right.$$
$$+J_t+J_W\left(r\,W(t)-C(t)\right)+J_W\left[w_1(t)\left(\alpha_1-r\right)+w_2(t)\left(\alpha_2-r\right)\right]W(t)$$
$$\left.+\frac{1}{2}J_{WW}W^2(t)\left[w_1^2(t)\sigma_1^2+w_2^2(t)\sigma_2^2+2w_1(t)w_2(t)\sigma_1\sigma_2\eta_{12}\right]\right\}$$

For the case of n stocks inclusive distinction between crashes and explosions, we get by analogy:

(A1.4) $$0 = \underset{C,\mathbf{w}}{\mathrm{Max}}\left\{e^{-\rho t}U\left[C(t)\right]\right.$$
$$+\sum_{j=1}^{n}\lambda_{Ej}\left(E_t\left\{J\left[(1+w_j(t)\,\tilde{\varphi}_{Ej}(t))W,t\right]\right\}-J\left[W,t\right]\right)$$
$$+\sum_{j=1}^{n}\lambda_{Cj}\left(E_t\left\{J\left[(1-w_j(t)\tilde{\varphi}_{Cj})(t)W,t\right]\right\}-J\left[W,t\right]\right)$$
$$+J_t+J_W\left(r\,W(t)-C(t)\right)+J_W\,\mathbf{w}^T(t)\left(a\,-\mathbf{1}\,r\right)W(t)$$
$$\left.+\frac{1}{2}J_{WW}W^2(t)\mathbf{w}^T(t)W\,\mathbf{w}(t)\right\}$$

2 Numerical analysis

table of the assets' basic data:

stock 1		stock 2		riskless asset
$\alpha_1 = 0.11$	$\sigma_1 = 0.2$	$\alpha_2 = 0.12$	$\sigma_2 = 0.22$	$r = 0.08$

table of the portfolio weights:

solely "normal" risk					
$\gamma = 0$			$\gamma = -9$		
$\eta_{12} = 0.7$	$\eta_{12} = 0$	$\eta_{12} = -0.7$	$\eta_{12} = 0.7$	$\eta_{12} = 0$	$\eta_{12} = -0.7$
$w_1 = 0.2228$ $w_2 = 0.6847$ $w_0 = 0.0925$	$w_1 = 0.75$ $w_2 = 0.8265$ $w_0 = -0.5765$	$w_1 = 2.7184$ $w_2 = 2.5563$ $w_0 = -4.2747$	$w_1 = 0.0223$ $w_2 = 0.0685$ $w_0 = 0.9093$	$w_1 = 0.075$ $w_2 = 0.0827$ $w_0 = 0.8424$	$w_1 = 0.2718$ $w_2 = 0.2556$ $w_0 = 0.4725$

"normal" risk and additional crashes					
$\gamma = 0$			$\gamma = -9$		
$\eta_{12} = 0.7$	$\eta_{12} = 0$	$\eta_{12} = -0.7$	$\eta_{12} = 0.7$	$\eta_{12} = 0$	$\eta_{12} = -0.7$
with jump probability = 0.5 % and amplitude = -1 %					
$w_1 = -0.6538$ $w_2 = 1.2425$ $w_0 = 0.4113$	$w_1 = 0.2987$ $w_2 = 0.8265$ $w_0 = -0.1251$	$w_1 = 1.8197$ $w_2 = 1.9844$ $w_0 = -2.8041$	$w_1 = -0,0654$ $w_2 = 0.1243$ $w_0 = 0.9411$	$w_1 = 0.0299$ $w_2 = 0,0827$ $w_0 = 0.8875$	$w_1 = 0.1820$ $w_2 = 0.1985$ $w_0 = 0.6196$
with jump probability = 0.5% and amplitude = -50 %					
$w_1 = -8.3239$ $w_2 = 6.1235$ $w_0 = 3.2004$	$w_1 = -5.3623$ $w_2 = 0.8265$ $w_0 = 5.5358$	$w_1 = -7.0411$ $w_2 = -3.6542$ $w_0 = 11.6953$	$w_1 = -0.4836$ $w_2 = 0.3904$ $w_0 = 1.0932$	$w_1 = -0.3582$ $w_2 = 0.0827$ $w_0 = 1.2765$	$w_1 = -0.4104$ $w_2 = -0.1785$ $w_0 = 1.5890$
with jump probability = 5% and amplitude = -1 %					
$w_1 = -7.9508$ $w_2 = 5.8861$ $w_0 = 3.0648$	$w_1 = -3.5939$ $w_2 = 0.8265$ $w_0 = 3.7674$	$w_1 = -5.6345$ $w_2 = -2.7592$ $w_0 = 9.3937$	$w_1 = -0.7931$ $w_2 = 0.5873$ $w_0 = 1.2057$	$w_1 = -0.3592$ $w_2 = 0.0827$ $w_0 = 1.2765$	$w_1 = -0.5624$ $w_2 = -0.2753$ $w_0 = 1.8376$
with jump probability = 5% and amplitude = -50 %					
$w_1 = -28.6063$ $w_2 = 19.0305$ $w_0 = 10.5759$	$w_1 = -19.8474$ $w_2 = 0.8265$ $w_0 = 20.0210$	$w_1 = -27.3474$ $w_2 = -16.5765$ $w_0 = 44.9239$	$w_1 = -0,933858$ $w_2 = 0,6769179$ $w_0 = 1,25694$	$w_1 = -0.7752$ $w_2 = 0.0827$ $w_0 = 1.6926$	$w_1 = -0.8798$ $w_2 = -0.4772$ $w_0 = 2.3570$
with jump probability = 50% and amplitude = -1 %					
$w_1 = -56.2484$ $w_2 = 36.6209$ $w_0 = 20.6275$	$w_1 = -33.0674$ $w_2 = 0.8265$ $w_0 = 33.2410$	$w_1 = -54.4211$ $w_2 = -33.8051$ $w_0 = 89.2262$	$w_1 = -5.2615$ $w_2 = 3.4309$ $w_0 = 2.8306$	$w_1 = -3.2069$ $w_2 = 0.0827$ $w_0 = 4.1243$	$w_1 = -5.0959$ $w_2 = -43.1602$ $w_0 = 9.2560$
with jump probability = 50% and amplitude = -50 %					
$w_1 = -92.8264$ $w_2 = 59.8978$ $w_0 = 33.9286$	$w_1 = -65.7100$ $w_2 = 0.8265$ $w_0 = 65.8835$	$w_1 = -91.5752$ $w_2 = -57.4485$ $w_0 = 150.0238$	$w_1 = -1,5209$ $w_2 = 1.0505$ $w_0 = 1.4704$	$w_1 = -1.3240$ $w_2 = 0,0827$ $w_0 = 2.2414$	$w_1 = -1.4771$ $w_2 = -0.8573$ $w_0 = 3.3344$

"normal" risk and additional explosions					
$\gamma = 0$			$\gamma = -9$		
$\eta_{12} = 0.7$	$\eta_{12} = 0$	$\eta_{12} = -0.7$	$\eta_{12} = 0.7$	$\eta_{12} = 0$	$\eta_{12} = -0.7$
with jump probability = 0.5 % and amplitude = 1 %					
$w_1 = 1.0956$ $w_2 = 0.1292$ $w_0 = -0.2249$	$w_1 = 1.1947$ $w_2 = 0.8265$ $w_0 = -1.0211$	$w_1 = 3.5703$ $w_2 = 3.0985$ $w_0 = -5.6688$	$w_1 = 0.1096$ $w_2 = 0.0129$ $w_0 = 0.8775$	$w_1 = 0.1195$ $w_2 = 0.0827$ $w_0 = 0.7979$	$w_1 = 0.3570$ $w_2 = 0.3098$ $w_0 = 0.3332$
with jump probability = 0.5% and amplitude = 50 %					
$w_1 = 8.5703$ $w_2 = -4.6274$ $w_0 = -2.9429$	$w_1 = 6.2227$ $w_2 = 0,8265$ $w_0 = -6.0491$	$w_1 = 10.0443$ $w_2 = 7.2183$ $w_0 = -16.2626$	$w_1 = 0.4986$ $w_2 = -0.2347$ $w_0 = 0.7360$	$w_1 = 0.4156$ $w_2 = 0,0827$ $w_0 = 0,5018$	$w_1 = 0.5964$ $w_2 = 0.4622$ $w_0 = -0.0581$
with jump probability = 5% and amplitude = 1 %					
$w_1 = 8.3652$ $w_2 = -4.4969$ $w_0 = -2.8683$	$w_1 = 5.0343$ $w_2 = 0.8265$ $w_0 = -4.8608$	$w_1 = 10.6898$ $w_2 = 7.6290$ $w_0 = -17.3190$	$w_1 = 0.8343$ $w_2 = -0.4483$ $w_0 = 0.6140$	$w_1 = 0.5030$ $w_2 = 0.0827$ $w_0 = 0.41438$	$w_1 = 1.0655$ $w_2 = 0.7607$ $w_0 = -0.8261$
with jump probability = 5% and amplitude = 50 %					
$w_1 = 28.8366$ $w_2 = -17.5241$ $w_0 = -10.3125$	$w_1 = 20.6327$ $w_2 = 0.8265$ $w_0 = -20.4592$	$w_1 = 30.1574$ $w_2 = 20.0174$ $w_0 = -49.1745$	$w_1 = 0.9445$ $w_2 = -0.5184$ $w_0 = 0.5739$	$w_1 = 0.8143$ $w_2 = 0.0827$ $w_0 = 0.1031$	$w_1 = 1.0105$ $w_2 = 0.7257$ $w_0 = -0.7363$
with jump probability = 50% and amplitude = 1 %					
$w_1 = 56.5759$ $w_2 = -35.1764$ $w_0 = -20.3995$	$w_1 = 34.2657$ $w_2 = 0.8265$ $w_0 = -34.0921$	$w_1 = 58.4167$ $w_2 = 38.0007$ $w_0 = -95.4173$	$w_1 = 5.2912$ $w_2 = -3.2845$ $w_0 = -1.0067$	$w_1 = 3.3209$ $w_2 = 0,0827$ $w_0 = -2.4035$	$w_1 = 5.4580$ $w_2 = 3.5559$ $w_0 = -8.0139$
with jump probability = 50% and amplitude = 50 %					
$w_1 = 93.0516$ $w_2 = -58.3882$ $w_0 = -33.6634$	$w_1 = 66.4711$ $w_2 = 0.8265$ $w_0 = -66.2976$	$w_1 = 94.3225$ $w_2 = 60.8498$ $w_0 = -154.1723$	$w_1 = 1.5293$ $w_2 = -0.8905$ $w_0 = 0.3613$	$w_1 = 1.3540$ $w_2 = 0.0827$ $w_0 = -0.4366$	$w_1 = 1.5796$ $w_2 = 1.0879$ $w_0 = -1.6675$

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