# A comparison of analytical VaR methodologies for portfolios that include options

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**Abstract**

It is the main objective of this paper to compare different approaches to analytically calculate value-at-risk (VaR) for portfolios that include options. We focus on approaches that are based on a second order Taylor-series approximation of the nonlinear option pricing relationship. The main difficulty common to all these methods is the estimation of the required quantile of the profit and loss distribution, since there exists no analytical representation of this distribution. In our analysis we examine different moment matching approaches and methods to directly approximate the required quantile. For this purpose, we perform a backtesting procedure based on randomly generated risk factor returns which are multivariate normal. The VaR-numbers calculated by a specific methodology are then compared to the simulated actual losses. We conclude that the accuracy of methodologies that rely only on the first four moments of the profit and loss distribution is rather poor. The inclusion of higher moments, e.g. through a Cornish-Fisher expansion seems to be appropriate.

## 1 Motivation

During the last decade value-at-risk (VaR) has become one of the most important risk measurement tools in financial institutions as well as in other corporations facing considerable market risk. One of the major features that make VaR attractive to risk managers across different institutions is its analytical tractability. Although numerical methods to calculate VaR have been developed leading to more accurate results depending on less restrictive assumptions, many institutions seem to still rely on analytical methodologies. The most important advantage of analytical methods over their numerical counterparts - the saving of computing time that makes real-time calculations possible - seems to outweigh their disadvantages for many practical applications. The motivation of this paper is thus based on the need to improve the accuracy of analytical VaR methodologies and simultaneously make the underlying assumptions less restrictive.

The VaR of a portfolio is defined as the maximum loss that will occur over a given period of time at a given probability level. The calculation of VaR numbers requires some assumptions about the distributional properties of the returns of the portfolio components. The common delta-normal approach originally promoted by JP Morgan´s RiskMetrics software is based on the assumptions of normally distributed returns of prespecified risk factors. In the case of a strictly linear relationship between the returns of the risk factors and the market value of the portfolio under consideration there exists a simple analytic solution for the VaR of the portfolio. This simple analytic solution does not hold for portfolios that include financial instruments with non-linear payoffs like options. Since the relationship between the normally distributed returns of the risk factors (underlyings, interest rates, etc) and the value of the options is nonlinear, the distribution of the portfolio value is no longer normal. It can be shown that for portfolios with a high degree of nonlinearity this distribution shows extremely high skewness and excess kurtosis. This makes a reasonable VaR-calculation using the delta-normal approach impossible.

A first step to solve this problem is to include the quadratic term of a Taylor-series expansion of the option pricing relation, i.e. the gamma matrix, in the VaR calculation framework. The inclusion of quadratic terms implies a distribution of portfolio values that may be described as a linear combination of non-central $\chi^2$-distributed random variables. Fortunately, this distribution was shown to be equivalent to the distribution of a random form in normally distributed random variables for which at least the moment-generating function exists (see Mathai and Provost (1992)). There are several attempts presented in the literature to incorporate higher moments of this distribution in approximation procedures to calculate the required quantile of the distribution. This paper will focus on this class of methods to incorporate nonlinearity in the VaR calculation.

In a first attempt Zangari (1996a) suggested to use the Cornish-Fisher approximation to directly calculate the quantile of a distribution with known skewness and kurtosis. Other approaches try to find a moment matching distribution for which the quantiles can be calculated. This class of approaches contains Zangari (1996b) who suggested to use the Johnson family of distributions to match the first four moments, Britten-Jones and Schaefer (1997) who suggested to use a central $\chi^2$-distribution to match the first three moments, and a simplifying approach that uses the normal distribution to match the first two moments (for a discussion of this approach, see El-Jahel, Perraudin, and Sellin (1999)).

The latter approach might be justified by applying the central limit theorem for portfolios with a gamma matrix of very large dimension. However, based on a simplified setting Finger (1997) argues that this application will only hold for uncorrelated risk factors. We provide additional analytic results for more general cases where the distribution of the portfolio value does not ‚converge‘ to a normal distribution even for uncorrelated risk factors , whereas we can show that there are cases where the distribution of the portfolio value ‚converges‘ to a normal distribution even for correlated risk factors. It depends on the structure of the gamma-matrix rather than the structure of the covariance matrix whether a ‚convergence‘ is achieved or not. Since it is hard to generalize these analytic results, this approach is included in our numerical analysis.

It is the main objective of this paper to compare the approaches mentioned above to calculate VaR for portfolios that include options. We perform a backtesting procedure based on randomly generated risk factor returns which are multivariate normal. These returns are used to calculate a simulated time-series of profits and losses given the portfolio composition determined by an

N-dimensional vector of deltas and an N×N dimensional matrix of gammas. The VaR-number calculated by a specific methodology is then compared to the simulated actual losses. The perfomance of the different methodologies is measured by the amount of deviation of the percentage of cases where the simulated actual loss exceeds the VaR from the required probability. Additionally, we provide likelihood ratio statistics to test for significance of our results.

In a recent paper, El-Jahel, Perraudin, and Sellin (1999) presented a methodology to calculate the moments of the portfolio‘s profit and loss distribution even for nonnormal risk factors. Under fairly general conditions, the knowledge of the moments of the distribution of the risk factors is sufficient to calculate the moments of the distribution of the portfolio. This leads to the same situation where the quantiles of this distribution have to be calculated. We have to stress the fact that the results of our paper are not limited to the case of normally distributed risk factors but are also relevant for nonnormal risk factors as long as the calculation scheme is based on a quadratic approximation of the nonlinear pricing relationship.

The outline of this paper is as follows: Section 2 describes problems arising for VaR methodologies when options are included and shows the distributional properties of a quadratic Taylor-series approximation of the portfolio‘s profit and loss distribution. Section 3 gives an overview over differnet methodologies that were developed to calculate quantiles of this distribution. Section 4 describes the Monte Carlo backtesting procedure used to evaluate the different approaches and summarizes the results of this evaluation procedure. Section 5 concludes the paper.

## 2 Analytic VaR for portfolios that include options

The delta-normal approach originally promoted by JP Morgan´s RiskMetrics software is based on two major assumptions.

*Assumption 1 (linearity):* The change in the value of the portfolio over a given interval of time is linear in the returns of $N < \infty$ risk factors.

Let $W$ denote the market value of the portfolio under consideration at a fixed point of time and $S_k$ , $k = 1, ...., N$, the contemporaneous value of the $k$-th risk factor, then assumption 1 can be formalized as

$$\Delta W = \sum_{k=1}^{N} \delta_k \cdot \frac{\Delta S_k}{S_k}, \qquad \delta_k = \frac{\partial W}{\partial S_k} S_k , \tag{1}$$

where $\delta_k$ denotes the factor sensitivity of the portfolio with respect to factor $k$. Using matrix notation we have $\Delta W = \delta^T R$, where $\delta$ denotes the $N \times 1$-vector of factor sensitivities and $R$ denotes the $N \times 1$-vector of factor returns ($\Delta S_k / S_k$ ).

*Assumption 2 (normality):* The returns of the risk factors follow a multivariate normal distribution.

We have $R \sim N(\mu, \Sigma)$, where $\Sigma$ denotes $N \times N$ covariance matrix of factor returns and $\mu$ denotes the $N \times 1$-vector of expected factor returns. Note, that in many applications the additional assumption $\mu = 0$ is made. In order to simplify the notation and the interpretation of our results the analysis of this paper follows this assumption.

These assumptions imply that the distribution of $\Delta W$ itself is normal and the VaR of the portfolio given $\mu = 0$ can be written as

$$VaR = -\sigma_{\Delta W} \cdot \Phi(\alpha),$$

where $\Phi(\alpha)$ denotes the $\alpha$-quantile of the standard normal distribution and $\sigma_{\Delta W}$ denotes the standard deviation of the distribution of $\Delta W$. The standard deviation $\sigma_{\Delta W}$ expressed in units of cash is given by

$$\sigma_{\Delta W} = \sqrt{\delta^T \cdot \Sigma \cdot \delta}\,.$$

The linearity assumption is crucial to preserve he normality property of the distribution of $\Delta W$. A nonlinear relationship between $\Delta W$ and the factor returns will in general lead to a nonnormal distribution of $\Delta W$. Numerical examples with portfolios containing positions in one single option or straddle show that this distribution can show extreme skewness and kurtosis. This makes a purely linear approximation of nonlinear instruments questionable.

It is a common approach in option markets as well as in the academic literature to try to approximate the nonlinearity through a quadratic Taylor-series expansion. We have to emphasize that the impact of higher order terms cannot be neglected in some cases. However, the analysis of the influence of third or fourth order terms is beyond the scope of this paper.

The quadratic Taylor-series expansion replaces (1) by the approximation

$$\Delta W = \delta^T R + \frac{1}{2} R^T \Gamma R, \qquad (2)$$

where $\Gamma$ denotes the $N \times N$ matrix of gamma terms

$$\Gamma_{ij} = \frac{\partial^2 W}{\partial S_i \partial S_j} S_i S_j\,.$$

For $N = 1$, the distribution of (2) is a noncentral $\chi^2$. For $N \geq 2$, the distribution can be shown to be the distribution of a quadratic form in normally distributed random variables (see Britten-Jones and Schaefer (1997)). Unfortunately, there exists no analytical expression for its density or distribution function. However, since the moment generating function of this distribution is known, one can easily calculate its moments given $\delta$, $\Gamma$, and $\Sigma$.

## 3 Methods to calculate quantiles of $\Delta W$

Since a direct calculation of the quantiles of $\Delta W$, e.g. through numerical integration, is not possible, two different approaches have been suggested to find approximations of the desired quantiles. The first approach is based on approximating the distribution of $\Delta W$ by finding a distribution which quantiles can be calculated using a moment matching procedure. The second approach employs Cornish-Fisher expansions (see Johnson and Kotz (1970)) to directly find approximations for the required quantiles.

It is common to both approaches that the moments of the distribution of $\Delta W$ have to be known. Given the portfolio structure, $\delta$ and $\Gamma$, and the distribution of the factor returns, $\Sigma$, $(\mu = 0)$, we have (see Mathai and Provost (1992))

$$E(\Delta W) = \frac{1}{2} tr[\Gamma\Sigma] \qquad VAR(\Delta W) = \delta^T \Sigma \delta + \frac{1}{2} tr[\Gamma\Sigma]^2, \tag{3}$$

i.e., an ‚adjustment term‘ is added to the expected value and the variance of $\Delta W$ compared to linear cases where $\Gamma = 0$. The trace of the matrix $\Gamma\Sigma$ is the sum of the $N$ eigenvalues of $\Gamma\Sigma$, the trace of $(\Gamma\Sigma)^2$ equals the sum of the squared eigenvalues of $\Gamma\Sigma$. Letting

$$X = \frac{\Delta W - E(\Delta W)}{\sqrt{VAR(\Delta W)}}$$

denote the standardized values of $\Delta W$, the higher moments of $X$ with $r \geq 3$ are given by

$$E(X^r) = \frac{\frac{1}{2} r! \, \delta^T \Sigma [\Gamma\Sigma]^{r-2} \delta + \frac{1}{2} (r-1)! \, tr[\Gamma\Sigma]^r}{VAR(\Delta W)^{\frac{r}{2}}} \tag{4}$$

For $r = 3$ we have the skewness and $r = 4$ gives the kurtosis of $\Delta W$.

Based on the fact that $\Sigma$ is symmetric and positive semidefinite it can be shown that there exists an orthogonal transformation $\Sigma^* = C^T \Sigma C$ that diagonalizes $\Sigma$, where $C$ denotes the matrix of eigenvectors of $\Sigma$, i.e. there exists a set of uncorrelated risk factors explaining $\Delta W$. Assuming without loss of generality that $\Sigma$ is positive definite, there exists a transformation $\Sigma^{**} = \Lambda \Sigma^* \Lambda$, where $\Lambda$ is an $N \times N$ diagonal matrix with elements $\Lambda_{ii} = \lambda_i^{-0.5}$, $\lambda_i$ denoting the $i$-th eigenvalue of $\Sigma$, where $\Sigma^{**}$ is the $N \times N$ identity matrix. Since one has free choice of the risk factors in (1), we conclude that the distributional properties of $\Delta W$ solely depend on the structure of the transformed vector of factor sensitivities, $\delta^* = B^T \delta$, and the transformed gamma matrix, $\Gamma^{**} = B \Gamma B^T$, where $B^T B = \Sigma$ denotes the Cholesky decomposition of $\Sigma$. The economic interpretation of this result is the following: It is always possible to find an orthogonal basis of risk factors being uncorrelated to each other and having unit variance. Of course, due to a different choice of risk factors the first and second order derivatives used by the Taylor series expansion change as well. In order not to complicate the economic interpretation of the influence of the portfolio structure on the perfomance of different estimation methods, we use the original inputs $\delta$, $\Gamma$, and $\Sigma$. In some cases where a closer inspection of the distributional properties of $\Delta W$ seems to be necessary, we refer to the transformed inputs $\delta^{**}$, $\Gamma^{**}$, and $\Sigma^{**}$.

### 3.1 Moment matching approaches

A first simple attempt to calculate quantiles of $\Delta W$ is to approximate the distribution of the quadratic form by a normal distribution with paramters given by (3). This approach is critically discussed in Finger (1997) and El-Jahel, Perraudin und Sellin (1999). This approach might be justified by applying the central limit theorem for portfolios with a gamma matrix of very large dimension. However, note that properties like ‚convergence to a normal distribution‘ can only be defined with respect to a specific sequence of $\delta$, $\Gamma$, and $\Sigma$ with growing dimension. In this rather heuristic context, ‚convergence to a normal distribution‘ means ‚convergence of the moments of a distribution to the moments of a normal distribution with $N \to \infty$ with respect to a specific sequence of $\delta$, $\Gamma$, and $\Sigma$‘. Finger (1997) argues that this ‚convergence‘ can only apply in the case of uncorrelated risk factors. In contrast to his findings we have shown above that the distributional properties of $\Delta W$ depend on $\delta^{**}$ and $\Gamma^{**}$ and, therefore, are independent of $\Sigma$. It can be shown analytically that the ‚convergence‘ to the moments of a normal distribution only depends on the sequence of eigenvalues of $\Gamma^{**}$ with growing $N$. The following examples should illustrate this result: (i) Consider a *full* gamma matrix $\Gamma^{**}$ with all elements equal to a nonzero constant. In this case all but one eigenvalues are zero (due to the obvious linear dependence) and this implies that the moments given by (4) do not converge to zero. It can be shown that Finger‘s analysis relies on a similar structure which is only a special case in a more general setting. (ii) Consider a *diagonal* gamma matrix $\Gamma^{**}$ with all *diagonal* elements equal to a nonzero constant with changing sign. In this case all eigenvalues are nonzero (due to the obvious linear independence) and sum up to zero for $N$ even. This implies that the moments given by (4) do converge to zero even if the original risk factors are correlated. This contradicts Finger‘s conclusion. There always exists a feasible structure of $\Gamma^{**}$ and thus of $\Gamma$ that ‚diversifies away‘ optionality, in the sense that the distribution of a sufficiently ‚large‘ portfolio ‚converges‘ to a normal distribution. Since it is hard to generalize these analytic results to arbitrary structures of $\Gamma$, this approach is included in our numerical analysis.

Britten-Jones und Schaefer (1997) suggested to approximate the distribution of $\Delta W$ through a central $\chi^2$-distribution matching the first three moments. The moment matching procedure requires to solve a nonlinear system of equations that leads to numerically instable solutions in a relevant number of cases. Besides this numerical disadvantages this approach is not able to take into account the kurtosis of the distribution which might be a desirable property. As a consequence, this approach is not included in our numerical analysis.

Zangari (1996b) presented an approach which takes into account the first four moments of the distribution (see also RiskMetrics, Technical Document, Chpt. 6.3.3). Given the expected value, the variance, the skewness, and the kurtosis, a member of the Johnson family of distributions is chosen to approximate the original distribution. More specifically $\Delta W$ is approximated by a transformed standard normal variable $Y = f^{-1}(Z)$, $Z \sim N(0,1)$. The specific choice of the transformation function depends on the ratio of the square root of the skewness and the kurtosis of $\Delta W$ (see Johnson and Kotz (1970), Chpt. 12.4.3). The following family of three transformation functions is sufficient to cover all possible combinations of the first four moments:

$$Z = a + bf\left(\frac{Y - c}{d}\right), \qquad f(u) = \begin{cases} \ln(u) & \dots \; \textit{lognormal} \\ \sinh^{-1}(u) & \dots \; \textit{unbounded} \\ \ln\left(\frac{u}{1-u}\right) & \dots \; \textit{bounded} \end{cases}$$

The parameter values for *a*, *b*, *c*, and *d* are chosen through a moment matching algorithm that has an analytical solution only in the lognormal case. In the unbounded and bounded cases we make use of the iterative algorithm by Hill, Hill and Holder (1976), which is shown to be convergent in all possible cases. Based on the calculation of the parameter values the $\alpha$-quantile of the Johnson distribution $J(\alpha)$ can easily be obtained through

$$J(\alpha) = c + d f^{-1}\left(\frac{\Phi(\alpha) - a}{b}\right).$$

There is no moment matching approach established in the financial literature that takes into account higher moments than the kurtosis. A possible extension in this direction could be the use of maximum entropy distributions which allow the incorporation of an arbitrary number of moments. However, there are a lot of numerical issues to be solved. Thus, we leave this analysis to a later version of this paper.

### 3.2 Direct quantile approximation

This approach was first suggested in the risk management literature by Zangari (1996a). The direct approximation of the required quantiles of the distribution of $\Delta W$ is based on the Cornish-Fisher expansion around the quantile of a standard normal distribution. This approach leads to an analytic approximation of the quantile as long as the moments of the distribution are known. As an example, the Cornish-Fisher approximation taking into account the first four moments yields

$$\alpha\text{-}quantile \approx \Phi(\alpha) + \frac{1}{6}\left[\Phi(\alpha)^2 - 1\right]E(X^3) +$$

$$+ \frac{1}{24}\left[\Phi(\alpha)^3 - 3\Phi(\alpha)\right]E(X^4) - \frac{1}{36}\left[\Phi(\alpha)^3 - 5\Phi(\alpha)\right]E(X^3)^2,$$

where $E(X^3)$ denotes the skewness and $E(X^4)$ denotes the kurtosis of $\Delta W$, respectively. Zangari‘s original suggestion was to use the first four moments. In order to examine the effects of higher moments we also analyse an extension of this approach where we take into account the first six moments (for details see Johnson and Kotz (1970), Chpt. 12.5).

## 4 Monte Carlo backtesting procedure

### 4.1 Test procedure

In this paper, the numerical evaluation of the different approaches is based on a Monte Carlo backtesting procedure. In a prior step we define a set of scenarios, i.e. specifications of the inputs $N$, $\delta$, $\Gamma$, and $\Sigma$. For each scenario the VaR is estimated using the method under consideration and compared to a sequence (length of sequence: 10,000) of simulated profits and losses, i.e. realisations of $\Delta W$. The simulation of $\Delta W$ is performed using the relationship $\Delta W = \delta^T R + \frac{1}{2} R^T \Gamma R$, where $R \sim N(0, \Sigma)$. In contrast to a full valuation Monte Carlo, where the original option pricing formula is used to simulate $\Delta W$, our approach may be described as a ‚Quadratic Form Monte Carlo‘ approach. The advantage of this approach is that it enables us to examine the ability of different methodologies to estimate quantiles of a quadratic form Taylor series expansion separated from effects caused by higher order terms of that expansion. If one uses a full valuation Monte Carlo technique, one will not be able to distinguish the effects of quantile estimation and delta-gamma approximation.

The backtesting procedure counts the number of cases in a specific scenario where the simulated loss exceeds the estimated VaR. An exact methodology is expected to yield a percentage of cases where the simulated loss exceeds the estimated VaR equal to $\alpha$. We refer to this percentage as *Pctg* in the subsequent analysis. To test the null hypothesis $H_0$: $Pctg = \alpha$, we perform a likelihood ratio test (LR) suggested by Kupiec (1995) and, for illustration purposes, a simple binomial test in spirit of the Basle Committee‘s backtesting proposal, where in a ‚traffic light approach‘ the red zone is assigned when the null hypthesis is rejected, the green zone is assigned when the null hypothesis is accepted, and the yellow zone indicates an indifferent result. The probability level $\alpha$ is set equal to 0.01 throughout the analysis.

In addition to the statistical tests we provide descriptive statistics that include the average percentage of cases where the simulated loss exceeds the estimated VaR of a given methodology accross a set of scenarios, the number of scenarios where $Pctg > \alpha$, and the mean absolute deviation $|Pctg - \alpha|$ of a methodology accross a set of scenarios. Finally, we compute the average relative VaR defined as the average ratio of the VaR estimated using a given methodology and the mean VaR accross all methods under consideration.

This paper contains the analysis of the following methodologies to estimate the quantile of the distribution of $\Delta W$ described in chapter 3.

| **Methodology** | **Description** |
|---|---|
| Delta | Delta-normal approach neglecting $\Gamma$ |
| Normal | Moment matching to a normal distribution |
| Cornish4 | Cornish-Fisher expansion with four moments |
| Cornish6 | Cornish-Fisher expansion with six moments |
| Johnson | Moment matching to a Johnson distribution |

**Table 1:** Methodologies to estimate the quantile of the distribution of $\Delta W$.

In the first part of the numerical analysis we make use of the original inputs $\delta$, $\Gamma$, and $\Sigma$. We decided to examine the following 144 scenarios, which include three different inputs for $\delta$, six different inputs for $\Gamma$, four different inputs for $\Sigma$, and to catch possible ‚convergence‘ effects two different inputs for $N$. All but the random matrices have nonzero elements equal to a given constant (e.g. all elements of $\Gamma$ are set equal to -100). Some scenarios have diagonal inputs for $\Gamma$ and/or $\Sigma$, again with equal elements on the diagonal. The random matrices use random elements drawn from a uniform distribution. All variances are set equal to unity. The structure of $\Sigma$ is described by the structure of the correlation matrix.

| $\delta$ | $\Gamma$ | Correlation matrix | $N$ |
|---|---|---|---|
| 0 | -1000, diagonal | diagonal | 10 |
| 100 | -10, diagonal | 0.15 | 100 |
| -100 | -1000, full | 0.8 | |
| | -10, full | random between -1 and +1 | |
| | random between -1000 and 0 | | |
| | random between -1000 and +1000 | | |

**Table 2:** Scenarios used in the first part of the analysis.

## 4.2 Results

The first step of our analysis is motivated by the fact that risk managers have special interest in measuring effects caused by negative gamma, especially when gamma is large relative to delta. Table 3 summarizes the results for the 120 cases where all elements of $\Gamma$ are less or equal to zero. As expected, the delta approach leads to inferior results (simulated losses exceeded VaR in 72.85% of all cases!). The normal approach performs relatively better but leads to very poor results in terms of the LR test. There are only four scenarios where this approach has not to be rejected at the 5%-confidence level indicating that the 3.18% percentage where the simulated loss exceeds VaR is too high. Since the relative VaR is only 76% compared to other approaches, we conclude that the normal approach tends to underestimate VaR for scenarios with negative gamma. A closer inspection of this approach follows below.

The results of the other approaches are not distinguishable. A common conclusion is that all these approaches perform sufficiently well. However, the Cornish4 approach seems to overestimate VaR. Detailed experiments not presented in this paper show that this result holds in general: The Cornish4 approach overstimates (underestimates) VaR when the distribution of $\Delta W$ has negative (positive) skewness. Note, that a gamma matrix with all elements less or equal to zero implies a negative skewness and vice versa.

| $\Gamma \leq 0$ | Delta | Normal | Cornish4 | Cornish6 | Johnson |
|---|---|---|---|---|---|
| Average *Pctg* | 72.85% | 3.18% | 0.88% | 0.94% | 0.97% |
| MAD in % | 71.85% | 2.19% | 0.16% | 0.12% | 0.10% |
| *Pctg* > 1% | 100.00% | 99.17% | 18.33% | 25.83% | 36.67% |
| LR-accept | 0.00% | 3.33% | 66.67% | 85.00% | 88.33% |
| Red | 100.00% | 95.83% | 0.00% | 0.00% | 0.00% |
| Yellow | 0.00% | 1.67% | 1.67% | 5.83% | 6.67% |
| Green | 0.00% | 2.50% | 98.33% | 94.17% | 93.33% |
| Relative VaR | 0.16 | 0.76 | 1.06 | 1.04 | 1.03 |

**Table 3:** Results for 120 scenarios with all elements of Γ less or equal to zero.

| | |
|---|---|
| Average *Pctg* | Average percentage where the simulated loss exceeds VaR. |
| MAD in % | Mean absolute deviation $\|Pctg - 0.01\|$. |
| *Pctg* > 1% | Percentage of scenarios where $Pctg > 0.01$. |
| LR-accept | Percentage of scenarios where $H_0$: $Pctg = 0.01$ is accepted using a likelihood ratio test. |
| Red | Percentage of scenarios where the red zone is assigned using the binomial test proposed by the Basle Committee. |
| Yellow | Percentage of scenarios where the yellow zone is assigned using the binomial test proposed by the Basle Committee. |
| Green | Percentage of scenarios where the green zone is assigned using the binomial test proposed by the Basle Committee. |
| Relative VaR | Average ratio of the VaR estimated using a given methodology and the mean VaR across all methods. |

Table 4 summarizes the results where $\Gamma$ is random and not restricted to be less or equal to zero. The delta-normal approach performs slightly better. This is due to the fact that positive gamma leads to an overstimation of VaR by a purely linear approximation. There are better results for the normal approach compared to the case where $\Gamma \leq 0$. However, without a more detailed analysis these findings do not indicate that the normal approximation produces sufficiently accurate results. The Cornish-Fisher approximation with four moments is remarkably less accurate (MAD = 1.00%, 33.33% acceptance rate) than the Cornish-Fisher approximation with six moments (MAD = 0.31%, 54.17% acceptance rate), whereas it is again hard to distinguish this approach from an approximation by a Johnson distribution. It turns out that the Cornish-

Fisher approximation with six moments leads to the most accurate results, although worse compared to the negative gamma case. A detailed analysis of those scenarios with very high positive gamma, i.e. where the positive eigenvalues of $\Gamma$ are much greater than the negative eigenvalues, shows that all methodologies are less accurate than in other scenarios. This is caused by the fact that large positive eigenvalues imply extreme positive skewness. Since the slope of a density function with extreme positive skewness might be very steep at the left tail, the quantile estimation is very sensitive in general and higher moments become more important. This explains the relatively good accuracy of the Cornish-Fisher approximation with six moments.

| **Γ random (between -1000 and +1000)** | **Delta** | **Normal** | **Cornish4** | **Cornish6** | **Johnson** |
|---|---|---|---|---|---|
| Average *Pctg* | 24.00% | 1.59% | 1.85% | 0.74% | 1.54% |
| MAD in % | 22.58% | 1.33% | 1.00% | 0.31% | 0.57% |
| *Pctg* > 1% | 100.00% | 37.50% | 58.33% | 25.00% | 79.17% |
| LR-accept | 0.00% | 4.17% | 33.33% | 54.17% | 50.00% |
| Red | 100.00% | 37.50% | 41.67% | 0.00% | 41.67% |
| Yellow | 0.00% | 0.00% | 8.33% | 4.17% | 8.33% |
| Green | 0.00% | 62.50% | 50.00% | 95.83% | 50.00% |
| Relative VaR | 0.13% | 1.06% | 0.84% | 1.13% | 0.84% |

**Table 4:** Results for 24 scenarios with random elements of Γ between -1000 and +1000.

| | |
|---|---|
| Average *Pctg* | Average percentage where the simulated loss exceeds VaR. |
| MAD in % | Mean absolute deviation $\|Pctg - 0.01\|$. |
| *Pctg* > 1% | Percentage of scenarios where $Pctg > 0.01$. |
| LR-accept | Percentage of scenarios where $H_0$: $Pctg = 0.01$ is accepted using a likelihood ratio test. |
| Red | Percentage of scenarios where the red zone is assigned using the binomial test proposed by the Basle Committee. |
| Yellow | Percentage of scenarios where the yellow zone is assigned using the binomial test proposed by the Basle Committee. |
| Green | Percentage of scenarios where the green zone is assigned using the binomial test proposed by the Basle Committee. |
| Relative VaR | Average ratio of the VaR estimated using a given methodology and the mean VaR across all methods. |

In order to illustrate the effect of growing dimension of $\Gamma$ on the ‚convergence‘ behaviour of the normal approximation we calculate the skewness and the kurtosis of the distribution of $\Delta W$ for four different scenarios with growing $N$. Note, that for all scenarios there is a fixed sequence of input matrices with growing $N$. Although, different sequences may lead to different results, one can gain valuable insight from this experiment. All scenarios have zero delta. The structure of the other input matrices $\Gamma$ and $\Sigma$ are summarized in table 5.

| **Scenario** | $\Sigma$ | $\Gamma$ |
|---|---|---|
| 1 | identity matrix | diagonal with elements = -100 |
| 2 | identity matrix | full with elements = -100 |
| 3 | off diagonal elements = 0.15 | diagonal with elements = -100 |
| 4 | off diagonal elements = 0.15 | diagonal with elements of changing sign = -100, 100,... |

**Table 5:** Sequence of input matrices.

In figure 1 the skewness and the kurtosis in the first scenario are plotted against $N$. Obviously, convergence is achieved rather quickly. This is due to the fact, that the sequence of transformed gamma matrices (which are identical to $\Gamma$ in this case) implies a sequence of constant eigenvalues which in turn leads to converging moments. This result is in line with Finger‘s (1997) findings. The second scenario (figure 2) shows that - in contrast to Finger‘s results - there is no convergence for the case of a sequence of full gamma matrices even for uncorrelated risk factors. $\Gamma^{**}$ possesses only one nonzero eigenvalue and thus does not lead to convergence. The third scenario (figure 3) shows - again in line with Finger - that for this specific diagonal structure of $\Gamma$ there is no convergence when risk factors are correlated. In fact, although this sequence of gamma matrices implies a sequence of nonzero eigenvalues, there is one large eigenvalue that dominates the others. Finally, in the fourth scenario (figure 4) there is convergence even in the case of correlated risk factors, which contradicts Finger‘s findings. The specific sequence of gamma matrices in this example implies a sequence of eigenvalues with the property that the sum of eigenvalues is zero for $N$ even, which is sufficient for this kind of ‚convergence‘. Note, that this structure of $\Gamma$ is not irrelevant, since it more or less means that there are straddle positions on different underlyings of opposite sign and comparable size.

## 5 Conclusion

The approximation of the distribution of $\Delta W$ through a moment matching normal distribution leads to very inaccurate results for a small number of risk factors. For large $N$ and specific structures of the gamma matrix this approach is sufficiently accurate. However, the accuracy of the normal approximation depends in general on the distribution of eigenvalues of the transformed gamma matrix (second derivatives with respect to uncorrelated standard normal risk factors).

In cases with negative skewness, where positions with negative gamma dominate those with positive gamma, all other methodologies (Cornish-Fisher approximation with four or six moments and approximation through a Johnson distribution) perform equally well. In cases with positive skewness the accuracy of all methods is worse due to the effect of higher moments. In these scenarios the Cornish-Fisher approximation with six moments leads to the most accurate results. Since this method is very easy to implement, we suggest to use a Cornish-Fisher approximation using at least the first six moments at least for practical purposes.

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