Preference-Based Asset Liability Management

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Abstract

Asset and liability management (ALM) for private investors is a demanding task, since a proper management process is based on advanced economic theory, finance and optimization theory. The goal of the present paper is to present an approach close to economic theory solely based on observable information provided by investors. That is, a main goal is to determine the investor's preferences and risk preferences. The approach proposed is novel since liabilities and assets are given their full weight in the process and the number of liabilities and assets is not restricted. The resulting dynamic optimization problems are analyzed and examples highlight the features of the model.

Keywords: Asset and liability management, stochastic optimization, preferences revelation
JEL-Classification: G11, G20, C61, D11, D81

1 Introduction

State-of-the-Art institutional investors apply asset and liability management (ALM) systems to maximize their shareholders' wealth over time. The goal of this paper is to show that the ALM approach employed by institutions can be applied to the special needs of individuals. In this paper we stress the economic requirements and show how the induced problems can be solved. The relevance of private ALM for the banking sector can be seen by the 1998 profit of 10 billion Swiss francs which the three largest Swiss banks earned in private banking. The analysis below implies strong evidence that preference-based asset and liability management can not be done in a consistent and optimal way if there is no ALM tool to support the banks' employees in their counseling process. Therefore, we assume that a necessary condition for successful ALM for private investors is the realization of an electronic tool. This forces us to solve the problem theoretically and to ensure that the solution is also feasible in practice.

The most important requirement is *economic meaningfulness* of the philosophy behind the ALM tool. This means that the subjective needs of the investors and the bank must be fulfilled given the objective characteristics of the financial markets and of the tax/legal system. Therefore, we first consider the requirements due to the two agents' preferences and goals.

The investor's goal is to find an optimal allocation, given his preferences and restrictions. Individuals expect a dynamically optimized plan, which makes the consideration of time preferences necessary. Since the world of financial securities (which finance the plans) is stochastic in nature, risk preferences matter too. The utility function represents the utility from daily consumption and large investment projects, which we call liabilities. Avoiding any unnatural restriction of the investor, all three types of preferences have to be part of a ALM tool.

Besides preferences, objective and subjective restrictions are the second main input into the ALM tool. Objective restrictions might be due to tax or legal restrictions.

Subjective restrictions might be investment restrictions imposed by the investor on certain markets or on specific securities. Summarizing, the investor wishes that a dynamic, stochastic optimization program under various restrictions is solved.

On the other hand, the bank requires a system with the following properties:

1. The system should be *simple* enough in order to compute answers in reasonable time.

2. The system should be *flexible* enough to cover most situations arising in practice.

3. All inputs required should either be *observable* or *assessable*.

4. The system should be *realistic* in that it will not do silly things.

5. The asset dynamics should be *robust* such that small misspecifications of the model have only a weak impact on the allocation outcome.

6. The tool should be *easily* implementable in the bank.

Given this long list of requirements it is not clear a priori whether an ALM tool meeting this standard exists. Before discussing some problems and how they are solved, a *fundamental institutional problem* which follows from the above requirements is considered.

We claim that consistent planning is not possible for most investors within a single investor-bank relationship. This is simply due to a *consolidation problem*, i.e. investors typically possess accounts at different banks and hesitate to fully inform the bank which provides the counseling. Therefore, nothing can be said about the accuracy of any planning if this problem is not mastered. A solution can only be found if a third party enters the process such that (i) the customer has no incentive to hide any assets or liabilities and (ii) it is in the banks' interest to supply the required data to this third party.

Can the requirements of the investor and those of the banks be met?

The requirements induce four major problems, which have to be solved for a successful ALM:

Complexity

This mathematical problem is largely due to four factors: The investor's multi-objective dynamic optimization problem, the appropriate definition of a global liability-based risk measure, the stochastic nature of the asset dynamics and finally the incompleteness of the market, caused e.g. by investment restrictions.

Lack of information

The elicitation of the investor's risk preferences poses some serious problems. The main problem is truthful telling. An impressive body of research in experimental economics and psychology shows that without controlling for the right incentives, the answers of the individuals are of very low value. The best way to induce truthful telling is by using a playful approach, where the investor has to put at stake a small amount of real money. Although for banking practitioners this suggestion may seem strange, it will pay once the investor is convinced of its information content. Information about utilitarian and time preferences is also asymmetrically distributed so that this same problem has to be overcome.

Possible nonconformity of the investor's and the banks' preferences

Whether the preferences of the investor or those of the banks have more weight is basically a question of the distribution of the respective bargaining power.

Institutional shortcomings

The consolidation problem was already discussed.

This paper will focus on the first two problems mentioned above. We furthermore concentrate on the theoretically challenging problems of the ALM process and do not discuss IT-problems or straightforward issues such as monitoring of portfolios.

Are there tools which are able to handle all those problems? The first generation of tools determined an asset allocation where neither preferences nor risk preferences mattered. In second generation tools, more effort was put to consider investor's preferences and risk preferences. Therefore, these tools were closer to economic principles[1]. At present, we know only one tool which considers risk preferences in full depth, where the number and type of liabilities is not restricted and where the number and type of securities can be freely chosen by the bank.

In the next section we argue how standard finance theory has to be transformed in order to be applied successfully to ALM for private investors. In Section 3 we present the ALM model in an abstract framework. This allows us to strengthen the logic behind the process. In Section 4 the abstract model is specified and examples illustrate the content of the model. In the last section we summarize and discuss the results.

2 Can We Apply Finance Theory to Real Life ALM?

A basic intertemporal consumption and investment problem can be written in the form

$$\begin{aligned} J(w(0), S(0)) &= \max_{c(\cdot),\pi(t)} E[\int_0^T e^{-\rho t} u(c) dt] \qquad (1) \\ dw &= \sum_i \pi_i w b_i dt - c dt + dy + \sum_i \pi_i w \sigma_i dB_i \ , w(0) \text{ given} \\ dS_i &= S_i b_i dt + S_i \sigma_i dB_i, \ S(0) \text{ given} . \end{aligned}$$

In this model, w represents wealth, π_i the fraction of wealth invested in the i^{th} asset, S is the price vector of the risky assets, c is daily consumption and the utility function u is strictly increasing and concave. Clearly, today there are more refined models where for example investment restrictions are considered. But these refinements in the quantitative literature do not touch the methodology of how preferences are modeled.

[1]See Berger and Mulvey (1998), Cario et al. (1994) and the Financial Engine Investment Advisor, http://www.financialengines.com. A recent overview of ALM is Ziemba and Mulvey (1994).

Our basic question is: can such models be used for real life ALM for private investors? To answer this question in the affirmative, we first need to be convinced that the methodology of the model (1) is a priori meaningful to real life ALM and second, that all information needed in a model such as (1) can be revealed in real life.

Evidence from experimental economics and psychology tells us that the functional form of preferences is too poor to capture human behavior. For example, humans are often more sensitive to how an outcome differs from some reference outcome than to the absolute level of the outcome. That is, reference levels matter. As a second example, we note that in a wide variety of domains people are more averse to losses than they are attracted to same-sized gains. This suggests that besides risk aversion, loss aversion is a driving force in human decision making. Finally, the willingness to trade one object for another depends on which object they begin with. This fact is known as the status quo bias[2].

We note that all these problems can be dealt with theoretically, i.e. the function u is replaced by a new function which takes into account these three features and further ones we did not mention. But from a practitioner's point of view, these extended models are very hard to implement. This is due to various path dependencies which enter the models and which are very difficult to quantify. For example, the reference levels are state variables which are built up by past consumption and social variables. Summarizing, generalizations of the simple preferences as in (1) are needed to remain in line with human behavior but there is *no* theoretical reason why a rigorous approach as (1) can not be used for practical purposes.

Besides the preferences, there are other parts of model (1) which need to be generalized or which are even missing: transaction costs in the financial markets, multiple classes of securities with their appropriate modeling (bonds and options for example), the possibility to outside finance and legal/tax restrictions.

[2]The survey article by Rabin (1998) gives a good overview for readers interested in the relationship between Economics and Psychology. Carbone and Hey (1998), O'Donoghue and Rabin (1997, 1998) are further papers which discuss individual decision making under uncertainty.

We show in the next sections how (1) has to be generalized and how the information needed to solve the generalized model is generated.

3 Model

We present a simplified model which captures the main features of the problem[3]. To clarify the logic we remain on a rather abstract level in this section and delegate specifications and applications to the next section. We prefer to work in a continuous time setup but the whole model can be translated into a discrete time framework without difficulty.

Assumption 1 *The explicit utility function is private information to any investor.*

This assumption - although evident - is crucial since it bears the chance of a succesful construction of such a function but also the risk of failing to take appropriate care of this fact. The variety of approaches and ALM tools observed reflects this asymmetric information between the investor and the bank. Since there are different approaches to deal with this basic problem we can observe the variety of tools. The assumption 1 is meant in the strict sense. That is, we exclude any possibility for the bank to find out a utility function which could be the basis of the optimization program (1). This implies that a optimization of the form (1) for the private investor is *not* feasible. But this does not imply that the solution proposed to the investor will be arbitrary. The approach we propose is "close" to the rigorous optimization model (1) in the following sense: We define an abstract sequential approach which uses as inputs the investors data, the properties of the financial markets, outside financing possibilities and other features described below. The properties of this sequentially defined functions are defined by appealing to economic meaningfulness. Once this procedure is set up, the abstract functions are specified and the ALM problem for the investor is solved. The approach is close to (1) since the logic is the same: We single out a portfolio for the investor which respects the preferences of the investor and the dynamic laws of the financial market as well as investment restrictions which we describe below. The difference between our approach is the lack of an explicit

[3] Tax aspects are not considered in this paper since they vary from country to country.

objective function of the investor which is instead approximated by a set of the investor's data[4].

We start with the description of the abstract sequential approach with the investor's preferences.

Assumption 2 (Investor's Preferences) *The private investor is characterized by the following data: the k investment projects with realization prices $c_j(t)$ and realization intervals[5] $t_j, j = 1, 2, \ldots, k$, and the expected daily "consumption" $d(t)$. Each investment project has an associated priority $\alpha_j \in N$. The investment projects, the daily consumption and the priorities are the elements of the preference set $\mathcal{C}$. Furthermore, a set $\mathcal{R}$ exists, which consists of the risk preferences of the investor.*

We distinguish between daily consumption and investment in large projects because for the latter one outside finance is often possible. Examples of investment projects are real estate, children education and retirement planning. The daily "consumption" function $d(t)$ reflects recurring payments such as for nutrition, rent and so on. The structure of the set of risk preferences will be discussed below. The next assumption considers the income and wealth data of the private investor:

Assumption 3 (Investor's income/wealth) *The private investor's income and wealth are characterized by the following data: initial wealth w_0, its diversification on the financial market $\mathcal{F}$ (see definition below) and the expected future income $y(t)$. This data are known to the bank. Then $\mathcal{Y}_{w_0} = \mathcal{Y}$ is the set of all possible expected incomes parametrized by the intial wealth (portfolio).*

It follows from the last two assumptions that all data are deterministic. This is an essential simplification but stochastic prices for the projects for example would increase the complexity of the model to a point where all analytical tractability is lost.

[4]Since a mathematical presentation of the model goes far beyond the standard size of a scientific publication, we have chosen to present the model in a descriptive form. A technical report can be ordered by one of the authors Paolo Vanini, paolo.vanini@ecofin.ch.

[5]We consider small intervals in order to avoid "sets of measure zero" problems in the continuous time framework. The price of an investment project is the integral over this small interval.

How can the data be gained? A reasonable and efficient way is to use the internet technology where the investors interactively draw their expected plans, their expected incomes and so on. Using such a technology, a large number of consequences of the investor's decision can be immediately displayed. Therefore using this technology is much appealing for the investor and efficient for the bank.

The input data which characterize the investor are $\mathcal{C}, \mathcal{R}, \mathcal{Y}$ (we avoid personal data or other trivial categories). By definition, the liability structure is the set of all plans where each project in each plan is characterized by price, realization time and priority of the project. For simplicity we assume that there is only one currency.

The next step is to define the financial market $\mathcal{F}$.

Assumption 4 (Financial Market) *The financial market $\mathcal{F}$ consists of a vector of stocks S, a vector of zero-coupon bonds B and a vector of money accounts M. The money accounts are locally risk free, the bonds are described by a stochastic differential equation and the stocks follow a geometric Brownian motion. The market is assumed to be liquid and there are no transat ction costs. The covariance matrix has constant rank for an investor in the respective time horizon.*

For simplicity we exclude default risk, funds and options. Summarizing, there are real assets - stocks, bonds - whose prices are described by stochastic differential equations. There are furthermore virtual, riskless assets which are introduced to maintain the dimension of the overall covariance matrix constant in time. The assumption that for each project realization time there exists a bond models the possibility to finance risk free (i.e. bonds are kept until expiration) or risky (investing in stocks) any of the projects.

We define $\mathcal{L}$ to be the set representing the outside finance possibilities and $\mathcal{I}$ the set of objective and subjective investment restrictions. We note that the set $\mathcal{L}$ depends on the specific investor and the assumed financial market, i.e. $\mathcal{L}$ is a function of $\mathcal{C}, \mathcal{Y}, \mathcal{F}$. With this assumptions and definitions we can define an *ALM-function* m as a mapping

$$m : \mathcal{C} \times \mathcal{R} \times \mathcal{Y} \times \mathcal{F} \times \mathcal{L} \times \mathcal{I} \to \Pi \ .$$

Therefore, an ALM-function associates a portfolio $\pi \in \Pi$ to an investor's preferences and income/earning while respecting the laws of the financial market and other restrictions mentioned above.

The next task is the specification of m. That for, we decompose $m = o \circ f$ into two functions, where f determines the feasible set of income/earnings and o is the portfolio optimization on the feasible set. Consider f first. This function is itself a composition of the *financial feasibility function* f_{fin}, the *outside financing function* f_{out} and the *subjective/objective investment restriction function* f_{rest}, i.e.

$$f = f_{rest} \circ f_{out} \circ f_{fin} \ .$$

In order to define f_{fin} we set $w^B(t)$ for a representative bond investment and $w^S(t)$ for a representative stock investment[6]. $y \hookrightarrow x$ denotes that income y is fully invested in the security x. Then

$$\underline{y}(t) \ = \ \min_{y \in \mathcal{Y}} \{y(t) \hookrightarrow S(t) | E[w^S(t)] \geq c_j(t) + d(t), \ \forall t, j\} \tag{2}$$

$$\bar{y}(t) \ = \ \max_{y \in \mathcal{Y}} \{y(t) \hookrightarrow B(t) | E[w^B(t)] \leq c_j(t) + d(t), \ \forall t, j\} \tag{3}$$

defines the minimum of income needed to finance in the expected value sense the liabilities if investment takes place into the risky asset $S(t)$, whereas the other limit income function is mutis mutandis the same quantity for an exclusive bond investment. The domain of f_{fin}, F_{fin} is defined as

$$F_{fin} = \{y \in \mathcal{Y} | \underline{y} \leq y \leq \bar{y}\} \ .$$

We note that the set is parameterized by initial wealth w_0.
The next function f_{out} characterizes the outside financing possibilities (see Figure 1 for an illustration of the various functions).
To define the domain of f_{out}, we first solve the two optimization problems

$$\bar{y}^* \ = \ \text{opt}_{y \in \mathcal{L}} g(y, \bar{y}) \tag{4}$$

$$\underline{y}^* \ = \ \text{opt}_{y \in \mathcal{L}} g(y, \underline{y}) \ . \tag{5}$$

[6]By a representative stock we mean a stock with mean return of 8 percent and a volatility of 20 percent. The representative bond is a vector of zero coupon bonds with maturities at the respective investment project realization times.

The intuition is as follows: We economically value the difference between the distortion of the set F_{fin} due to the outside financing possibilities and the case where these possibilities are neglected (see Figure 1). The function g measures the value of any such distortion and we choose the optimum for either a pure bond or a pure stock investment, respectively. In the next section we provide an explicit example of the abstract optimization problem. It will prove, that the extent of outside financing the investor's projects can be economically valued and optimized. If the expected income path $y(t)$ is not an element of $F_{out} = \{y \in \mathcal{Y} | \underline{y}^* \leq y \leq \bar{y}^*\}$, the plans can not be financed, not even by the most risky investment. Therefore, the plans of the investor have to be modified, either by reducing the number of projects, postponing their realization or reducing the size of the projects for example. A succesful tool analyzes the criticial point(s) in the investor's plan, i.e. the project(s) which are responsible for $y(t) \notin F_{out}$. Defining and implementing such an analysis is straightforward and we omit its detailed discussion.

The function f_{rest} captures the subjective investment restrictions of the investor in the financial market $\mathcal{F}$ ("no investment in Russia") and objective restrictions (which may be investor-specific) such as the impossibility of short selling. This function f_{rest} splits income $y(t)$ at each time t into a number of classes which equals the number of future projects $c_s, s > t$, and which considers all those past projects which for the investor still has to pay credits back. Each class is parametrized by (i) the subjective - and (ii) the objective restrictions.

The optimization function o is defined on F_{rest} and is itself decomposed into three functions

$$o = o_{div} \circ o_{match} \circ o_{global} \ .$$

The three functions arise because we face the following problems:

1. On the behalf of which project with realization time later than t should a one dollar cash flow be invested at time t

2. How much of this dollar should be invested in risky and how much in risk-free assets?

If the multi-characteristic utility function of the investor was known, the standard approach of dynamic control theory could be applied. Since in our case this function is not known, we have to find a different approach to the optimal portfolio selection problem. Our approach is split into the following steps:

1. Consider one dollar given at time s to finance a project c_j with priority α_j and realization time s_j. How much of this dollar should be invested in the risky asset and how much in the riskless asset? For this distribution problem all other projects matter. We call this the global splitting problem under objective risk and return. As we do not know the investor's utility function in its functional form, we rely on the known information given by the projects. The function o_{global} characterizes the splitting problem.

2. How should a project c_j be financed, i.e. how much of each income before time s_j is used to finance c_j and in which projects are the remains invested, given that the splitting of each dollar into risky and riskless asset is known? We call this the conditional optimal matching of income to projects. The function o_{match} characterizes the matching problem and we end up with an income stream for each project. That is, in the steps 1. and 2. we decompose the original income stream $y(t) \in F_{rest}$ into k components (the number of projects to be financed) and we determine how much of each dollar available is invested in stocks and bonds, respectively.

3. Given 1. and 2., the fractions of income allocated to risky assets are diversified in the financial markets, using the techniques of incomplete financial markets in continuous time (this defines o_{div}).

Summarizing, in the first step the feasible set F_{rest} of income streams is defined, where for each element $y \in F_{rest}$ the projects of the investor can be financed by a combination of risk free and risky investment. The boundary of the feasible set where either no risk or maximum risk is considered is then deduced. In a next step, the obtained set is distorted by considering the possibilities of outside financing the projects and finally, the obtained set is again restricted since objective and subjective investment restrictions are allowed. All income functions in F_{rest} can finance the

Type	Symbol	Main Property	Definition
Adm.	f_{fin}	$F_{fin} \subset \mathcal{Y}$	$f_{fin} : \mathcal{Y} \times \mathcal{C} \to \mathcal{Y}$
Adm.	f_{out}	$F_{out} \subset \mathcal{Y}$	$f_{out} : F_{fin} \times \mathcal{L} \to \mathcal{Y}$
Adm.	f_{rest}	$F_{rest} \subset \mathcal{Y}$	$f_{rest} : F_{out} \times \mathcal{I} \to \mathcal{Y}$
Opt.	o_{global}	(y^S, y^B, y^M)	$o_{global} : \hat{y} \times \mathcal{F}_0 \times \mathcal{C} \to \{(y^S, y^B, y^M)\}, \hat{y} \in F_{rest}$
Opt.	o_{match}	$(\vec{y}^S, \vec{y}^B, \vec{y}^M)$	$o_{match} : (y^S, y^B, y^M) \times \mathcal{F}_0 \times \mathcal{C} \to \{(\vec{y}^S, \vec{y}^B, \vec{y}^M)\}$
Opt.	o_{div}	(π^S, π^B, π^M)	$o_{div} : (\vec{y}^S, \vec{y}^B, \vec{y}^M) \times \mathcal{F} \times \mathcal{R} \to (\pi^S, \pi^B, \pi^M)$

Table 1: There are two classes "Admissibility" and "Optimization". The function $\hat{y}$ is the expected income function of the investor. (y^S, y^B, y^M) is the splitting of expected income into the three representative asset "stock", "bond" and "money account". $(\vec{y}^S, \vec{y}^B, \vec{y}^M)$ is the distribution of the splitting (y^S, y^B, y^M) on the k projects which are symbolized by the vector array. (π^S, π^B, π^M) is the optimal portfolio arising from the diversification. Each component is itself a vector with dimension equal to the asset number of the respective class. The set $\mathcal{F}_0$ is the statistical set of securities with the following elements: The representative stock with the long-run return and risk characteristics, the representative bonds with maturities equal to the project realization time and a money account.

projects, given they are diversified on the financial market. In the next step, the expected income of the investor, which is an element of F_{rest}, is invested optimally in the financial market with respect to individual risk preferences and the objective risk and return characteristics of the securities. The constructed ALM function

$$m : \underbrace{\mathcal{C} \times \mathcal{R}}_{\text{approx. prefer.}} \times \underbrace{\mathcal{Y}}_{\text{bud. restr.}} \times \underbrace{\mathcal{F}}_{\text{fin. mark.}} \times \underbrace{\mathcal{L}}_{\text{out. fin.}} \times \underbrace{\mathcal{I}}_{\text{inv. restr.}} \to \Pi \ .$$

is defined over the same sets as the choice variables in the model (1). Therefore, it is possible to circumvent the asymmetric information problems at least approximately and to remain in line with methodologies of economic theory. Table 1 summarizes the main properties of the various functions which define the ALM function m.

$$\text{The ALM function } m = o_{div} \circ o_{match} \circ o_{global} \circ f_{rest} \circ f_{out} \circ f_{fin}$$

4 Applications

4.1 Risk Preferences

Risk preferences $\mathcal{R}$ constitute the second main piece of information needed to carry out the asset optimization. Typically, either too little of an effort is made to elicit risk preferences and/or the methods applied are not appropriate. We state the basic finding of experimental economics which deals with human decision making under optimal control of the circumstances:

> In order to give the individual an incentive to tell truthfully, his decisions have to induce consequences.

Therefore it is optimal for both the investor and the bank to implement a risk preference revelation process where the investor's decisions have monetary consequences. That is, real money (a small amount is sufficient) is put at stake in the revelation process. Such incentives are typically best implemented in an interactive game where in the background the answers of the investor are evaluated immediately. Contrary to questionnaires, such games can even be fun! Nevertheless, there are situations where a game is difficult to implement. In this case, questions regarding risk preferences should be adapted to the investor's asset and liability situation and not be of a general form.

The set $\mathcal{R}$ is split into two parts; the parameter set $\mathcal{R}_{par}$ and the strategy set $\mathcal{R}_{strat}$. The parameter set represents the investor's risk aversion parameter in the sense of Arrow-Pratt. The difficulty is to invent a situation, where at least a proxy for this measure can be deduced. That for, we apply an extension of the methodology presented by Berg et al. (1986). Since in our ALM setting we are interested not solely in the control of the risk-preferences but also in the Arrow-Pratt risk measure, we had to enlarge the Berg et al. approach. The other part of subjective restrictions considers the asset allocations. For each allocation, the investor can set lower or upper bounds. For example, a German investor may bound his investment of stocks in the U.S. market between 20 and 30 percent of the total investment in risky assets. Although, such risk preferences seem innocent to implement, they are hard to work with since they alter the unrestricted stochastic optimization problems (such as Merton's model for example) in a non-trivial way.
In a second part $\mathcal{R}_{strat}$, the investor's strategies for financing his liabilities are revealed under various scenarios for future realizations of the portfolio. For example, the investor is asked to fix the maximum acceptable deviation from the planned

volume of every project, so defining the "indifference bounds" of the projects. Still more challenging is the determination of the strategy to be applied if the realized portfolio value for a specific project no longer lies within these "indifference bounds". Will the portfolio destined to finance another project be restructured? If yes, which assets should be transferred?

4.2 Outside financing - an economic analysis

We refer to the equations (4) and (5) and we set

$$\begin{aligned}
&\mathrm{opt}_{y\in\mathcal{L}} g(y,\underline{y}) = \max_{y\in\mathcal{L}}(E(K)+E(K-L))\\
&E(K) := \mathrm{RAV}_{P\in\mathcal{P}} E_P[\int_0^T S(t)(y(t)-\underline{y}(t))dt)]\\
&L := -\mathrm{RAV}_{P\in\mathcal{P}} E_P[-c_t - d_t + y_t^S]\\
&K - L = -\mathrm{RAV}_{P\in\mathcal{P}} E_P[-\underline{y}_t + E(K)y_t]\ .
\end{aligned}$$

with $\mathcal{P}$ the set of scenarios (i.e. probability laws). The term $E(K)$ represents the expected capital gains for the investor if the income function varies relative to the boundary function $\underline{y}$. The operator $\mathrm{RAV}_{P\in\mathcal{P}}$ denotes that this variation in capital are calculated by running different scenarios for the evolution of the asset dynamics and a robust average (RAV) is calculated. The second factor $E(K-L)$ measure the expected capital-adjusted losses. Hence, the criterium in the outside financing step is to find the optimal trade-off between additional expected income earned through outside financing of the projects and the expected increase in financing risk of the projects.

4.3 Portfolio optimization

We characterize the three optimization functions of the model and illustrate their properties by considering examples.

4.3.1 Global splitting problem under objective risk and return

The function o_{global} associates to each dollar the fraction invested in the stock and the bond, respectively, where the characteristics of the projects and their interrelationship are taken into consideration. The function o (we omit the index in this section) is by definition a measure of the trade-off between the investment in the stock and the investment in the bond based on the target of the project. Assume that $o^j_{s,j}$ is the fraction of 1 dollar which is invested in the stock at time s for a project which is realized at time t_j with priority α_j where no other project matter. A simple equation which considers the mentioned trade-off is

$$o^j_{s,t}\frac{E[B(s,t)]}{R(B(s,t))}a^j = (1-o^j_{s,t})\frac{E_s[S(j)]}{R_s(S(j))} \tag{6}$$

where $E_s[S(t_j)]$ ($E_s[B(t,t_j)]$) is the expected value of the stock (bond) at time t_j from vista s (with maturity $T_j = t_j$). The risk measures of the stock $R_s[S(t_j)]$ and of the bond $R_s[B(t,t_j)]$ are additional ingredients of $o_{s,j}$. The number $a_j = \frac{\alpha_j}{\frac{1}{k}\sum_{n=1}^{k}\alpha_n}$ guarantees that the weight of the project under consideration with respect to the average priority of all projects matters when splitting a dollar into the stock and the bond part. If we put more weight on the interrelationship between the projects, we define $o^j_{s,j|j-1}$ to be the fraction of 1 dollar which is invested in the stock at time s for a project which is realized at time t_j with priority α_j when the project with a greater priority α_{j-1} is taken into account. Then, we set

$$O_{s,j} = \frac{1}{m}\sum_{m=0}^{k} o^j_{s,j|j-m} \tag{7}$$

which represents the fraction of 1 dollar which is invested in the stock at time s for a project which is realized at time t_j with priority α_j where all projects with a larger priority are considered.

Example: We choose the transition probabilities for the risk measures and we assume that both the bond and the stock satisfy a geometric Brownian motion with different expected means and volatilities. We further assume that the realization time of the projects is far away from the vista time, i.e. the boundary effects in the term structure equation can be neglected. The transition probabilities for the stock $p_S(x,y,t)$ and for a perpetual bond $p_B(x,y,t)$ are

$$p_S(x,y,t) = \frac{\left(e^{\frac{1}{2}(1-\gamma)\sigma t} - e^{\frac{(\log(x)-\log(y))^2}{2\sigma^2 t}}\right)\ \sigma^2(xy)^{2(1-\gamma)}}{\sqrt{2\pi}\sqrt{\sigma^2 t}}, \gamma = \frac{2\mu}{\sigma} \tag{8}$$

$$p_B(x,y,t) \;=\; \frac{\left(e^{\frac{1}{2}(1-\gamma_B)\sigma_B t} - e^{\frac{(\log(x)-\log(y))^2}{2\sigma_B{}^2 t}}\right)\sigma_B{}^2 (xy)^{2(1-\gamma_B)}}{\sqrt{2\pi}\sqrt{\sigma_B{}^2 t}}, \gamma = \frac{2\mu_B}{\sigma_B}\;. \qquad (9)$$

$o^j_{s,j}$ is the fraction of 1 dollar which is invested in a project with realization time j from vista time s. Solving (6) implies

$$o^j_{s,j} = \frac{e^{\mu(j-s)-\mu_B(j-s)} p_B(j-s)}{p_S(j-s)\left(a^j + \frac{e^{\mu(j-s)-\mu_B(j-s)} p_B(j-s)}{p_S(j-s)}\right)}\;, \qquad (10)$$

where μ and μ_B are the expected returns of the stock and of the bond with maturity j, respectively, σ and σ_B are the volatility of the stock and of the bond and $x = e^{\mu . j}, y = 1$. We further assume that the priority of the projects starts with 1 for the most important project and increases in unit steps to the maximum number of projects under consideration.

We choose the following values for the parameters to see the implications of (10):

$$\mu = 0.08, \sigma^2 = 0.3, \sigma_B^2 = 0.15$$

and for the term structure ($t_1 = 3y, t_2 = 5y, t_3 = 10y, t_4 = 20y$):

$$\mu_B(3y) = 0.03, \mu_B(5y) = 0.033, \mu_B(10y) = 0.04, \mu_B(20y) = 0.05\;.$$

The numbers in the table represent the respective investments in the stock, given 1 dollar from vista $s = 0$ for projects which are realized at $t_j = 3, 5, 10, 20$ years.

Increasing priority of the project $\Longrightarrow$

Maturity $20y$	0.84	0.72	0.64	0.57	0.51	0.47	0.43
Maturity $10y$	0.83	0.71	0.63	0.56	0.5	0.45	0.42
Maturity $5y$	0.84	0.72	0.63	0.56	0.5	0.46	0.42
Maturity $3y$	0.7	0.53	0.43	0.36	0.31	0.28	0.25

The case of 7 projects.

Increasing priority of the project $\Longrightarrow$

Maturity $20y$	0.82	0.7	0.6	0.53	0.48	0.43
Maturity $10y$	0.81	0.69	0.59	0.52	0.47	0.42
Maturity $5y$	0.82	0.69	0.6	0.53	0.47	0.43
Maturity $3y$	0.67	0.5	0.4	0.33	0.29	0.25

The case of 6 projects.

	Project 1	Project 2	Project 3	Project 4
Realization time (years)	2	3	6	7
Price of projects (USD.)	20	30	80	60
Priority of the projects	2	3	4	1
Zero coupon bonds returns (%)	2	3	5	6
Optimal spliting of w_0 (USD)	18.1	27.4	60.7	33.7
Optimal risky asset fraction (%)	13.7	19	39.7	27.5
Optimal zero coupon bonds (%)	9.9	22	57.9	10.2
Optimal risky asset fraction (%)	84.6	77.4	73.3	91.5
Optimal zero coupon bonds (%)	15.4	22.6	26.7	8.5
Project price financed by ZCB (%)	90.5	91.4	75.9	56.1

Table 2: The "Optimal risky asset fraction (%)" describes the distribution of the part of the initial portfolio value, which is invested in the risky assets, across the projects. The "Optimal risky asset fraction (%) for project j " is that fraction of the portfolio, which is used to finance project j, which is invested in the risky asset.

Two observations follow from this example: First, the more time lies between today and the realization of a project, the more is invested in the stock; and second, the more projects there are, the less risk is taken for the important ones and the more risk is undertaken in financing the projects with very low priority.

We consider the following second example where four projects are to be financed. The financial market characteristic is the same is the same than before and the value of the initial portfolio is $w_0 = 140$ (USD).

What happens if the project prices are interchanged? We consider for example the project prices of project 2 and 3 interchanged. The results are collected in Table 3.

The results are economic meaningful since they obey the following rules: Since more mass is shifted to the present all the other variables unchanged, there must be more investment in the risky asset. This is confirmed by the last row of Table 3. Second, the much larger price of project two implies an increase in the risky asset financing of the project. Other features of the model which are desirable are the time-scale

	Project 1	Project 2	Project 3	Project 4
Price of projects (USD.)	20	**80**	**30**	60
Optimal spliting of w_0 (USD)	17.4	67	22.4	33
Optimal risky asset fraction (%)	13.7	19	39.7	27.5
Optimal ZCB (%)	9.9	22	57.9	10.2
Optimal risky asset fraction (%)	87.7	84.4	74.5	93.3
Optimal ZCB fraction (%)	12.3	15.6	25.5	6.7
Project price financed by ZCB (%)	87.3	83.8	74.7	55

Table 3:

invariance and the asset-liability invariance. The former means that shifting all project realization times by the same factor leaves the allocation invariant and the other invariance states that the multiplication of all asset prices (wealth and income) and of all the liability prices by the same factor also leaves the allocation invariant.

4.3.2 Conditional optimal matching of income to projects

We denote with $A_{s,t}$ the expected value of investing 1 dollar at time s to finance a project at time t, where a fraction $o_{s,t}$ is invested in the stock and $1 - o_{s,t}$ in the corresponding zerobond, i.e.

$$A_{s,t} = o_{s,t} E_s[S(t)] + (1 - o_{s,t}) E[B(s,t)] \ . \tag{11}$$

Let c^1 be the price of the project with the highest priority (to be realized at s_1) and c^k the project with priority k. Then in order to finance the projects, the following conditions have to hold:

$$\begin{aligned}
c^1 &= \tilde{y}_0^1 A_{0,s_1} + \tilde{y}_1^1 A_{t_1,s_1} + \ldots + \tilde{y}_{m_1}^1 A_{t_{m_1},s_1} \ , \ t_{m_1} < s_1 \\
c^2 &= \tilde{y}_0^2 A_{0,s_2} + \tilde{y}_1^2 A_{t_1,s_2} + \ldots + \tilde{y}_{m_2}^2 A_{t_{m_2},s_2} \ , \ t_{m_2} < s_2 \\
c^3 &= \tilde{y}_0^3 A_{0,s_3} + \tilde{y}_1^3 A_{t_1,s_3} + \ldots + \tilde{y}_{m_3}^3 A_{t_{m_3},s_3} \ , \ t_{m_3} < s_3 \\
\vdots &= \vdots \quad \vdots \quad \vdots \quad \vdots \\
c^k &= \tilde{y}_0^k A_{0,s_k} + \tilde{y}_1^k A_{t_1,s_k} + \ldots + \tilde{y}_{m_k}^k A_{t_{m_k},s_k} \ , \ t_{m_k} < s_k
\end{aligned} \tag{12}$$

under the feasibility conditions

$$\sum_{m=1}^{k} \tilde{y}_n^m \leq y_n \ , \ n = 1, 2, \ldots, k \ . \tag{13}$$

With

$$d^l := \sum_{r=1}^{m_l} \tilde{y}_r^l A_{t_r, s_l} \tag{14}$$

we are able to

1. define an optimization problem which
2. determines for any income y how much of this income is used to finance each project realized later than the income; i.e. the fractions $\tilde{y}$ are found.

We set:

$$\max_{\substack{\{\tilde{y}_r^t\} \\ \{r=1,2,\ldots,\sum_{w=1}^{k} m_w, t=1,2,\ldots,k\}}} \sum_{q=1}^{k} d^q \tag{15}$$

$$s.t.$$

$$\sum_{m=1}^{k} \tilde{y}_n^m \leq y_n \ , \ n = 1, 2, \ldots, k \tag{16}$$

$$c^m \leq d^m \ , \ m = 1, 2, \ldots, k \ . \tag{17}$$

4.3.3 Portfolio diversification

In the last section we determined how much income is invested in a representative stock. This section considers diversification of risky investments by investing in an arbitrary number of financial markets, where investment restrictions can be fixed by the investor, and where risk preferences matter. We assume that the revelation of the risk preferences - independent of any project risk as defined in 3.1. - leads to a classification of the investors which can be described by a risk aversion parameter $p \in (0, 1)$. Further we assume that all m assets follow a geometric Brownian motion with σ the covariance matrix such that the non-degeneracy condition for an $\epsilon > 0$

$$(\sigma(t)x)'\sigma^t(t)x \geq \epsilon ||x||^2 \ , \ \forall (t, x) \in [0, T] \times R^m. \tag{18}$$

holds. T is the time horizon of the investor. The covariance matrix may depend on time but not on the risky assets. The non-degeneracy of the covariance matrix is necessary for no-arbitrage to hold and for the completeness of the markets (see

Karatzas and Shreve (1998)). Beside the m risky assets we also introduce a riskless asset (the money account), i.e.

$$dM(t) = M(t)r(t)dt \ , \ M(0) = 1 \tag{19}$$

with a interest rate process $r(t)$. We allow the investors to impose the following "rectangular restrictions" on each asset:

$$I_i = [a_i, b_i] \ , \ i = 1, 2, \ldots, m \ . \tag{20}$$

So, for each asset the investor can choose the maximum and/or minimum amount he is willing to invest in the respective asset[7]. Clearly, the bank can set some of these variables by default. Implementing $a_i = 0$ for all i implies that short-selling of all assets is prohibited. We assume that utility of the agents is given by final wealth of the type $\frac{w^p}{p}$ where the absolute risk aversion parameter is determined in the risk preferences module. The optimization, i.e. the determination of optimal fractions invested in each asset at each time, is maximizing expected utility given (i) the risky asset dynamics, (ii) the bank account dynamics and (iii) the convex constraints on the assets. Since the Hamilton-Jacobi-Bellman equation of the primal model is non-linear, but the respective optimal equation for the dual problem is linear, the problem is transformed into its dual form. This implies the following result:

$$\pi(t) = \frac{1}{1-p}(\sigma(t)\sigma^t(t))(b(t) - r(t)1 + \lambda(t)) \tag{21}$$

where the only difference to the mutual fund theorem (see Merton (1969)) is the occurrence of the optimal dual process $\lambda(t)$ which is due to the convex restrictions. The vector $b(t)$ is the drift vector of the risky assets, 1 is a m-dimensional unit vector and $\pi(t)$ is the optimal portfolio process, i.e. the sum of all components of π minus wealth at time t defines the amount $\pi_0(t)$ which is invested in the bank account.

Formula (21) is theoretically appealing but there are two remarks which are important at this point. First, if a large number of assets is involved, the calculation of the optimal fractions invested in the assets becomes a formidable task if restrictions

[7] If there are no restrictions, the values a_i, b_i are assumed to be infinite.

are possible (i.e. if λ is not zero). Second, transaction costs are not considered in the derivation of (21). If the covariance matrix is changing over time, the resulting transaction costs are not tolerable for an investor. We show by an example the former problem and do not further discuss the later one.

Example:

Suppose that the utility function is $\log(w)$ of final wealth, that there are two risky assets which are not correlated (i.e. the covariance matrix is the 2x2 unit matrix) to simplify the calculations and that we impose the following restrictions:

1. Short selling is prohibited.
2. There are constraints on borrowing in the form specified below.

The convex set $K = \{\pi \in R^2 | \pi_i, \pi_2 \geq 0, \pi_1 + < pi_2 \leq a\}$ represents the two restrictions with a a positive real number. If $a = 1$ we exclude borrowing. The first step is to determine the "dual" set, i.e. first the support function has to be calculated:

$$\delta(x) := sup_{\pi \in K} - \pi' x \ . \tag{22}$$

It follows, that the effective domain $\tilde{K} = \{x \in R^2 | \delta(x) < \infty\}$ is a convex cone; the "barrier cone" of $-K$. In our example the support function is easily calculated to be

$$\delta(x) = a \max(\min(x_1, 0), \min(x_2, 0)) \ . \tag{23}$$

The next step is to calculate λ, i.e. the function which minimizes the following expression over $\tilde{K} = R^2$, i.e.

$$\lambda(t) = \text{argmin}_{x \in \tilde{K}} \left\{ \frac{1}{2} || \theta(t) + \sigma^{-1}(t) x ||^2 + \delta(x) \right\} \ . \tag{24}$$

The function $\theta(t) = \sigma^{-1}(b(t) - r(t)1)$ is the market price of risk. Due to the assumptions the inverse covariance matrix is the unit matrix. The optimal portfolio is

$$\pi(t) = \theta(t) + \lambda(t). \tag{25}$$

For different coordinates $\theta(t) = (\theta_1, \theta_2)$ of the market price of risk, different optimal values for the dual process and therefore different optimal portfolio allocations follow.

1. Assume that both market price of risk components are negative, i.e. the return on the riskless asset exceeds the drift of the two risky assets. Then, it is optimal to choose λ equal to $-\theta$ which implies that $\pi = (0, 0)$, i.e. invest all in the bank account.

2. Suppose $\theta_1 \geq 0, \theta_2 \leq 0, a \geq \theta_1$. It follows that $\lambda = (0, -\theta_2), \pi = (\theta_1, 0)$. Therefore, do not invest in the asset whose rate is smaller than the interest rate and invest the proportion $w\theta_1$ of wealth w in the asset with a rate larger than the interest rate.

If there are m assets under consideration and restrictions are imposed in the minimization of (24) the term $\delta(x)$ typically is a nasty one, i.e. it is a combination of multiple min-max operators. In order to carry out the minimization one rewrites the expression in term of the indicator function and then uses the calculus of distributions (generalized functions) to find the optimum.

5 Discussion

This papers shows that successful ALM for private investors is feasible if a number of requirements are met. The major requirement is to find a good proxy for the investors preferences since by assumption the utility function is private information of the investor. We have shown how such a reasonable proxy is found, both for the preferences and the risk preferences. Given that we only have partial information about the individuals preferences, i.e. we possess a huge number of the investor's preferences but no explicit utility function is available, we then defined the ALM function. This function sets up the procedure which leads from the investor's input to an asset allocation. The sequence of functions which define the different steps are "parameterized" by several restrictions/features which matter for preference based ALM: The objective properties of the financial market, the investment restrictions, the outside financing possibilities and tax considerations (which we did not considered in this paper).

After the construction of the abstract setup some particular parts of the whole model were analyzed in a more concrete form. That is, for the abstract functions which define the ALM function explicit representatives were chosen and the results of this examples highlighted some properties of our approach.

After this huge bulk of theory, we return to the original requirements stated in the introduction which an ALM tool should possess and we discuss how close or far away the model led us from these requirements. We fully took care about the requirement that the investor's goal is to find an optimal allocation given his preferences, risk preferences and restrictions. The approach also met the demand that the analysis is dynamic and stochastic in nature. The various sets which parameterize the ALM function reflect that the investor meets various outside financing possibilities, investment restriction and legal restrictions. Although we showed how these features/restriction are considered we are at present not in the position to model these restrictions comprehensively and in a consistent way. An extreme situation arises for the legal restrictions since the legal system is not a theory and maybe even not a consistent system of rules of thumb. The system which is based on the model is flexible enough to cover most situations, all inputs are observable and it is realistic. Although the answers are computed in reasonable time, it is not a simple system, i.e. many procedures need to run in black boxes for the bank's employees or effort in the training and education of the bank's staff is required.

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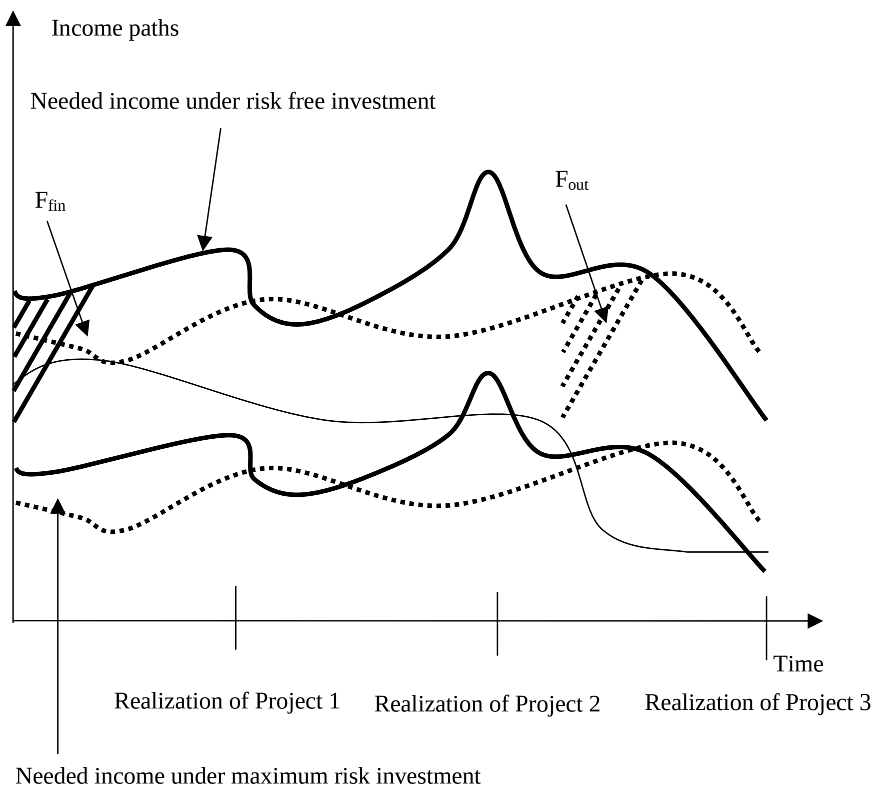

Income paths
Needed income under risk free investment
F_{fin}
F_{out}
Time
Realization of Project 1
Realization of Project 2
Realization of Project 3
Needed income under maximum risk investment

Figure 1