Testing copulas to model financial dependence

Erik Kole * Kees Koedijk†
Marno Verbeek‡

Dept. of Financial Management, RSM Erasmus University, Rotterdam, The Netherlands

May 25, 2005

*Corresponding author. Address: P.O.Box 1738, 3000 DR Rotterdam, The Netherlands. Tel. +31 (0)10 408 13 58. e-mail: ekole@rsm.nl.

†Koedijk is also at CEPR.

‡Verbeek is also at the Econometric Institute, Erasmus University Rotterdam.

Testing copulas to model financial dependence

Abstract

Copulas offer economic agents facing uncertainty a powerful and flexible tool to model dependence between random variables and are preferable to the traditional, correlation-based approach. In this paper we show how standard tests for the fit of a distribution can be extended to copulas. Because they can be applied to any copula and because they are based on a direct comparison of a given copula with observed data, these tests are preferable to existing, indirect tests. We illustrate the tests by selecting a copula to manage the risk of a well diversified portfolio consisting of stocks, bonds and real estate. They provide clear evidence in favor of the Student's t copula, and reject both the correlation-based Gaussian and the extreme value-based Gumbel copula. A detailed inspection of the tails reveals that the Student's t copula accurately captures the risk of joint downside movements, while it is underestimated by the Gaussian and overestimated by the Gumbel copula. Because existing tests that focus on bivariate tail dependence fail to unambiguously select from these three alternatives, the results indicate the superiority of our approach to test and select copulas for modelling dependence.

Key words: financial dependence, copulas, distributional tests, tail dependence
JEL classification: G11, C12, C14

1 Introduction

Modelling dependence is of key importance to all economic fields in which uncertainty plays a large role. It is a crucial element of decision making under uncertainty and risk analysis. Consequently, an inappropriate model for dependence can lead to suboptimal decisions and inaccurate assessments of risk exposures. Traditionally, correlation is used to describe dependence between random variables, but recent studies have ascertained the superiority of copulas to model dependence, as they offer much more flexibility than the correlation approach. Clemen and Reilly (1999) discuss the application of copulas in decision marking, Frees and Valdez (1998) show the use of copulas in actuarial risk analyses, while Embrechts et al. (2002) advocate using copulas in finance. An important reason to consider other copulas than the correlation-implied Gaussian copula is its failure to capture dependence between extreme events. However, up to now no consensus has been reached on which copula to use in specific applications or on how to test the accuracy of a specific copula.

In this paper we propose a new procedure to determine which copula to use. Generally, theory offers little guidance in choosing a copula, making the selection an empirical issue. Since a copula is equivalent to a distribution, we consider traditional tests designed for the fit of a distribution on a sample. We show how modifications of the Kolmogorov-Smirnov test and the Anderson-Darling test can be applied. These tests are based on a direct comparison of the dependence implied by the copula with the dependence observed in the data. If dependence over the complete distribution is important, as in the case of investment decisions, the Kolmogorov-Smirnov tests can be chosen because of their focus on the fit in the distribution's center. If dependence of extreme values is of interest, as in the case of risk management, the Anderson-Darling tests are preferable, because they pay more attention to the tails. Using these direct tests of the fit of a copula has several advantages over alternative approaches proposed in the literature. First of all, it is applicable to any copula, not only to the Student's t and Gaussian copula. Second, it can be used for copulas of any dimension, not only for bivariate copulas. Third, it indicates whether a copula captures the observed dependence accurately, and not only whether it can be rejected against another specific copula. Finally, if the tests that we propose for selecting a copula are used, the decision is based on the complete dependence pattern, contrary to selection procedures that consider only part of the dependence pattern (i.e. dependence of extreme observations).

We apply the tests to select a copula for the risk management of an asset portfolio consisting of stocks, bonds and real estate, which are among the main asset classes available to investors. As investors are typically averse to downside risk, it is important to capture the risk of joint downside movements of the asset prices, without failing to exploit the diversification possibilities that the assets offer. This is particularly relevant for the allocation over the different asset classes, which is generally the first step of portfolio construction, because this step should realize most diversification advantages. Therefore, we consider the Gaussian, the Student's t and the Gumbel copula to model the dependence. The Gaussian copula is the traditional candidate for modelling dependence. The Student's t copula is a natural second candidate, because it can capture dependence in the tails without giving up flexibility to model dependence in the center. We include the Gumbel copula, because it is directly related to multivariate extensions of extreme value theory, which has gained popularity in risk management over the last decade (see e.g. Longin, 2000). To make proper inferences of extreme events, we base our analysis on six years of daily returns of US indexes that represent the asset classes. Our procedure provides clear evidence against the Gaussian and Gumbel copulas, but does not reject the Student's t copula. As a comparison, we apply the approach of Poon et al. (2004), which is based on bivariate dependence in the tails and show that it does not facilitate a decision. A detailed comparison of the tail behavior present in the data with the tail behavior of the copulas shows the importance of choosing an appropriate copula for risk management. While the Gaussian copula leads to a serious underestimation of the risk of joint downside movements and the Gumbel copula overestimates it, the Student's t copula captures this risk accurately. Moreover, these differences are statistically significant.

Our study adds to the ongoing debate on modelling dependence in finance, particularly regarding asset returns. This debate concentrates not only on which copula to use, but also on the methods used to make the selection. Different authors have proposed different methods, based on either dependence in the tails[1], likelihood ratio tests[2], correlations conditional on the size of returns[3] or regime switching models[4]. However, contrary to our approach, none of these methods tests directly the fit of a dependence model. Moreover, the recent methods based on dependence in the tails can only handle bivariate dependence and take only part of the dependence pattern into account. Likelihood ratio tests can only handle nested copulas, while the methods based on size-conditional correlations are sensitive to (corrections for) biases (see Corsetti et al., 2005). Regime switching models are generally difficult to combine with copulas. Finally, the evidence supplied by the different methods on relatively similar data sets is mixed. We try to overcome the drawbacks of these existing methods by proposing tests that relate directly to the fit of a dependence model. We extend the work of Malevergne and Sornette (2003), who only consider the Gaussian copula.

The remainder of this article is structured as follows. Section 2 discusses the tests and their application to the Gaussian, the Student's t and the Gumbel copulas. In Section 3 we show how to use the tests to select a copula to model dependence in a risk management application. We demonstrate the advantages of our approach by comparing it with the testing procedure in Poon et al. (2004). A detailed analysis of tail behavior shows the importance of choosing the right copula. Section 4 concludes.

[1]See Hartmann et al. (2004), Poon et al. (2004) and Longin and Solnik (2001).

[2]See Mashal et al. (2003) and Mashal and Zeevi (2002).

[3]See Ang and Chen (2002), Campbell et al. (2003), Campbell et al. (2002), Forbes and Rigobon (2002) and Loretan and English (2000).

[4]See Ang and Bekaert (2002), Edwards and Susmel (2001) and Ramchand and Susmel (1998).

2 A procedure to test the fit of copulas

In this section we explain how the Kolmogorov-Smirnov and Anderson-Darling tests can be implemented for copulas. We start with a short introduction on copulas.[5] In the second subsection we present the tests, followed by a discussion of the implementation in the third subsection.

2.1 Copulas

Dependence between random variables can be modelled by copulas. A copula enables the calculation of the joint probability of events from the marginal probabilities of each event. This makes copulas attractive, as the univariate marginal behavior of random variables can be modelled separately from their dependence. For a random vector X of size n with marginal cumulative density functions F_i, the copula with cdf $C(\cdot)$ gives the cumulative probability for the event x:

$$P(X \leq x) = C(F_1(x_1), \ldots, F_n(x_n)). \tag{1}$$

The applicability of copulas is wide, as Sklar (1959) proves that each multivariate distribution with continuous marginals has a unique copula representation. Moreover, any function $C : [0,1]^n \to [0,1]$ satisfying some regularity restrictions implies a copula.[6]

Tail dependence is an important property of copulas. It describes the behavior of copulas when the values of the marginal cdf F_i reach their bounds zero (lower tail dependence) or one (upper tail dependence) and is defined as the limiting probability that a subset of the variables in X has extreme values, given that the complement has extreme values.[7] If the limiting probability equals zero, a copula exhibits tail independence; if the probability exceeds zero, it exhibits tail independence.

[5] A more rigorous treatment of copulas can be found in Joe (1997) and Nelsen (1999). For a discussion applied to finance we refer to Cherubini et al. (2004) and Bouyé et al. (2000).

[6] See Definition 1 in Embrechts et al. (2002).

[7] Joe (1997) Sec. 2.1.10 gives a definition for the bivariate case, which is generalized by Schmidt and Stadmüller (2003) to $n > 2$ dimensions.

The traditional use of correlation to model dependence implies using the Gaussian copula which has cdf:

$$C_n^{\Phi}(u;\Omega) = \Phi_n(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n); \Omega), \tag{2}$$

where u is a vector of marginal probabilities, Φ_n denotes the cdf for the n-variate standard normal distribution with correlation matrix Ω, and Φ^{-1} is the inverse of the cdf for the univariate standard normal distribution. For imperfect correlated variables, the Gaussian copula implies tail independence.

Closely related to the Gaussian copula is the Student's t copula, with cdf:

$$C_n^{\Psi}(u;\Omega,\nu) = \Psi_n(\Psi^{-1}(u_1;\nu), \ldots, \Psi^{-1}(u_n;\nu); \Omega, \nu), \tag{3}$$

where Ψ_n denotes the cdf of an n-variate Student's t distribution with correlation matrix Ω and degrees of freedom parameter $\nu > 2$, and Ψ^{-1} is the inverse of the cdf for the univariate Student's t distribution with mean zero, dispersion parameter equal to one and degrees of freedom ν. The Gaussian and Student's t copula belong to the class of elliptic copulas. A higher value for ν decreases the probability of tail events. As the Student's t copula converges to the Gaussian copula for $\nu \to \infty$, the Student's t copula assigns more probability to tail events than the Gaussian copula. Moreover, the Student's t copula exhibits tail dependence (even if correlation coefficients equal zero). For $\nu > 30$, differences between the Student's t and Gaussian copula are negligible.

The third copula we consider in the paper is the Gumbel copula, which belongs to the class of Archimedean copulas. The Gumbel copula is an extreme value copula.[8] Its standard cdf is given by

$$C_n^{G}(u;a) = \exp\left(-\left(\sum_{i=1}^{n}(-\log u_i)^a\right)^{1/a}\right), \tag{4}$$

with $\alpha \geq 1$, where $\alpha = 1$ implies independence. Because this standard cdf implies the same dependence between all combinations of marginal variables u_i, we use the extension

[8] Joe (1997) provides a detailed, general discussion of extreme value theory in relation to copulas, while Bouyé (2002) discusses it from a risk management perspective.

of the standard Gumbel copula proposed by Bouyé (2002). He uses a recursive definition, in which the dependence of the marginal probability u_{i+1} with the preceding marginal probabilities $u_1, \ldots, u_i$ is characterized by a specific parameter a_i:

$$C_n^B(u_1, \ldots, u_n; a_1, \ldots, a_{n-1}) = \begin{cases} C_2^G(u_1, u_2; a_1) & \text{if } n = 2 \\ C_2^G(C_{n-1}^B(u_1, \ldots, u_{n-1}; a_1, \ldots, a_{n-2}), u_n; a_{n-1}) & \text{if } n > 2, \end{cases} \tag{5}$$

with $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_{n-1} \geq 1$. $C_2^G()$ denotes the standard bivariate Gumbel copula as defined in Eq. 4. The restrictions on the α's impose a descending dependence order: the dependence between u_1 and u_2 is at least as strong as the dependence between u_1 and u_2 on the one hand and u_3 on the other. The ordering of the variables is therefore important. The Gumbel copula exhibits upper tail dependence but lower tail independence, which can be reversed by using the survival copula.[9]

2.2 Test statistics for the fit of copulas

The tests we propose belong to the large class of test statistics for the fit of distributions. Suppose we want to test whether a specific distribution for a random variable accurately fits the corresponding observations. Under the hypothesis that this is the case, the empirical cumulative distribution of the observations F_E will converge to the hypothesized cumulative distribution F_H almost surely, as stated by the Glivenko-Cantelli theorem (see Mittelhammer, 1996, p. 313). Therefore, we can use the deviations of the empirical distribution from the hypothesized distribution to test the fit. Let x_t be a realization of the

[9]The cumulative joint probability of events u is calculated by the survival copula: $P(U \leq u) = \bar{C}(\iota_n - u)$, where $\bar{C}$ denotes the joint survival function. For a random vector X with (multivariate) density function $F(x)$ (not necessarily a copula) the joint survival function is defined as $\bar{F}(x) = P(X \geq x)$. Joe (1997) (p. 10, item 39) gives the general formula that relates $\bar{F}$ to F (e.g. for the two dimensional case $\bar{F}(x_1, x_2) = 1 - F_1(x_1) - F_2(x_2) + F(x_1, x_2)$, where F_i denotes a marginal distribution).

random variable X out of sample of T realizations. We propose the following four statistics:

$$D_{KS} = \max_t |F_E(x_t) - F_H(x_t)|; \tag{6}$$

$$D_{\overline{KS}} = \int_x |F_E(x) - F_H(x)| dF_H(x); \tag{7}$$

$$D_{AD} = \max_t \frac{|F_E(x_t) - F_H(x_t)|}{\sqrt{F_H(x_t)(1 - F_H(x_t))}}; \tag{8}$$

$$D_{\overline{AD}} = \int_x \frac{|F_E(x) - F_H(x)|}{\sqrt{F_H(x)(1 - F_H(x))}} dF_H(x). \tag{9}$$

The first distance measure is commonly referred to as the Kolmogorov-Smirnov distance, of which the second is an average. The third distance measure is known as the Anderson-Darling distance after Anderson and Darling (1952), and the fourth is again an average of it. The Kolmogorov-Smirnov distances are more sensitive to deviations in the center of the distribution, whereas the Anderson-Darling distances give more weight to deviations in the tails. The average measures are used to limit the influence of outliers. To further limit the influence of outliers in the Anderson-Darling distances, we follow Malevergne and Sornette (2003) by replacing the original $(F_E(x_t) - F_H(x_t))^2$ term by $|F_E(x_t) - F_H(x_t)|$. The distributions of the statistics under the null hypothesis are nonstandard. Moreover, the parameters for the hypothesized distribution are often estimated on the same data. Therefore, simulations are necessary to evaluate the statistics.

One way to test the fit of a specific copula is to derive the test statistics directly, by transforming each observation to the corresponding marginal probabilities, based on which the distance measures are then calculated. The hypothesized and empirical copulas take the place of F_H and F_E, respectively. The empirical copula C_E based on a sample $\mathcal{X}$ gives the joint probability for a vector of marginal probabilities u as follows:

$$C_E(u; \mathcal{X}) = \frac{1}{T} \sum_t^T I(x_{1,t} \leq x_1^{\lfloor u_1 \cdot T \rfloor}) \cdot \ldots \cdot I(x_{n,t} \leq x_n^{\lfloor u_n \cdot T \rfloor}), \tag{10}$$

where $I(\cdot)$ is the indicator function, which equals 1 if the statement in parentheses is true and zero otherwise, and $x_j^{\lfloor u_j \cdot T \rfloor}$ is the k^{th} (ascending) order statistic, k being the largest integer not exceeding $u_j \cdot T$.

Inspired by Malevergne and Sornette (2003) we propose a slightly different approach that can be applied when dealing with elliptical copulas. As the cumulative distribution functions of elliptical distributions are generally not available in closed form, calculation of the hypothesized probabilities will be computationally demanding if the number of dimensions increases. We use a faster procedure and evaluate the fit of elliptical copulas in terms of the fit of a univariate random variable. We use the property that the density functions of elliptical distributions are constant on ellipsoids. Each elliptically distributed random variable implies a univariate random variable with a specific distribution that corresponds with the radii of the ellipsoids of constant density (see Fang et al., 1990, for a formal treatment). Instead of considering the observation itself we consider the squared radius of the ellipsoid of constant density that it implies. We compare the empirical distribution of the squared radii with their theoretical distribution, which are standard distributions in case of the Gaussian and Student's t copula.

For a random vector $U = (U_1, \ldots, U_n)'$ with marginal uniform distributions on $[0, 1]$ and dependence given by the Gaussian copula with correlation matrix Ω, we construct the squared radius as:

$$Z_\Phi = \tilde{U}'\Omega^{-1}\tilde{U}, \tag{11}$$

where $\tilde{U} = (\Phi^{-1}(U_1), \ldots, \Phi^{-1}(U_n))'$ and $\Phi^{-1}(\cdot)$ is the inverse of the standard normal cdf. The random variable Z_Φ has a χ^2_n-distribution. This follows easily upon realizing that $\tilde{U}$ has a normal distribution with correlation matrix Ω, which makes Z_Φ the sum of n squared random variables that are independently, standard normally distributed. So, starting with a sample having uniform marginal distributions, we transform each observation u to $z = \tilde{u}'\Omega^{-1}\tilde{u}$ and calculate its associated cumulative probability by the cdf of the χ^2_n-distribution.

For the Student's t copula we use a similar transformation. Let $V = (V_1, \ldots, V_n)'$ be a random vector with each V_i uniformly distributed on $[0, 1]$ and dependence given by the Student's t copula with correlation matrix Ω and degrees of freedom ν. Now we construct the squared radius as

$$Z_\Psi = \tilde{V}'\Omega^{-1}\tilde{V}/n, \tag{12}$$

where $\tilde{V} = (\Psi^{-1}(V_1;\nu), \ldots, \Psi^{-1}(V_n;\nu))'$ and $\Psi^{-1}(V_j;\nu)$ is the inverse function of the standard Student's t distribution with degrees of freedom parameter ν. The variable Z_Ψ is distributed according to an F-distribution with degrees of freedom parameters n and ν. Note that the variable $\tilde{V}$ has a Student's t distribution and can therefore be written as $W/\sqrt{S/n}$, with W being an n-dimensional normally distributed random variable with correlation matrix Ω and S being a univariate random variable with a χ^2_ν-distribution. Consequently, we can write

$$Z_\Psi = \frac{W'\Omega^{-1}W/n}{S/\nu},$$

which makes Z_Ψ the ratio of two χ^2-distributed variables divided by their respective degrees of freedom. Therefore, it has an $F_{n,\nu}$ distribution. So when we test the Student's t copula, we start with a sample having uniform marginal distributions, transform each observation v to $\tilde{v}'\Omega^{-1}\tilde{v}/n$ and calculate the cumulative probability with the cdf of the $F_{n,\nu}$ distribution.

2.3 The procedure

Suppose that we want to use a specific copula with cdf C and parameters θ to model the dependence of a random variable X for which we have a sample available of size T. The procedure that we propose to evaluate the fit of this copula consist of four steps:

Estimation step We estimate the parameters θ. In general two approaches can be used for estimating copula parameters. For our test procedure we advocate the inference functions for margins method (IFM) (see Joe, 1997, Ch. 10). In this two-step approach the parameters for the marginal models are estimated first. In the second step, the copula parameters are estimated with the marginal distribution parameters treated as given.[10] It is also possible to apply maximum likelihood to jointly estimate the parameters for the marginal models and the copula. Though IFM is less efficient than one-step maximum likelihood, it is computationally more attractive and allows larger flexibility in the estimation techniques for the marginal models.

[10] The resulting estimators $\hat{\theta}$ belong to the general class of sequential estimators (see Newey, 1984).

Evaluation step We evaluate the fit of the copula with the estimated parameters by calculating the four distance measures of the previous subsection. If the copulas belong to the elliptical family, we propose to base the calculation on a transformation of the uniform marginals. We use $\hat{d}_{KS}$, $\hat{d}_{\overline{KS}}$, $\hat{d}_{AD}$ and $\hat{d}_{\overline{AD}}$ to refer to the distance measures for the original sample.

Simulation step To test whether the distance measures provide evidence for rejecting the fit of the copula, we need to construct the distribution of the distance measures under the null hypothesis. Given the form of the distance measures and the fact that the parameters of the copula are not known but estimated, simulations should be used. For each simulation, we generate a random sample of size T from the copula with parameters $\hat{\theta}$.[11] We apply the estimation and evaluation step on this simulated sample (and find a new estimate for θ). Each simulation yields new values for the distant measures. Combined, the simulations result in a distribution of random variables corresponding to $\hat{d}_{KS}$, $\hat{d}_{\overline{KS}}$, $\hat{d}_{AD}$ and $\hat{d}_{\overline{AD}}$.

Test step Finally, we use the distribution that results from the simulation step to judge the values $\hat{d}_{KS}$, $\hat{d}_{\overline{KS}}$, $\hat{d}_{AD}$ and $\hat{d}_{\overline{AD}}$, by determining their p-value. p-values below the commonly used thresholds of 10%, 5% or 1% lead to rejection of the fit of the copula on that sample.

This procedure can be implemented straightforwardly. Note that the estimation step within the simulation step should be applied to the marginal parameters as well. If, for example, the empirical distributions are used to model the marginal distributions of the original sample, they should be used for the simulated sample too.

[11]Simulation techniques for copulas can be found in Bouyé et al. (2000). General simulation techniques are discussed in Devroye (1986). Aas (2004) discusses a specific simulation technique for Gumbel copulas.

3 A risk management application

In this section we consider three copulas to model the dependence between asset returns. The assets we consider are stocks, bonds and real estate. When investors determine their asset allocations, stocks, bonds and real estate are among the main asset classes available to them. The investor's objective is to construct a portfolio that has the optimal risk-return trade-off. The risk that a specific portfolio entails is directly related to the dependence between the portfolio's constituents. Consequently, the model used for the dependence is of key importance for the construction of an optimal asset allocation. Moreover, portfolios are often constructed by a top-down approach, in which the allocation to the different asset classes is determined first. Since dependence within asset classes is generally stronger than dependence between the different classes, this first step should ensure most diversification advantages.

Overwhelming evidence has been established that investors are sensitive to downside risk, implying that investors pay specific attention to extreme negative returns.[12] This makes it important to capture the risk entailed by the joint tail behavior of returns, without failing to exploit the diversification possibilities represented by the center of the return distribution. Therefore, we consider the Gaussian, the Student's t and the Gumbel copulas to model dependence. The Gaussian copula, the traditional method to model dependence, is most sensitive to the center of the distribution and implies tail independence. The Gumbel copula is most sensitive to tail dependence. Being an extreme value copula, it is an extension of the successful univariate extreme value theory techniques in risk management, as shown by Longin (2000) and Jansen et al. (2000). In their study of dependence of extreme returns Longin and Solnik (2001) and Poon et al. (2004) also use the Gumbel copula. Hennessy and Lapan (2002) consider portfolio allocations when Archimedean copulas are used to model dependence. The Student's t copula can capture

[12] See Kahneman and Tversky (1979) and Tversky and Kahneman (1991) for a general discussion. Benartzi and Thaler (1995) and Berkelaar et al. (2004) discuss the implications of downside risk aversion from a finance perspective.

both dependence in the center and the tails of the distribution, and has been proposed as alternative to the Gaussian copula by several authors including Glasserman et al. (2002), Campbell et al. (2003), Mashal et al. (2003), Valdez and Chernih (2003) and Meneguzzo and Vecchiato (2004). Though the evidence for tail dependence is actually mixed, as Longin and Solnik (2001) and Hartmann et al. (2004) find positive evidence for it in international asset returns while Poon et al. (2004) reject it by applying a different test, its importance for downside risk averse investors is large enough not to exclude it a priori.

Our procedure can be used to determine which copula should be used. If the Gaussian copula fits the data well, the center is the dominating factor and the correlation matrix suffices to describe dependence. If the Student's t fits the data well and the Gaussian does not, the Student's t captures dependence in the tails accurately while the Gaussian copula fails to do so. Since the Student's t converges to the Gaussian copula if the degrees of freedom parameter increases, a good fit of the Gaussian copula will necessarily imply a good fit of the Student's t copula for a sufficiently high value for the degrees of freedom parameter. If the Gaussian copula fits the data well, it should be preferred to the Student's t copula, as it is more parsimonious. Finally, if the Gumbel copula fits the data well, dependence in the tails is the dominating factor.

In the next subsection we introduce the data. We briefly discuss how the marginal distributions for each return can be modelled. In the second subsection we test the fit of the Gaussian, the Student's t and the Gumbel copula. We compare the outcome of the selection with the outcome from the selection method proposed by Poon et al. (2004). In the last subsection we analyze the dependence in the tails in more detail and discuss its implications for risk management.

3.1 Data and marginal models

We use indexes to proxy for the returns on stocks, bonds and real estate: Standard & Poor's 500 Composite Index (stocks), JP Morgan's US Government Bond Index (bonds) and the NAREIT All Index (real estate). Because we think it is important to pay attention to dependence in the tails of the distribution, a reasonable number of tail observations should be included. Therefore, we calculate daily total returns for all indexes. We collect data from DataStream over the period January 1, 1999 to December 17, 2004. Excluding non-trading days the sample consist of 1499 returns. Panel (a) in Table 1 presents summary statistics on the returns in the sample. Our data exhibit the well-known stylized facts: asymmetry as indicated by nonzero skewness and fat tails as indicated by excess kurtosis.

[Table 1 about here.]

We propose to model the marginal distributions by the semi-parametric method by Daníelsson and de Vries (2000), who model the center of the distribution by the empirical distribution function and rely on univariate extreme value theory to model the tails. This enables us to combine the good approximation to the center of the actual distribution offered by the empirical distribution, and the statistical rigor from extreme value theory to model the tails of the distribution. Central in extreme value theory is the tail index α, which characterizes the limiting behavior of a density. A distribution is fat tailed if the hypothesis $1/\alpha = 0$ is rejected in favor of the alternative $1/\alpha > 0$. In that case, the tail of the distribution can be modelled by the Pareto distribution. Tail index estimation is commonly based on the Hill-estimator (Hill, 1975). We use the modified Hill-estimator developed by Huisman et al. (2001) because of its unbiasedness.[13]

Table 1(b) reports estimates for the left tail index (α_l) and for the right tail index (α_r), and estimates for the tail indices under the assumption that the left and right tail index are equal. The hypothesis $1/\alpha = 0$ is rejected in all cases. The hypothesis of equal left and right tails cannot be rejected for stocks and bonds. For real estate, this hypothesis is marginally rejected with a p-value of 0.088. We model the tails separately, the left tail applying to cumulative probabilities below 0.01 and the right tail applying to cumulative probabilities above 0.99.

[13] Other methods for tail index estimation can be found in Daníelsson et al. (2001) and Drees and Kaufmann (1998). Brooks et al. (2005) conclude that the modified Hill-estimator by Huisman et al. (2001) outperforms other methods for tail index estimation when applied in Value-at-Risk calculations.

3.2 Selecting a copula

We use the procedure outlined in Section 2.3 to select from the Gaussian, Student's t and Gumbel copulas. The copula parameters are estimated following Joe (1997)'s IFM method, with the marginal distributions based on the semi-parametric method of Daníelsson and de Vries (2000) and maximum likelihood estimation in the second step. We calculate the distance measures and evaluate them by constructing their distributions under the null hypothesis of an accurate fit by simulating 10,000 samples based on the parameters that resulted from the estimation step.

The outcomes of this analysis are reported in Table 2. The parameter estimates in panel (a) for the Gaussian copula reveal that the correlations are negative and close to zero for stocks and bonds, and bonds and real estate, and moderate and positive for stocks and real estate. For an investor these estimates offer an attractive perspective, as they indicate large diversification possibilities. However, this conclusion is premature, since the test statistics indicate that the Gaussian copula does not capture the actual dependence well. For 3 out of 4 statistics, p-values are below 0.05, rejecting the hypothesis of an accurate fit. These include the average Kolmogorov-Smirnov distance and the average Anderson-Darling distance, which are less sensitive to outliers than the other two measures, and can be considered more reliable.

[Table 2 about here.]

The Student's t copula performs better as the p-values for the distance measures reported in Table 2(b) are substantially above 0.05. Its estimates for the correlation coefficients are largely equal to the estimated correlation coefficients for the Gaussian copula, but the degrees of freedom parameter is relatively low. This indicates that extreme events have a stronger tendency to occur jointly than captured by the Gaussian copula. Consequently, the different categories stocks, bonds and real estate still offer ample diversification opportunities, but because of the dependence in the tails diversifying downside risk becomes more difficult. The latter effect will be stronger for investors with a stronger aversion to downside risk.

The results for the Gumbel copula in panel (c) show that basing a dependence model on tail dependence does not lead to good results. We use Bouyé (2002)'s extension of the standard Gumbel copula, which makes the ordering of the variables important. To determine that order, we estimate a bivariate copula for the three possible combinations. We put the two variables with the highest α estimate first (being stocks and real estate) and the remaining one last (bonds). Since we focus on downside risk, we use the survival copula in order to allow for lower tail dependence. The estimate for α_1 shows dependence and lower tail dependence between stocks and real estate, but the α_2 estimate being equal to one indicates independence of bond returns from the returns on stocks and real estate. Three out of four tests (including the average Kolomogorov-Smirnov and average Anderson-Darling statistic) provide evidence against the Gumbel copula.

We conclude that the followed procedure provides a clear positive advise for selecting the Student's t copula. None of the four distance measures indicates rejection, while both for the Gaussian and the Gumbel copula three out of four distance measures lead to a negative advice. Both the parameter estimates and the test results indicate that dependence in the tails is not accurately captured by the Gaussian copula. However, the Gumbel copula fails to capture the dependence in the center. The likelihood ratio test proposed by Mashal et al. (2003) and Mashal and Zeevi (2002) confirms the preference for the Student's t copula over the Gaussian copula, but the outcome of their test alone does not indicate that the Student's t copula fits the data well.[14] A selection based on AIC or BIC also leads to a preference of the Student's t copula, but again, this does not imply by itself a good fit for the Student's t copula.[15]

[14]Because the Student's t copula converges to the normal copula for $\nu \to \infty$, a likelihood ratio test for the restriction that degrees of freedom is high (say $\nu = 10,000$) can be used to test the Gaussian copula versus the Student's t copula. We estimate a log likelihood value of 218.47 for a Student's t copula with $\nu = 10,000$ degrees of freedom, which results in an adjusted likelihood ratio statistic of $-2 \cdot (218.47 - 230.47)/2 = 12$. The original statistic is halved to take the estimation of the parameters of the marginal models into account when comparing with the usual critical value. The p-value of 0.00053 leads to rejection.

[15]Using the value for the log likelihood function in Table 2, we find for the AIC values 430.88 (Gaussian), 452.94 (Student's t) and 363.10 (Gumbel); and for BIC 196.50 (Gaussian), 201.22 (Student's t) and 168.92

3.3 Using tail dependence to select a copula

In the literature, alternative procedures have been proposed to select copulas. Because of the problems with the correction for biases in the size-conditional correlation approach pointed out by Corsetti et al. (2005), selection based on tail dependence seems most promising. If tail dependence is found, all copulas exhibiting tail independence can be eliminated, and vice versa. In a two dimensional setting this approach is appealing, but for realistic problems with more dimensions, basing a selection on tail dependence becomes problematic, since different forms of tail dependence can be defined (see Schmidt and Stadmüller, 2003). One approach would be to base the decision on pairwise comparisons. Another drawback is that only choices between a copula with and a copula without tail dependence can be made.

To compare the outcome of a selection based on tail dependence with the outcomes of the previous subsection, we apply the method proposed by Poon et al. (2004). In a bivariate setting, measures for tail dependence are derived from the limiting behavior of one random variable, conditional on the other being more and more extreme. Coles et al. (1999) construct two tail dependence parameters, either describing the behavior of two asymptotic dependent random variables, or of two asymptotic independent random variables. Both dependence parameters can be estimated directly and used to test for dependence or independence, but Ledford and Tawn (1996) show that tests based on the parameter estimate that describes the behavior of asymptotic dependent random variables are biased towards rejecting independence, which is why Poon et al. (2004) use the second measure.

Poon et al. (2004) define the tail dependence measure $\bar{\chi}$, describing the behavior of asymptotically independent random variables X_1 and X_2 as:

$$\bar{\chi} = \lim_{s \to \infty} \frac{2 \log \Pr(S_2 > s)}{\log \Pr(S_1 > s, S_2 > s)} - 1, \tag{13}$$

where $S_i = -1/\log F_i(X_i)$, and F_i is the marginal cdf for X_i. By construction, $-1 \leq \bar{\chi} \leq 1$, while $\bar{\chi} = 1$ indicates that the two variables are asymptotically dependent. Rejection of the

(Gumbel).

hypothesis $\bar{\chi} = 1$ leads to copulas exhibiting tail independence, and failure to reject it leads to copulas exhibiting tail dependence. Poon et al. construct their estimate for $\bar{\chi}$ based on the estimated (right) tail index α for the variable $S_{\min} = \min(S_1, S_2)$ by $\bar{\chi} = 2/\alpha - 1$.[16]

One way to extend their approach to a setting with more than two dimensions, is to apply it to each possible bivariate combination of the random variables. This results in three analyses. For consistency we construct the marginal distributions in the same way as for the estimation of the copula parameters. We measure tail dependence both for the left tails and for the right tails of the distribution. The results in Table 3 are mixed, however: for three combinations of stock, bond and real estate returns, tail dependence is clearly rejected, for two combinations, the tests clearly fail to reject, and for one combination (right tail-independence for bond and real estate returns) the hypothesis is rejected at the 5% level but not at the 2.5% level. Based on these results, we cannot decide which copula to use.

[Table 3 about here.]

The tests that we propose consider directly the fit of the copulas on the observed data, instead of being based on pairwise analyses. Moreover, they yielded a clear preference for the Student's t copula. As a trivariate Student's t copula implies bivariate Student's t copulas for each combination of two out of the three variables, the results of Poon et al.'s approach lead to different conclusions than our results. However, as their approach considers variables two by two, it is less efficient, which can influence the outcomes, in particular for dependence models. Furthermore, Coles et al. (1999) remark that the estimation of $\bar{\chi}$ can also be subject to biases. Finally, contrary to the tests we apply Poon et al.'s approach is based on asymptotic theory; its finite finite sample behavior is not known. To get more insight in the actual tail behavior we investigate the tails in more detail in the next subsection.

[16] This approach is based on Ledford and Tawn (1996, 1998).

3.4 Tail behavior

Figure 1 presents the tail behavior of the return series and of the Gaussian, Student's t and Gumbel copula for the parameter estimates. As a starting point we take returns with a marginal cumulative probability of 0.10, which gives -1.54% for stocks, -0.40% for bonds and -0.81% for real estate. We can use the different copulas to calculate the joint (cumulative) probability of these returns. Under the assumption of independence, the joint probability of three returns with a marginal cumulative probability of 0.10 simply equals $0.10^3 = 0.001$ or one day per 48 months. Using the Gaussian copula, this probability becomes 0.0015 (one day per 30 months), for the Student's t copula it increases to 0.0024 (one day per 20 months), while it equals 0.0042 (one day per 11 months) for the Gumbel copula.

In Figure 1(a) we show how the joint probability of the returns changes, if the return for stocks is reduced, while the returns for bonds and real estate remain at the 10% marginal probability level of -0.40% and -0.81%. In Figure 1(b) we present the effect of reducing the returns for bonds while keeping the other two unchanged, while Figure 1(c) does this for real estate. The choice between the copulas has a large impact on the joint probabilities. If the events get more extreme (i.e. the returns are reduced), the probabilities implied by the Gaussian copula decrease much faster than those implied by the Student's t or Gumbel copula, and consequently, the average waiting periods implied by the Gaussian copula increase much faster. Because the Student's t copula and Gumbel copula both exhibit tail dependence, their decrease is comparable, though the Gumbel copula always deems the joint downward movements more likely than the Student's t copula.

[Figure 1 about here.]

[Figure 2 about here.]

To show the significance of the difference between the probabilities from the different copulas, we graph 95%-confidence intervals based on the estimated variance of the parameter estimates. Because the confidence intervals are (mostly) non-overlapping, we conclude that those difference are significant. Because the probabilities implied by the empirical copula fall largely in the confidence interval for the Student's t probabilities, this shows once again that the Gaussian and Gumbel copulas differ significantly from the empirical dependence patterns, while the Student's t copula does not, exactly what our procedure indicated. The Student's t copula provides an accurate estimate of the risk of joint downside movements. On the contrary, the Gaussian copula significantly underestimates this risk, while the Gumbel copula overestimates it.

This analysis can be linked directly to stress tests for risk management. In a stress test, a risk manager analyzes a portfolio of assets for extreme events taking place (see Longin, 2000; Kupiec, 1998). Berkowitz (2000) argues that the probability for those extreme events should be included in the evaluation of the results of the stress tests to retain consistency with other elements of the risk management system. A stress test for a portfolio for which the prices of stocks, bonds and real estate are the risk factors would hence consist of specifying extreme events, i.e. returns below a threshold, calculating the probability of the event and analyzing the impact on the portfolio. Our analysis shows that the weight given to a stress test is largely influenced by the chosen copula, as extreme events have considerably different probabilities of occurrence depending on which copula is used.

4 Conclusions

In this paper we have considered copula selection. Because accurately modelling dependence is crucial to many fields in finance, a dependence model should be selected prudently. Both recent theoretical and empirical evidence have cast doubt on the accuracy of the Gaussian copula that is implied by using correlations. We have discussed how traditional tests for distributional assumptions, being the Kolmogorov-Smirnov and Anderson-Darling tests, can be implemented to determine the accuracy of the Gaussian and alternative copulas, such as the Student's t and Gumbel copula. These tests are preferable to existing tests in the literature, as they compare directly the fit of the copula on observed dependence, while the existing tests only use indirect comparisons. Moreover, they can be applied more generally, while several existing tests can only be used in bivariate cases or for elliptical copulas. Finally, while the choice of test leaves some flexibility - the Kolmogorov-Smirnov-based tests are more sensitive to fit in the center and the Anderson-Darling-based tests more to fit in the tails - the complete dependence pattern is taken into account, contrary to approaches that focus exclusively on dependence of extreme returns.

We have applied the tests to choose between the Gaussian, Student's t and Gumbel copula to model the dependence between three broad indexes for stocks, bonds and real estate. Since investors are typically averse to downside risk, the dependence model that they use should not only capture dependence in the center, but also the dependence in the tails, to accurately incorporate the risk of joint downside movements. The Gaussian copula, which focuses on dependence in the center and exhibits tail independence, and the Gumbel copula, which focuses mostly on dependence in the tails, are clearly rejected, while the Student's t copula, which can capture both central and tail dependence is not. In contrast, we show that the selection procedure proposed by Poon et al. (2004) does not lead to unambiguous results. As this procedure is based on bivariate tail dependence, this comparison demonstrates the disadvantages of using a procedure based on an analysis of pairwise dependence. In a detailed inspection of the tails we have found that the Student's t copula captures the empirical tail behavior accurately, while the Gaussian copula underestimates the risk of joint downward movements and the Gumbel copula overestimates this risk. While this result has a direct impact on stress tests in a risk management system, it can also influence investor's optimal allocation, in particular when downside risk aversion is taken into account.

References

Aas, K. (2004). Modelling the dependence structure of financial assets: A survey of four copulas. Research report SAMBA/22/04, Norwegian Computing Center.

Anderson, T. W. and Darling, D. A. (1952). Asymptotic theory of certain “goodness of fit” criteria based on stochastic processes. *Annals of Mathematical Statistics*, 23(2):193–212.

Ang, A. and Bekaert, G. (2002). International asset allocation with regime shifts. *Review of Financial Studies*, 15(4):1137–1187.

Ang, A. and Chen, J. (2002). Asymmetric correlations of equity portfolios. *Journal of Financial Economics*, 63(3):443–494.

Benartzi, S. and Thaler, R. (1995). Myopic loss aversion and the equity premium puzzle. *Quarterly Journal of Economics*, 110(1):73–92.

Berkelaar, A. B., Kouwenberg, R., and Post, T. (2004). Optimal portfolio choice under loss aversion. *Review of Economics and Statistics*, 86(4):973–987.

Berkowitz, J. (2000). A coherent framework for stress testing. *Journal of Risk*, 2(1):1–11.

Bouyé, E. (2002). Multivariate extremes at work for portfolio risk management. Working paper, City University Business School, London, UK.

Bouyé, E., Durrleman, V., Nikeghbali, A., Riboulet, G., and Roncalli, T. (2000). Copulas for finance: A reading guide and some applications. Groupe de Recherche Opérationelle, Crédit Lyonnais, Paris, France.

Brooks, C., Clare, A., Dalle Molle, J., and Persand, G. (2005). A comparison of extreme value theory approaches for determining Value-at-Risk. *Journal of Empirical Finance*, 12(2):339–352.

Campbell, R. A., Koedijk, C. G., Forbes, C. S., and Kofman, P. (2003). Diversification meltdown or the impact of fat tails on conditional correlation? Working paper, University of Technology, Sydney, Australia.

Campbell, R. A., Koedijk, C. G., and Kofman, P. (2002). Increased correlation in bear markets. *Financial Analysts Journal*, 58(1):87–94.

Cherubini, U., Luciano, E., and Vecchiato, W. (2004). *Copula Methods in Finance.* Wiley Series in Financial Engineering. Wiley, Chichester, UK.

Clemen, R. T. and Reilly, T. (1999). Correlations and copulas for decision and risk analysis. *Management Science*, 45(2):208–224.

Coles, S., Heffernan, J., and Tawn, J. (1999). Dependence measures for extreme value analyses. *Extremes*, 2(4):339–365.

Corsetti, G., Pericoli, M., and Sbracia, M. (2005). 'Some contagion, some interdependence' more pitfalls in tests of financial contagion. *Journal of International Money & Finance*, forthcoming.

Daníelsson, J., de Haan, L., Peng, L., and de Vries, C. G. (2001). Using a bootstrap method to choose the sample fraction in tail index estimation. *Journal of Multivariate Analysis*, 76(2):226–248.

Daníelsson, J. and de Vries, C. G. (2000). Value-at-risk and extreme returns. *Annales d' Économie et de Statistique*, 60:239–270.

Devroye, L. (1986). *Non-Uniform Random Variate Generation.* Springer, New York, NJ, USA.

Drees, H. and Kaufmann, E. (1998). Selecting the optimal sample fraction in univariate extreme value estimation. *Stochastic Processes and their Applications*, 75(2):149–172.

Edwards, S. and Susmel, R. (2001). Volatility dependence and contagion in emerging equity markets. *Journal of Development Economics*, 66(2):505–532.

Embrechts, P. M. E., McNeil, A. J., and Straumann, D. (2002). Correlation and dependency in risk management: Properties and pitfalls. In Dempster, M., editor, *Risk Management: Value at Risk and Beyond*, pages 176–223. Cambridge University Press, UK.

Fang, K.-T., Kotz, S., and Ng, K. W. (1990). *Symmetric Multivariate and Related Distributions*, volume 36 of *Monographs on Statistics and Applied Probability*. Chapman and Hall Ltd, London, UK.

Forbes, K. and Rigobon, R. (2002). No contagion, only interdependence: Measuring stock market co-movements. *Journal of Finance*, 57(5):2223–2261.

Frees, E. W. and Valdez, E. A. (1998). Understanding relationships using copulas. *North American Actuarial Journal*, 2(1):1–25.

Glasserman, P., Heidelberger, P., and Shahabuddin, P. (2002). Portfolio value-at-risk with heavy -tailed risk factors. *Mathematical Finance*, 12(3):239–269.

Hartmann, P., Straetmans, S., and de Vries, C. G. (2004). Asset market linkages in crisis periods. *The Review of Economics and Statistics*, 86(1).

Hennessy, D. A. and Lapan, H. E. (2002). The use of archimedean copulas to model portfolio allocations. *Mathematical Finance*, 12(2):143–154.

Hill, B. M. (1975). A simple general approach to inference about the tail of a distribution. *The Annals of Statistics*, 3(5):1163–1174.

Huisman, R., Koedijk, C. G., Cool, C. J., and Palm, F. C. (2001). Tail-index estimates in small samples. *Journal of Business & Economic Statistics*, 19(2):245–254.

Jansen, D. W., Koedijk, K. G., and de Vries, C. G. (2000). Portfolio selection with limited downside risk. *Journal of Empirical Finance*, 7(3-4):247–269.

Joe, H. (1997). *Multivariate Models and Dependence Concepts*, volume 73 of *Monographs on Statistics and Applied Probability*. Chapman & Hall, London, UK.

Kahneman, D. and Tversky, A. (1979). Prospect theory: An analysis of decision under risk. *Econometrica*, 47:263–290.

Kupiec, P. H. (1998). Stress testing in a value at risk framework. *The Journal of Derivatives*, 6(1):7–24.

Ledford, A. W. and Tawn, J. A. (1996). Statistics for near independence in multivariate extreme values. *Biometrika*, 83(1):169–187.

Ledford, A. W. and Tawn, J. A. (1998). Concomitant tail behavior for extremes. *Advances in Applied Probability*, 30(1):197–215.

Longin, F. M. (2000). From value at risk to stress testing: The extreme value approach. *Journal of Banking & Finance*, 24(7):1097–1130.

Longin, F. M. and Solnik, B. (2001). Extreme correlation of international equity markets. *The Journal of Finance*, 56(2):649–676.

Loretan, M. and English, W. B. (2000). Evaluating "correlation breakdowns" during periods of market volatility. International Finance Discussion Papers Nr. 658, Board of Governors of the Federal Reserve System.

Malevergne, Y. and Sornette, D. (2003). Testing the gaussian copula hypothesis for financial assets dependences. *Quantitative Finance*, 3(4):231–250.

Mashal, R., Naldi, M., and Zeevi, A. (2003). Comparing the dependence structure of equity and asset returns. *Risk*, 16(10):83–87.

Mashal, R. and Zeevi, A. (2002). Beyond correlation: Extreme co-movements between financial assets. Working paper: Columbia University, USA.

Meneguzzo, D. and Vecchiato, W. (2004). Copula sensitivity in collateralized debt obligations and basket default swaps. *The Journal of Futures Markets*, 24(1):37–70.

Mittelhammer, R. C. (1996). *Mathematical Statistics for Economics and Business*. Springer-Verlag, New York, USA.

Nelsen, R. B. (1999). *An Introduction to Copulas*, volume 139 of *Lecture Notes in Statistics*. Springer, New York, NJ, USA.

Newey, W. K. (1984). A method of moments interpretation of sequential estimators. *Economics Letters*, 14:201–206.

Poon, S.-H., Rockinger, M., and Tawn, J. (2004). Extreme value dependence in financial markets: Diagnostics, models and financial implications. *The Review of Financial Studies*, 17(2):581–610.

Ramchand, L. and Susmel, R. (1998). Volatility and cross correlation across major stock markets. *Journal of Empirical Finance*, 5(4):397–416.

Schmidt, R. and Stadmüller, U. (2003). Nonparametric estimation of tail dependence. Working paper, London Business School, London, UK.

Sklar, A. (1959). Fonctions de repartition a n dimensions et leurs marges. *Publications de l'Institut de Statistique de l'Université de Paris*, 8:229–231.

Tversky, A. and Kahneman, D. (1991). Loss aversion in riskless choice - a reference dependent model. *Quarterly Journal of Economics*, 106(4):1039–1061.

Valdez, E. and Chernih, A. (2003). Wang's capital allocation formula for elliptically contoured distributions. *Insurance: Mathematics and Economics*, 33(3):517–532.

(a) *Summary statistics*

	stocks	bonds	real estate
mean	0.012	0.022	0.065
volatility	1.26	0.34	0.81
skewness	0.18	-0.34	-0.33
kurtosis	4.60	3.83	7.34
minimum	-5.83	-1.38	-5.19
maximum	5.73	1.08	4.68

(b) *Tail indices*

α_l	5.31	4.62	2.60
α_r	4.44	4.96	4.23
$\alpha_l = \alpha_r$	4.51	5.89	3.27

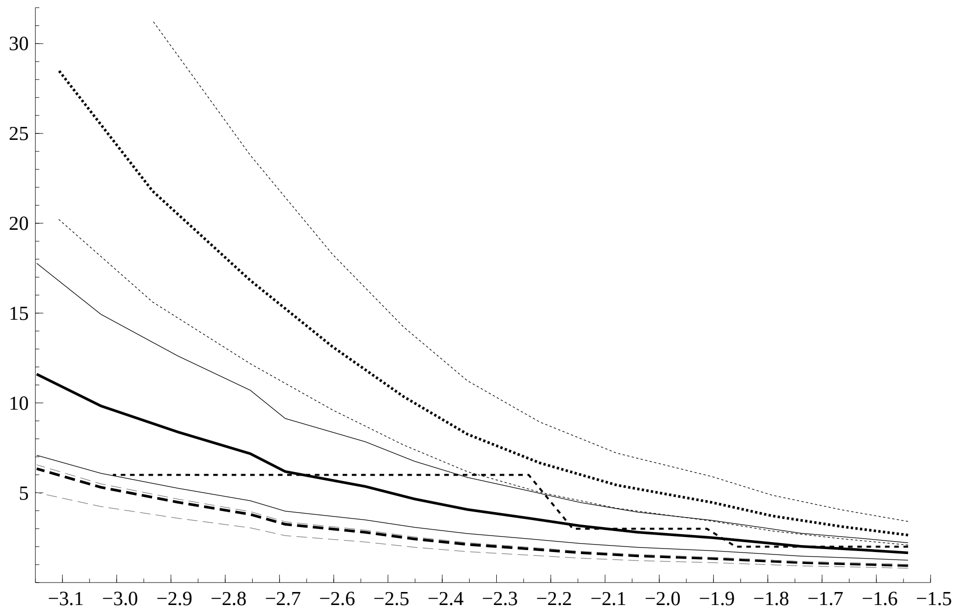
Table 1: Summary statistics and tail indices. Panel (a) reports summary statistics for the three index return series (in %) in our sample: S&P 500 Composite Index (stocks), JP Morgan Government Bond Index (bonds) and NAREIT All Index (real estate). The series consist of 1499 returns from January 1, 1999 to December 17, 2004. Panel (b) reports estimates for the left and the right tail indices (α_l and α_r respectively) and the estimates for the tail indices under the restriction that the left and right tail indices are equal ($\alpha_l = \alpha_r$). The tail indices are estimated by Huisman et al. (2001)'s modified Hill-estimator, with the maximum number of observations used (κ) equal to 149.

(a) *Gaussian copula*		(b) *Student's t copula*		(c) *Gumbel copula*	
$\rho_{s,b}$	-0.200 (0.024)	$\rho_{s,b}$	-0.195 (0.026)	α_1	1.42 (0.028)
$\rho_{s,r}$	0.471 (0.018)	$\rho_{s,r}$	0.471 (0.020)	α_2	1.000 (0.038)*
$\rho_{b,r}$	-0.073 (0.026)	$\rho_{b,r}$	-0.074 (0.027)		
		ν	12.1 (2.76)		
$\log L$	218.44	$\log L$	230.47	$\log L$	183.55
$\hat{d}_{KS}$	0.026 [0.013]	$\hat{d}_{KS}$	0.0095 [0.98]	$\hat{d}_{KS}$	0.035 [$< 5 \cdot 10^{-5}$]
$\hat{d}_{\overline{KS}}$	0.012 [0.0006]	$\hat{d}_{\overline{KS}}$	0.0024 [0.9980]	$\hat{d}_{\overline{KS}}$	0.0082 [0.0003]
$\hat{d}_{AD}$	0.058 [0.34]	$\hat{d}_{AD}$	0.044 [0.69]	$\hat{d}_{AD}$	0.28 [0.33]
$\hat{d}_{\overline{AD}}$	0.030 [0.00073]	$\hat{d}_{\overline{AD}}$	0.0065 [0.9993]	$\hat{d}_{\overline{AD}}$	0.026 [0.0016]

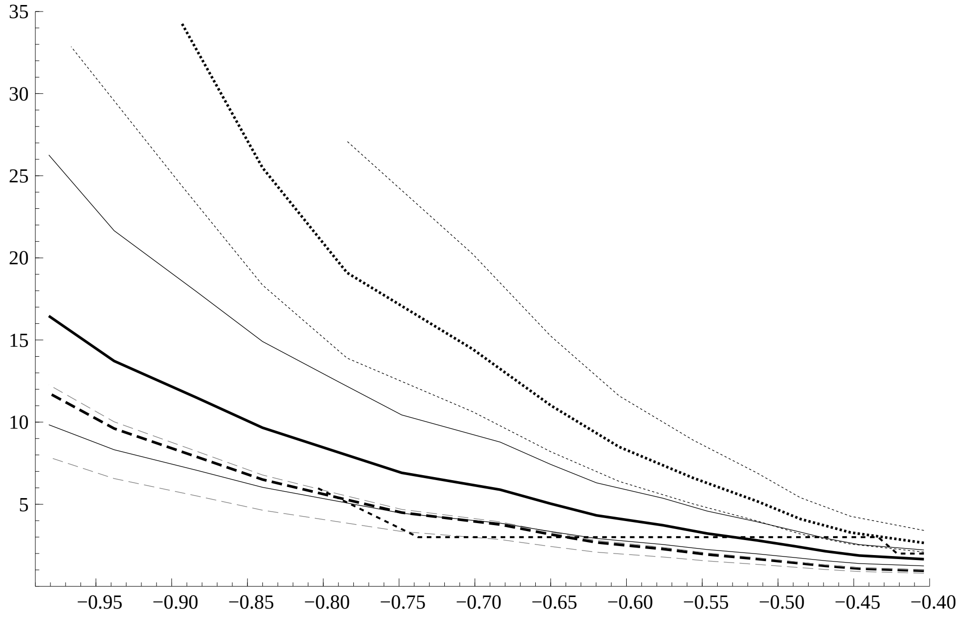
Table 2: Estimation and test results for the Gaussian, Student's t and Gumbel copula. Panels (a) to (c) report the parameter estimates, log likelihood values and distance measures. The copulas are estimated on daily returns from the S&P 500 Composite Index, the JP Morgan Government Index and NAREIT All Index from January 1, 1999 to December 17, 2004 using the IFM method (Joe, 1997). The marginal distributions are constructed using the semi-parametric method by Daníelsson and de Vries (2000), with cut-off probabilities 0.01 and 0.99 for the left and right tail respectively, and tail indices estimated using the modified Hill-estimator by Huisman et al. (2001) (see Table 1). For both the Gaussian and the Student's t copula we report the correlation coefficients for stocks and bonds ($\rho_{s,b}$), stocks and real estate ($\rho_{s,r}$) and bonds and real estate ($\rho_{b,r}$). For the Student's t copula we include the degrees of freedom parameter ν. The parameters for the Gumbel copula refer to Bouyé (2002)'s extension of the standard Gumbel copula, applied to the survival copula. Parameter α_1 refers to the dependence between stocks and real estate; α_2 to the dependence between stocks and real estate on the one hand, and bonds on the other. Standard errors are reported in parentheses. (In the estimation $\alpha_2 = 1 + a^2$ is used; the standard error marked with * corresponds with a.) The lower part reports the distance measures resulting from the test procedure. The values for the distance measures result from the evaluation step, applying the transformation in Eq. (11) for the Gaussian copula and in Eq. (12) for the Student's t copula. The p-values, based on 10,000 simulations as described in the simulation step, are reported in brackets.

	stocks and bonds	stocks and real estate	bonds and real estate
$\bar{\chi}_l$	0.0049 (0.19)	0.69 (0.32)	0.36 (0.26)
	[$< 10^{-3}$]	[0.17]	[0.007]
$\bar{\chi}_r$	0.019 (0.20)	0.619 (0.31)	0.48 (0.29)
	[$< 10^{-3}$]	[0.11]	[0.035]

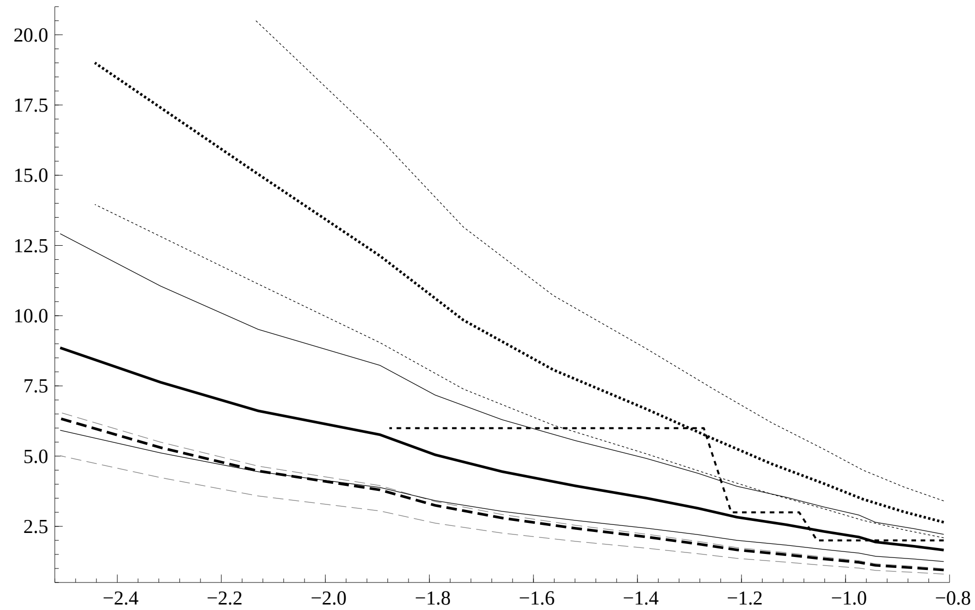
Table 3: Estimates for the asymptotic independence parameter $\bar{\chi}$. We report the estimated asymptotic independence parameters for both the left tails and the right tails of the different combinations of the returns on stock, bond and real estate indexes. The estimates are based on the tail index estimate for the right tail resulting from Huisman et al. (2001)'s modified Hill-estimator, which is applied to the series that results from taking the minimum of each couple of transformed observations. The observations are the returns on the S&P 500 Composite Index, the JP Morgan Government Index and NAREIT All Index from January 1, 1999 to December 17, 2004. All observations x_i are transformed to $-1/\log F_i(x_i)$ for the left tail-independence parameter and $-1/\log(1 - F_i(x_i))$ for the right tail-independence parameter. The marginal distributions are constructed by the semi-parametric method by Daníelsson and de Vries (2000) (see Table 1). Standard errors are reported in parentheses. In brackets the p-values for the hypothesis $\bar{\chi} = 1$ are reported.



(a) stocks



(b) bonds



(c) real estate

Figure 1: Behavior of the Gaussian, Student's t and empirical copula for extreme negative returns. This figure presents the expected waiting time (in years) on the y-axis for the joint occurrence of returns below thresholds (in %) on the x-axis. The expected waiting time is calculated as the inverse of the joint probability. The basic thresholds are selected as those returns with a marginal cumulative probability of 0.10, which gives -1.54% for stocks, -0.40% for bonds and -0.81% for real estate. For each of the three categories, the corresponding subfigure shows the expected waiting time if the corresponding threshold is reduced, while the others remain at their basic level. We plot waiting times for the Gaussian (dotted), Student's t (solid), Gumbel (long dashed) and empirical (dashed, piecewise linear) copulas. The thick lines show the point estimates, the thin lines show the 95% confidence intervals. The parameter estimates are reported in Tables 1 and 2. The confidence intervals are based on 200 parameter drawings based on the estimated Hessian matrix.