The Impact of a Drift on VaR

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We consider a financial market where basic risk factors are given by a multidimensional geometric Brownian motion. In a Monte Carlo framework we empirically investigate the problem whether there are substantial differences calculating VaR when assuming that the returns of the risk factors have a zero mean or when using the estimated mean. The portfolios under inspection consist of several assets including puts and calls. We see that there are significant differences.

1 Introduction

The requirement of measuring risk that is due to changes in the economic value of a portfolio of traded assets is raising with the growing volatility in financial markets and the increasing complexity of financial products. One such widely used risk measure is Value at Risk (VaR).[1] Value at Risk is that loss due to decreases in the market value of a given portfolio over a specified time horizon that will not be exceeded with a given probability, see Picoult (1998). The use of VaR is not restricted to quantify risk for internal purposes. It can also be applied in the assessment of regulatory capital of market risk: When using internal models to quantify market risk, regulatory capital is - under certain restrictions given by regulations - proportional to the VaR, calculated with the internal model.

Different models for calculating VaR are possible. One concept is the methodology RiskMetrics[2] developed by J.P. Morgan. In RiskMetrics (1996) it is basically assumed that returns of the basic risk factors are conditionally normally distributed[3] with zero mean. Now consider a portfolio consisting of exactly one asset, the return of which is normally distributed with mean μ. μ is or is not equal to zero. Calculating VaR under consideration of μ will result in a smaller VaR if $\mu > 0$.

In the present paper we examine the question whether there are differences in the values for VaR for more general portfolios, when VaR is calculated either under the assumption that the returns of the risk factors have a zero mean or when using the

[1] A detailed work on VaR is for instance the monography of Jorion (1997). A short introduction is given by Smithson and Minton (1996).

[2] See RiskMetrics (1996).

[3] They are normally distributed at each point in time.

estimated mean μ. We examine portfolios consisting of several assets including puts and calls.[4] The basic risk factors are given by a multidimensional geometric Wiener process.[5] With the help of Monte Carlo simulation, we simulate the development of risk factors and calculate corresponding changes in the market values of the assets and derivatives. We will see that there are remarkable differences in calculating VaR including an estimated, non-zero mean μ. We will also see that it is dependent on the portfolio whether the inclusion of a non-zero μ will result in a higher or a smaller VaR.

The paper is organized as follows: In Section 2 we briefly recall the basic concept underlying VaR. In Section 3, the simulation procedure and the test we have used, the Welch test, are introduced. We present the basic risk factors and their parameters in the assumed lognormal model. The results are discussed in Section 3.4.

2 The Concept behind VaR when Using Simulation

For a given portfolio, VaR measures that loss that is due to changes in the prices of the portfolio's assets over a specified time horizon T. Usually, basic risk factors are selected which determine the value of the portfolio. Denote the risk factors by $X^1, \ldots, X^n$, $n \in \mathbb{N}$, and the value of the portfolio by $(V_t)_{t\in[0,T]}$. Each asset is a function $f_i(X^j, j = 1, \ldots, n)$ of these risk factors. The portfolio V can therefore be written as a sum of functions of $X^1, \ldots, X^n$:

$$V = \sum_i w_i f_i(X^j, j = 1, \ldots, n),$$

where w_i denotes the portfolio weight of the ith asset, and the sum is taken over the number of assets in the portfolio. Different observations for $X^1, \ldots, X^n$ at time T will generate different values of the portfolio, say $V_T(\omega)$, ω taken from the state space Ω. Using the empirical distribution function

$$F(x) = \frac{1}{|\Omega|} \sum_{\omega} 1_{\{\frac{V_T - V_0}{V_0}(\omega) \le x\}}$$

[4] To compare different concepts for Value-at-Risk limits, Beeck et al. (1999) consider portfolios consisting of one stock, distinguishing between a non-zero and a zero drift, too.

[5] We made the assumption of constant parameters as we do not focus on the corresponding dynamical problem.

with corresponding probability measure P^F, the potential relative loss in T which will not be exceeded by a given confidence level can be determined: Given $p > 0$ determine $c \in \mathbb{R}$ such that

$$P^F\left\{\frac{V_T - V_0}{V_0} \le c\right\} \le 1 - p.$$

$-V_0 \cdot c$ is the VaR of the portfolio.[6]

Using Monte Carlo Simulation for Calculating VaR

In the following we focus on Monte Carlo simulation on the assumption that market factors are lognormally distributed, a widely used assumption in risk management. Nevertheless the following steps apply similarly to other distributions. Calculating VaR in a Monte Carlo framework[7] consists of the following steps:

- Choose relevant market factors.
- Specify the parameters of their joint distribution. On the assumption of log-normally distributed market factors, n drifts and $\frac{n(n-1)}{2}$ correlations have to be estimated.
- Transform independent normally distributed random variables into a sequence of correlated random variables according to the parameters estimated in the previous step.
- Simulate values of the market factors at the end of the holding period T in accordance to the specified distribution and evaluate the new portfolio value.
- Choose a confidence level.
- Calculate VaR for a given confidence level p by using the empirical distribution function of these simulated portfolio's values.

[6] See for instance Jorion (1996).
[7] A detailed description of calculating VaR by Monte Carlo simulation is given in Picoult (1998).

3 The Impact of a Drift

3.1 The Simulation

In our simulation we consider different securities including put and call options. Basic risk factors are chosen which are assumed to be jointly lognormally distributed with parameter

Case 1 $\mu = (\mu_1, \ldots, \mu_n)$ and covariance matrix Σ.

Case 2 $\mu = 0$ and covariance matrix Σ.

That is, the logarithm of the risk factor X^i is given by the stochastic differential equation

$$d\ln(X^i) = \mu_i dt + \sigma_i dB_t^i, \quad i = 1, \ldots, n,$$

where $(B^1, \ldots, B^n)$ denotes an n-dimensional correlated Wiener process with correlations according to the covariance matrix Σ, and $\sigma_i^2 = \Sigma_{ii}$ is the instantaneous variance of $\ln(X^i)$. The dynamics of the risk factor X^i are given by the stochastic differential equation

$$dX^i = (\mu_i + \tfrac{1}{2}\sigma_i^2) X^i dt + \sigma_i X^i dB_t^i, \quad i = 1, \ldots, n.$$

It is very common to refer to $\tilde{\mu}_i = \mu_i + \frac{1}{2}\sigma_i^2$ as the drift of the risk factor X^i. On the other hand the parameter μ_i is the drift[8] of the logarithmic process $\ln(X^i)$. When using the terms zero drift and non-zero drift we will in the following always refer to μ.

In the second case under consideration, μ is assumed to be zero. Therefore the covariance matrix differs from the covariance matrix in Case 1, formally one should write $\Sigma(\mu)$.[9] In the following, we will nevertheless write Σ and σ for short.

By varying the portfolio weights of the underlying assets, we generate different portfolios. With the help of Monte Carlo simulation, we investigate how VaR of each portfolio differs in Case 1 from Case 2.

[8] A definition of the drift process can be found in Arnold (1973).

[9] This is analogous to the estimator in RiskMetrics (1996).

For calculating VaR, we meet the following standards according to the Basle conventions[10], see Picoult (1998):

- Volatilities and correlations are derived by a sample period of one year.
- We choose a confidence level of 99%.
- The portfolio is static.
- The time horizon T lasts 10 days.

The sample underlying the mean and covariance estimators consists of daily mid-price data over a period of one year. Denote by r_t^i the daily return of the ith market factor at time t, that is

$$r_t^i = \ln\left(\frac{X_{t+\Delta t}^i}{X_t^i}\right),$$

where Δt is the fraction of a day in a year, and denote by N the number of returns per asset. For each market factor the mean estimator is given by the following formula

Case 1 $\qquad \hat{\mu}_i = \frac{1}{N} \cdot \sum_{t=1}^{N} r_t^i,$

Case 2 $\qquad \hat{\mu} \equiv 0,$

and the covariance estimator in both cases by

$$\hat{\sigma}_{ij}^2 = \frac{1}{N-1} \cdot \sum_{t=1}^{N} (r_t^i - \hat{\mu}_i)(r_t^j - \hat{\mu}_j); \quad i, j = 1, \ldots, N.$$

3.2 The Welch Test

Denote by VaR_μ the value at risk in Case 1 (μ under consideration) and by VaR_0 the value at risk in Case 2 ($\mu \equiv 0$). We would like to solve the question whether $VaR_\mu > VaR_0$ or vice versa.

For calculating VaR for the portfolios under consideration, we use a number of 1,000 trials each. In both cases 1 and 2 we generate a sample of VaR calculations, each of size one hundred. Denote these realizations by

[10] The Basle conventions are discussed by Johanning (1996).

$$
\begin{array}{rl}
 & y_1(\mu); \ldots; y_{n_0}(\mu); \\
\text{resp.} & y_1(0); \ldots; y_{n_0}(0);
\end{array}
$$

and the corresponding random variables by

$$
\begin{array}{rl}
 & Y_1(\mu); \ldots; Y_{n_0}(\mu); \\
\text{resp.} & Y_1(0); \ldots; Y_{n_0}(0):
\end{array}
$$

The random variables $Y_i(\mu)$ and $Y_i(0)$ are approximately normally distributed[11] with mean VaR_μ and VaR_0, respectively. Equality of the variances is not given.[12] We therefore use the Welch test, which is suitable to test the significance of the difference in the means of two normally distributed samples with different variances. To test the hypothesis

$$H_0 : VaR_0 \leq VaR_\mu \tag{1}$$

against

$$H_1 : VaR_0 > VaR_\mu; \tag{2}$$

the statistic

$$W = \frac{\bar{Y}(0) - \bar{Y}(\mu)}{\sqrt{S^2(\mu) - S^2(0)}} \sqrt{n_0}; \tag{3}$$

is used, where $\bar{Y}(\cdot) = \frac{1}{n_0} \sum_{i=1}^{n_0} Y_i(\cdot)$ and $S^2(\cdot) = \frac{1}{n_0 - 1} \sum_{i=1}^{n_0} (Y_i(\cdot) - \bar{Y}(\cdot))^2$. Under $VaR_\mu = VaR_0$; W is approximately t-distributed with k degrees of freedom, where k is given by

$$k = \frac{(n_0 - 1)[S^2(\mu) + S^2(0)]^2}{S^4(\mu) + S^4(0)}:$$

For a given confidence level $1 - \alpha$ choose η such that

[11] The empirical quantile is approximately normally distributed, see Serfling (1980) for the precise statement.

[12] Although the empirical variances are very close in our samples.

$$1 - t_k(\eta) = \alpha;$$

where t_k denotes the distribution function of a t-distributed random variable. The null hypothesis is rejected if W is greater or equal to η. The case $VaR_1 > VaR_0$ is treated analogously.

3.3 The Portfolios

The portfolios under consideration consist of the following assets:

The Portfolios Assets
German Dax
Eurostoxx 50
German Government Bond, maturing 2008
German Government Bond, maturing 2004
Dow Jones Industrial Index
3-Months Call on Dax
3-Months Put on Eurostoxx 50

The composition of these assets is varied to obtain different portfolios. The positions are all held until the end of the period, according to the regulation.

The following securities serve as risk factors:

Risk Factors
USD/DEM Currency
German DAX
Eurostoxx 50
German Government Bond[13], maturing 2008
German Government Bond, maturing 2004
Dow Jones Industrial Index
3-Months-Euribor
VDAX

[13] In general non-static portfolios, in which a great varity of bonds has to be valued, interest rates themselves should be taken as risk factors. In our static portfolios, the government bonds are fixed and only their portfolio weights are changed. Therefore it is more natural to consider the bonds as risk factors in our case.

Portfolio changes for calculating VaR are obtained by simulating the risk factors according to the assumed distribution (case 1 or 2). Put and call options are analytically valued via the Black-Scholes formula, where the parameters are given by simulation. The volatility of the Eurostoxx put is not given by a risk factor, we assume it to be constant and equal to 25%.[14]

3.4 The Results

The results are not surprising. Whether $VaR_1 < VaR_0$ or vice versa clearly depends on the portfolio and the sign and size of the drift parameters.[15]

With the Welch test, see Section 3.2, the hypothesis $H_0 : VaR_0 \leq VaR_1$ is tested against $H_1 : VaR_0 > VaR_1$. In the reverse case, the hypothesis $H_0^0 : VaR_0 \geq VaR_1$ is tested against $H_1^0 : VaR_0 < VaR_1$. Remember, the statistic W is approximately t-distributed with k degrees of freedom, where k is depending on the standard errors of the samples. For all portfolios considered below, k is much larger than 100. Therefore for all portfolios, the null hypothesis is rejected at the 99 percent confidence level if the statistic $W \geq 2.33$.

The parameters are estimated by a sample, consisting of daily mid-price data over a period of one year. The first sample contains the daily returns from 18 September 1998 to 17 September 1999. This leads to the following annualized mean and volatility and the following correlations:

Case 1

	μ	σ	DM/$	DAX	STOXX	9Y-Bd	DJ	5Y-Bd	3M-EUR	VDAX
DEM/USD	0.110	0.097	1							
DAX	0.188	0.271	0.219	1						
STOXX	0.353	0.246	0.246	0.883	1					
9Y-BOND	-0.088	0.056	-0.127	0.083	0.055	1				
DJ	0.311	0.171	0.271	0.380	0.439	0.068	1			
5Y-BOND	-0.052	0.035	-0.106	0.060	0.041	0.941	0.075	1		
3M-EUR	-0.258	0.116	-0.016	-0.048	-0.040	-0.113	-0.046	-0.130	1	
VDAX	-0.914	0.773	-0.247	-0.666	-0.663	-0.096	-0.342	-0.091	0.007	1

[14] This assumption is made for simplicity. Alternatively, one should use historical implied volatility data.

[15] This is easy to see as already mentioned in Section 1 if a portfolio is considered consisting of exactly one lognormally distributed asset. In this case $VaR_1 < VaR_0$ iff $\mu > 0$.

Case 2

	1	¾	DM/$	DAX	STOXX	9Y-Bd	DJ	5Y-Bd	3M-EUR	VDAX
DEM/USD	0	0.097	1							
DAX	0	0.271	0.222	1						
STOXX	0	0.246	0.251	0.882	1					
9Y-BOND	0	0.056	-0.133	0.078	0.046	1				
DJ	0	0.171	0.276	0.382	0.445	0.057	1			
5Y-BOND	0	0.035	-0.112	0.055	0.032	0.942	0.064	1		
3M-EUR	0	0.116	-0.025	-0.053	-0.051	-0.098	-0.060	-0.115	1	
VDAX	0	0.773	-0.251	-0.667	-0.665	-0.088	-0.348	-0.084	0.017	1

For these parameters the null hypothesis is rejected at the 99 percent confidence level for all portfolios, see below. The following table lists some results for extreme cases, a portfolio consisting only of puts, one consisting only of calls and one which has about equal investments in the bonds and equity indices.[16] The last portfolio furthermore contains short calls on the DAX in the same number as the underlying and long puts on the Eurostoxx in the same number as the underlying.

Table 1

weights of assets in %	av. rel. VaR_1 (Std err)	av. rel. VaR_0 (Std err)	W		Diff. in %
(0,0,0,0,0,0,0,0,0,100)	0.8430	0.8026	17.34	reject H_0^{o}	+ 5%
	0.0155	0.0175			
(0,0,0,0,0,0,0,0,0,-100)	1.5247	1.7742	17.5	reject H_0	-14%
	0.0958	0.1056			
(0,0,0,0,0,0,0,0,100,0)	0.6712	0.6854	5.10	reject H_0	-2%
	0.0213	0.0178			
(0,0,0,0,0,0,0,0,-100,0)	1.7632	1.6512	7.20	reject H_0^{o}	+ 7%
	0.1197	0.0991			
(0,19,20,20,19,21.5,0,0,-1,0.5)	0.033	0.0366	15.8	reject H_0	-10%
	0.0017	0.0017			

Here a vector $(x_1; \dots; x_n)$ denotes the portfolio weights of the asset-vector (USD/DEM, DAX, Eurostoxx, 9Y-Bond, Dow Jones, 5Y-Bond, 3M-EUR, VDAX, DAX-Call, Eurostoxx-Put). For example the portfolio (0,0,0,0,0,0,0,0,0,100) consists at one hundred percent of put options on the Eurostoxx. The negative sign indicates a short position. The values av. rel. VaR_1 and av. rel. VaR_0 are obtained by taking the arithmetic average of one hundred VaR calculations each, expressed as a

[16] Some factors have zero weight in the portfolio as they just serve as risk factors.

proportion of the portfolio's value. W is the result of the Welch test. Remember the critical point is $´ = 2.33$. The last column indicates how much the average estimated VaR_1 relatively differs from the average estimated VaR_0.

The long put portfolio has a significantly higher VaR_1; the short put portfolio has a significantly smaller VaR_1 than VaR_0. This is due to the positive drift of the underlying in our case.[17] The reverse is true for the long call portfolio, as the underlying has a positive drift in our case[18], too. The last portfolio, which is more equally invested, has a significantly smaller VaR_1. In this case, the average estimated VaR_1 is 10 % smaller than the average estimated VaR_0.

The variability of the drift should be taken into account. The following table presents the means and standard deviations based on the period from 22 September 98 to 21 September 99:

	DM/$	DAX	STOXX	9Y-Bd	DJ	5Y-Bd	3M-EUR	VDAX
μ	0.109	0.109	0.275	-0.081	0.255	-0.049	-0.264	-0.675
σ	0.098	0.268	0.243	0.055	0.170	0.035	0.115	0.771

Using these parameters we have the following results[19]:

Table 2

weights of assets in %	av. rel. VaR_1 (Std err)	av. rel. VaR_0 (Std err)	W		Diff. in %
(0,0,0,0,0,0,0,0,0,100)	0.8353	0.7981	14.35	reject H_0^0	+5%
	0.0182	0.0204			
(0,0,0,0,0,0,0,0,0,-100)	1.5721	1.7842	13.48	reject H_0	-12%
	0.1069	0.1153			
(0,0,0,0,0,0,0,0,100,0)	0.6743	0.6766	0.84	not sign.	0%
	0.0182	0.0204			
(0,0,0,0,0,0,0,0,-100,0)	1.6971	1.6252	4.33	reject H_0^0	+4%
	0.1197	0.0991			
(0,19,20,20,19,21.5,0,0,-1,0.5)	0.0338	0.0370	14.18	reject H_0	-9%
	0.0017	0.0014			

[17] The reverse result is obtained by replacing the return r of the underlying Eurostoxx by $-r$ keeping the other parameter constant. Remember that a put is decreasing in the underlying.

[18] Remember that a call is increasing in the underlying.

[19] To compare the results at different points in time, the starting values of all risk factors are fixed.

In comparison with the results presented in Table 1, the differences between the average VaR_μ and the average VaR_0 are smaller, and the test statistic W has smaller values. In the case of the long Call the null hypothesis cannot be rejected.

When regarding the portfolios above and the differences between their estimated VaR_μ and VaR_0 numbers, an impact of the drift on VaR is in most cases present. The data required for estimating μ are also used for estimating variances and correlations. Therefore it is little extra work to estimate μ: The lognormal distribution we used is known not to fit extreme values of financial data very accurately.[20] But a time-dependent volatility can easily be incorporated by using exponentially weighted estimators. However, it is not obvious how to involve a time-dependent mean. We used the arithmetic average, because it is relatively stable in time. An exponentially weighted estimator will in general bring in more variability of the mean.

Historical simulation for calculating VaR would overcome the problem of ignoring the drift. But it is strongly path-dependent, so that probabilistic results require a large amount of historical data, which might not be available in all cases.

Similar results are also obtainable for a one-day holding period. The VaR numbers are smaller than in the case of a ten-day holding period. The statistic W has smaller values, that is to say the hypothesis H_0 is rejected at a smaller significance level.

Résumé: The differences in the value-at-risk numbers, when calculated under consideration of an estimated drift and on the assumption of a zero drift, respectively, are considerably in a 10-day holding period. Assumptions posed on the value and type of the drift parameters should therefore be handled with care. The variability of the drift forces a regular readjustment of the estimated parameters.

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