Dynamic Asset Allocation with Stochastic Income and Interest Rates*

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Abstract. We investigate the optimal investment and consumption choice of individual investors with uncertain future labor income operating in a financial market with stochastic interest rates. Since the present value of the individual's future income is a main determinant of the optimal behavior and this present value depends heavily on the interest rate dynamics, the joint stochastics of income and interest rates will have consequences beyond the separate effects of stochastic income and stochastic interest rates. We study both the case where income risk is spanned and there are no portfolio constraints and the case with non-spanned income risk and a constraint ruling out borrowing against future income. For the spanned, unconstrained problem we study a special case in which we obtain closed-form expressions for the optimal policies. For the unspanned, constrained problem we implement a numerical solution technique and compare the solutions to the spanned, unconstrained problem. We also allow for typical life-cycle variations in labor income.

Keywords. Portfolio management, labor income risk, interest rate risk, hedging, borrowing constraints, life-cycle

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1 Introduction

The household participation in financial markets creates a demand and a need for competent investment advice. The optimal investment strategy for a given individual will depend both on market characteristics, such as the dynamics of asset prices and interest rates, and on personal characteristics, such as risk aversion, time preference, and labor income. In this paper we will study the optimal strategies when both interest rates and the labor income rate of the individual are stochastic. Stochastic interest rates is the main source of shifts in the investment opportunity set, and the effect of interest rate uncertainty on the optimal strategies of an investor without labor income is by now relatively well-studied in the literature. There are also a number of studies of the effects of labor income uncertainty on the optimal strategies, but they all assume constant investment opportunities. (References are given below.) We argue and demonstrate in this paper that allowing jointly for stochastic interest rates and labor income will affect the optimal investment strategy beyond the separate effects of stochastic income and stochastic interest rates. In addition, the expected income rate in the near future is likely to depend on the interest rate level, which serves as a good proxy of the overall well-being of the economy. High interest rates typically reflect high growth rates of the economy which may lead to an upward pressure on wages, higher bonuses, fewer lay-offs, etc.

We first set up a quite general model of the financial market in which the interest rate dynamics is given by a one-factor model and several risky assets (bonds and stocks) are traded. The consumer-investor is assumed to have a time-additive utility for consumption exhibiting constant relative risk aversion and a risky labor income stream, which is correlated with both the interest rate and the risky asset prices. We calibrate the model to actual data on income and financial returns.

For the special case of the model, where the income risk can be fully hedged by appropriate financial investments and there are no portfolio constraints, we are able to obtain closed-form expressions for the value of the labor income stream (i.e. human wealth) and the optimal consumption and investment strategies, which allows us to perform a detailed analysis. While it is well-known from the literature that the labor income increases the optimal speculative investments due to a wealth effect, we show that the relative allocation to bonds and stocks can be significantly affected by the presence of uncertain labor income for several reasons. First, bonds and stocks can be differently correlated with labor income shocks so that bonds may be better for hedging income rate shocks than stocks or vice versa. Second, risk-averse investors want to hedge total wealth against shifts in investment opportunities. When the short-term interest rate captures the investment opportunities, the appropriate asset for this hedging motive is the bond. Third, since human wealth is defined as the discounted value of the future income stream, it will in general be sensitive to the interest rate level like a bond and, hence, the income stream represents an implicit investment in a bond, so that the explicit bond investment is reduced. As explained above, the expected growth of labor income may itself be sensitive to the level of interest rates, which will also affect hedging demand. We also give a detailed numerical analysis of this special case, illustrating the sign and magnitudes of various portfolio components and their sensitivity to key parameters. The optimal strategies will often involve extremely large positions in bonds and stocks and an extreme level of borrowing. In this unconstrained case, the investor is even allowed to borrow against future labor income so that his financial wealth in some situations will be negative.

The assumptions of spanned income and no portfolio constraints are certainly questionable. It is generally believed that labor income risk is not fully insurable and, due to moral hazard and adverse selection considerations, investors can only to a very limited extent borrow money using future income as implicit collateral. Introducing non-spanned income risk and portfolio constraints, we must solve the utility-maximization problem by a numerical method. We solve the dynamic programming equation associated with the problem with a finite difference technique. We show in numerical examples that imposing the constraint that financial wealth has to stay non-negative at all times reduces risk-taking considerably but will still involve a high degree of borrowing and large positions in the bond and stock. When we, in addition, prevent the individual from taking short positions (and hence from borrowing at all), we see that the optimal portfolio will often consist of 100% in the stock, but this result is sensitive to correlation parameters, the degree of risk aversion, and the investment horizon.

Let us briefly review the relevant literature for this study. As first noted by Merton (1971), long-term investors will generally hedge stochastic variations in the investment opportunity set, e.g. interest rates, excess returns, and volatilities. Several recent papers study optimal investments in concrete settings with interest rate uncertainty. Sørensen (1999) and Brennan and Xia (2000) consider interest rate dynamics as in the Vasicek (1977) model and assume complete financial markets and constant market prices of both interest rate risk and stock market risk. They find that the optimal investment strategy of an investor with power utility of terminal wealth only is a simple combination of the mean-variance optimal portfolio, i.e. the optimal portfolio assuming investment opportunities do not change, and the zero-coupon bond maturing at the end of the investment horizon. Liu (1999) provides similar insight using the one-factor square-root model of Cox, Ingersoll and Ross (1985). Furthermore, Liu shows that in a complete and "affine" market the appropriate interest rate hedge instrument for an investor with a time-additive power utility of consumption is some sort of coupon bond rather than a zero-coupon bond. Other studies of portfolio choice with uncertain interest rates include Brennan, Schwartz, and Lagnado (1997), Campbell and Viceira (2001), Detemple, Garcia, and Rindisbacher (2003), and Munk and Sørensen (2004). None of these papers take into account a labor income stream of the investor, although labor income is the main source of funds for most individuals.

On the other hand, several papers discuss how the presence of a labor income process affects the consumption and investment decisions of individual investors, but all in the context of constant financial investment opportunities. A deterministic income stream is equivalent to an implicit investment in the riskless asset and, hence, it is optimal to invest a higher fraction of financial wealth in the risky assets than in the no-income case; cf., e.g., Hakansson (1970) and Merton (1971). With stochastic income, but fully hedgeable income risk, the optimal unconstrained strategies can be deduced from the optimal strategies without labor income; see, e.g., Bodie, Merton, and Samuelson (1992). Duffie, Fleming, Soner, and Zariphopoulou (1997), Koo (1998), and Munk (2000) study (mostly by use of numerical methods) the valuation of income and the optimal consumption and investment strategies of an infinite-horizon, liquidity constrained power utility investor with non-spanned income risk. The presence of liquidity constraints can significantly decrease the individual's implicit valuation of the future income stream and, hence, dampen the quantitative effects of income on portfolio choice. Other recent papers on consumption and portfolio choice with stochastic income include Cocco, Gomes, and Maenhout (2005), Constantinides, Donaldson, and Mehra (2002), El Karoui and Jeanblanc-Picqué (1998), He and Pagès (1993), Heaton and Lucas (1997), and Viceira (2001). Besides working with constant investment opportunities, the concrete models with stochastic income in these papers assume a single risky asset, interpreted as the stock market index. Since different risky assets will have different correlations with the labor income of a given individual, this assumption is not without loss of generality. In our numerical examples, we will have two risky assets, namely a stock and a bond.

The outline of the rest of the paper is as follows. In Section 2 we set up the general model of the financial market and specify the preferences and income of the individual. Section 3 discusses the calibration of the model and the choice of benchmark parameter values. Section 4 focuses on the case with spanned income uncertainty and unconstrained investment strategies, where we derive, illustrate, and discuss the explicit solution to the decision problem of an investor with constant relative risk aversion. In Section 5 we then turn to the case with unspanned income uncertainty and liquidity constraints on the investment strategy, where again we provide relevant numerical examples. In Section 6 we study the effects of introducing a realistic life-cycle pattern in expected income growth. Section 7 gives some concluding remarks. The appendices contain proofs of propositions and lemmas and also a detailed description of the numerical method applied in the solution of the problem with unspanned income and liquidity constraints.

2 Description of the model

We model the intertemporal consumption and investment choice of a price-taking individual who can trade in stocks and bonds and receives a stochastic stream of income from non-financial sources, say labor income. We assume that the economy has a single perishable consumption good which serves as a numeraire so that all asset prices, interest rates, and income rates are specified in units of this good, i.e. in *real* terms.

2.1 Financial assets

We assume that the real short-term interest rate follows the Vasicek (1977) model,

$$dr_t = \kappa \left(\bar{r} - r_t\right) \, dt - \sigma_r \, dz_{rt}, \tag{1}$$

where κ, $\bar{r}$, and σ_r are positive constants, and $z_r = (z_{rt})_{t \geq 0}$ is a standard Brownian motion. We also assume that the market prices of interest rate risk, λ_r, is constant. The price of a zero-coupon bond paying one unit of account at some time $\bar{T}$ is then given by

$$B_t^{\bar{T}} \equiv B^{\bar{T}}(r_t, t) = e^{-a(\bar{T}-t)-b(\bar{T}-t)r_t}, \tag{2}$$

where

$$b(\tau) = \frac{1}{\kappa}\left(1 - e^{-\kappa\tau}\right), \tag{3}$$

$$a(\tau) = y_\infty[\tau - b(\tau)] + \frac{\sigma_r^2}{4\kappa} b(\tau)^2, \tag{4}$$

$$y_\infty = \bar{r} + \frac{\sigma_r \lambda_r}{\kappa} - \frac{\sigma_r^2}{2\kappa^2}. \tag{5}$$

Here y_∞ is the limit of the yield of a zero-coupon bond as maturity goes to infinity, i.e. the asymptotic long rate.

Any desired interest rate exposure can be obtained by combining deposits/loans at the short-term interest rate (interpreted as cash or the bank account) and a single default-free real bond.[1] The dynamics of the price B_t of such a bond is given by

$$dB_t = B_t \left[(r_t + \sigma_B(r_t,t)\lambda_r) \, dt + \sigma_B(r_t,t) \, dz_{rt} \right], \tag{6}$$

where $\sigma_B(r_t,t) > 0$ is the bond price volatility, which will generally depend on both the interest rate level and the time-to-maturity and hence on time. However, for a zero-coupon bond the volatility is $\sigma_r b(\bar{T}-t)$, which depends on the time-to-maturity $\bar{T}-t$, but not on the interest rate level. The bond price has a perfectly negative (instantaneous) correlation with the interest rate, $\rho_{Br} = -1$.

In addition to the bond, we assume that agents can invest in stocks. In the general analysis we will allow agents to trade in n stocks. The price of stock i at time t is denoted by S_{it} and is assumed to evolve as

$$dS_{it} = S_{it} \left[(r_t + \psi_i) \, dt + \sigma_i \left(\rho_{iB} \, dz_{rt} + \sum_{j=1}^{i} k_{ij} \, dz_{jt} \right) \right], \quad i = 1, \dots, n, \tag{7}$$

where $z_{St} = (z_{1t}, \dots, z_{nt})^\top$ is an n-dimensional standard Brownian motion independent of z_{rt}, ψ_i is the constant expected excess return, σ_i is the constant volatility, and $\rho_{iB} = -\rho_{ir}$ is the constant correlation between stock i and the bond. Finally, the constants k_{ij} will determine the correlations between the n stocks.[2] We gather the dynamics of stock prices in vector form as

$$dS_t = \mathrm{diag}(S_t) \left[(r_t \mathbf{1}_n + \psi) \, dt + \mathrm{diag}(\sigma_S) \left\{ \rho_{SB} \, dz_{rt} + K \, dz_{St} \right\} \right], \tag{8}$$

where $\mathrm{diag}(x)$ means the diagonal matrix with the vector x along the diagonal, $\mathbf{1}_n$ is an n-dimensional vector of ones, K is the lower triangular matrix $[k_{ij}]$, and we have introduced the vectors $\psi = (\psi_1, \dots, \psi_n)^\top$, $\sigma_S = (\sigma_1, \dots, \sigma_n)^\top$, and $\rho_{SB} = (\rho_{1B}, \dots, \rho_{nB})^\top$. Note that in order to focus on the effects of combining stochastic labor income and stochastic interest rates, we have chosen to assume constant expected excess returns and volatilities on the stocks, despite ample evidence that these quantities vary over the business cycle.

To simplify some of the following expressions, we introduce the vector $P_t = (B_t, S_t)^\top$ of prices of all $n+1$ risky assets. By combining the dynamics of B_t and S_t, we get

$$dP_t = \mathrm{diag}(P_t) \left[(r_t \mathbf{1}_{n+1} + \Sigma(r_t,t)\lambda) \, dt + \Sigma(r_t,t) \, dz_t \right], \tag{9}$$

[1] We assume that real, i.e. inflation-indexed, bonds are traded. Brennan and Xia (2002) and Munk, Sørensen, and Vinther (2004) study how real interest rate risk affects optimal investments when the traded bonds are nominal.

[2] The correlation between stock i and stock l (with $i \le l$) is $\rho_{il} = \rho_{iB}\rho_{lB} + \sum_{j=1}^{i} k_{ij}k_{lj}$. Since $\rho_{11} = 1$, we must have $k_{11} = \sqrt{1-\rho_{1B}^2}$. We have $\rho_{12} = \rho_{1B}\rho_{2B} + k_{11}k_{21}$ implying that $k_{21} = (\rho_{12} - \rho_{1B}\rho_{2B})/\sqrt{1-\rho_{1B}^2}$. Since $1 = \rho_{22} = \rho_{2B}^2 + k_{21}^2 + k_{22}^2$, we can then conclude that $k_{22} = \sqrt{1-\rho_{2B}^2 - k_{21}^2}$. Continuing this way, we can express all the k_{ij} constants in terms of the correlations between the individual stocks and the correlations between the stocks and the bond.

where $z = (z_r, z_S)^\top$ and $\Sigma(r_t, t)$ is the $(n+1) \times (n+1)$ matrix

$$\Sigma(r_t, t) = \begin{pmatrix} \sigma_B(r_t, t) & 0 \\ \operatorname{diag}(\sigma_S)\rho_{SB} & \operatorname{diag}(\sigma_S) K \end{pmatrix}.$$

Furthermore, $\lambda = (\lambda_r, \lambda_S)^\top$ where $\lambda_S = (\lambda_1, \ldots, \lambda_n)^\top$ is the n-dimensional vector

$$\lambda_S = K^{-1} \left[\operatorname{diag}(\sigma_S)^{-1} \psi - \rho_{SB} \lambda_r \right].$$

The vector λ has the interpretation as the market price of risk vector. For example, λ_i is the excess expected return per unit volatility on an asset which is only sensitive to z_i and not to any other random shocks.

2.2 The preferences and labor income of the individual

We assume throughout the paper that the individual has a time-additive utility function of consumption c_t and possibly terminal wealth W_T and seeks to maximize

$$\mathrm{E}\left[\int_0^T e^{-\delta t} U(c_t)\, dt + \varepsilon e^{-\delta T} U(W_T) \right],$$

where T is the time of death, assumed non-random, and ε is a 0-1 parameter indicating whether or not the individual has utility from leaving wealth to her heirs. Throughout the paper we use a power utility function

$$U(c) = \frac{1}{1-\gamma} c^{1-\gamma},$$

where $\gamma > 0$ is the constant relative risk aversion.

We set up a relatively simple model of income that is tractable and allows us to focus on the interaction between stochastic income and stochastic interest rates. We assume that the individual receives a continuous stream of non-negative income from non-financial sources throughout her life. The income rate at time t is denoted by y_t. We assume that y_t evolves as[3]

$$dy_t = y_t \left[(\xi_0(t) + \xi_1 r_t)\, dt + \sigma_y(t) \left\{ \rho_{yP}^\top\, dz_t + \sqrt{1 - \|\rho_{yP}\|^2}\, dz_{yt} \right\} \right], \tag{10}$$

where $z_y = (z_{yt})$ is a one-dimensional standard Brownian motion independent of z_r and z_S. The constant vector ρ_{yP} is defined as $(\rho_{yB}, \hat{\rho}_{yS})^\top$, where $\rho_{yB} = -\rho_{yr}$ is the instantaneous correlation between the income rate and the bond price, and the vector $\hat{\rho}_{yS}$ is given by $\hat{\rho}_{yS} = K^{-1} (\rho_{yS} - \rho_{SB}\rho_{yB})$, where ρ_{yS} is the vector of correlations between the income rate and the stocks and the other terms have been defined above. If $\|\rho_{yP}\|^2 = 1$, the income rate

[3]As most authors, we have modeled the income stream as an exogenously given process. Of course, in real life the individual can affect her labor income to some extent by choice of education and effort. To avoid further complications of the model we do not endogenize the labor supply decision. We refer the reader to Bodie, Merton, and Samuelson (1992) and Chan and Viceira (2000).

is spanned, i.e. only sensitive to the traded risks represented by z. If that is the case, and there are no portfolio constraints, the income process can be replicated by some dynamic trading strategy of the traded assets and hence valued as a traded asset.

The income process have the following features:

- The percentage drift $\xi_0(t) + \xi_1 r_t$ and volatility $\sigma_y(t)$ are allowed to depend on time in order to reflect the empirically relevant variations in expected income growth and uncertainty over the life of an individual, cf. e.g. Hubbard, Skinner, and Zeldes (1995) and Cocco, Gomes, and Maenhout (2005).

- In contrast to other studies, we allow the drift to depend on the interest rate in order to incorporate the plausible link between the expected growth in income and the overall well-being of the economy.

- For simplicity, we ignore retirement and assume that y describes the income rate until time T.

- Our model does not allow for jumps in income, although jumps reflecting for example lay-offs may be relevant. Results of Cocco, Gomes, and Maenhout (2005) indicate that even a small probability of a large sudden income reduction may substantially affect portfolio choice. However, for many individuals the possible unemployment periods are likely to be rather short and in many countries individuals can partly insure against temporary income losses due to unemployment. Hence, it may not be that important to explicitly allow for large drops in income.

- Several empirical studies suggest a link between expected excess stock returns and the expected growth and riskiness of labor income (see for example Jagannathan and Wang (1996), Julliard (2004), and Storesletten, Telmer, and Yaron (2004)), but we will not address the potential implications for portfolio choice in this paper.

2.3 Optimal strategies

The individual is to choose a consumption strategy $c = (c_t)$ and an investment strategy $\theta = (\theta_t)$. Here c_t is the rate at which goods are consumed at time t with the natural requirement that $c_t \geq 0$ at all times and in all states of the economy. Furthermore, θ_t is a vector $(\theta_{Bt}, \theta_{St})^\top$ of the amounts (i.e. units of the consumption good) invested at time t in the bond and the n stocks. With W_t denoting the financial wealth of the investor at time t, the amount invested in the bank account (held in "cash") is residually determined as $\theta_{0t} = W_t - \theta_{Bt} - \theta_{St}^\top \mathbf{1}_n$. Given a consumption strategy c and an investment strategy θ, the financial wealth of the individual W_t evolves as

$$dW_t = (r_t W_t + \theta_t^\top \Sigma(r_t, t)\lambda - c_t + y_t)\, dt + \theta_t^\top \Sigma(r_t, t)\, dz_t. \tag{11}$$

The consumption and investment strategies must satisfy some technical conditions for the wealth process to be well-defined. If there are no other restrictions on the strategies (except $c_t \geq 0$) we denote by $\mathcal{A}_t^{\text{unc}}$ the set of admissible consumption and investment strategies (c, θ) over the time interval $[t, T]$. We will also consider the case where it is not possible for the individual to borrow funds using future income as collateral so that her financial wealth W_t must stay non-negative at all times and in all states of the world.[4] In a continuous-time setting this is implemented by requiring that whenever the financial wealth hits zero, the investor must eliminate her positions in bonds and stocks. After she have received labor income, she may again enter the markets for risky securities. We denote the set of admissible strategies with this constraint by $\mathcal{A}_t^{\text{con}}$.

The indirect utility function of the individual is defined as

$$J(W, r, y, t) = \sup_{(c,\theta) \in \mathcal{A}_t} \mathrm{E}_t \left[\int_t^T e^{-\delta(s-t)} U(c_s)\, ds + \varepsilon e^{-\delta(T-t)} U(W_T) \right], \tag{12}$$

where the expectation is computed given the values of W, r, y at time t and given the strategy (c, θ). The set $\mathcal{A}_t$ is either equal to $\mathcal{A}_t^{\text{unc}}$ or $\mathcal{A}_t^{\text{con}}$. With the assumed CRRA utility function the marginal utility is infinite at zero consumption so that the non-negativity constraint on consumption is not binding. The Hamilton-Jacobi-Bellman (HJB) equation associated with this dynamic optimization problem is

$$\begin{aligned} \delta J = \sup_{c,\theta} \Big\{ & U(c) + J_t + J_W \left(rW + \theta^\top \Sigma \lambda - c + y \right) + \frac{1}{2} J_{WW} \theta^\top \Sigma \Sigma^\top \theta \\ & + J_r \kappa [\bar{r} - r] + \frac{1}{2} J_{rr} \sigma_r^2 + J_y y (\xi_0 + \xi_1 r) + \frac{1}{2} J_{yy} y^2 \sigma_y^2 \\ & - J_{Wr} \theta^\top \Sigma \mathbf{e}_1 \sigma_r + J_{Wy} y \sigma_y \theta^\top \Sigma \rho_{yP} + J_{ry} y \rho_{yr} \sigma_y \sigma_r \Big\}, \end{aligned} \tag{13}$$

where $\mathbf{e}_1 = (1, \mathbf{0}_n)^\top$, subscripts on J denote partial derivatives, and we have suppressed the arguments of the functions for notational simplicity. The terminal condition is $J(W, r, y, T) = \varepsilon U(W) = \varepsilon W^{1-\gamma}/(1-\gamma)$.

The first-order condition for consumption is the standard envelope condition

$$U'(c_t) = J_W(W_t, r_t, y_t, t) \quad \Rightarrow \quad c_t = \left[J_W(W_t, r_t, y_t, t) \right]^{-1/\gamma}. \tag{14}$$

For the unrestricted investment case the first-order condition for the portfolio θ implies that

$$\theta_t = -\frac{J_W}{J_{WW}} \left(\Sigma(r_t, t)^\top \right)^{-1} \lambda - \frac{J_{Wy}}{J_{WW}} y \sigma_y(t) \left(\Sigma(r_t, t)^\top \right)^{-1} \rho_{yP} + \frac{J_{Wr}}{J_{WW}} \frac{\sigma_r}{\sigma_B(r_t, t)} \mathbf{e}_1. \tag{15}$$

The first part corresponds to the standard mean-variance optimal portfolio, the second part is a hedge against changes in the income rate, while the third part is a hedge against changes

[4] This "hard" borrowing constraint is standard in the literature. A recent paper by Davis, Kubler, and Willen (2003) studies the portfolio choice under a "soft" borrowing constraint that allows individuals to borrow even with a negative current wealth although at a rate higher than the riskless interest rate. Their study assumes constant interest rates. To focus on the interaction between stochastic interest rates and stochastic labor income we stick to the "hard" borrowing constraint, which is easier to handle.

in the interest rate. The income hedge term reflects a position in the portfolio with relative weights given by $\left(\Sigma(r_t,t)^{\top}\right)^{-1}\rho_{yP}/\mathbf{1}_{n+1}^{\top}\left(\Sigma(r_t,t)^{\top}\right)^{-1}\rho_{yP}$. This is the portfolio with the maximal absolute correlation with the income rate of the individual, cf. Ingersoll (1987, Ch. 13). This maximal correlation equals $\|\rho_{yP}\|$ so if the income rate is spanned, this correlation will equal 1. Since the bond price is perfectly negatively correlated with the interest rate, the interest rate is hedged by a position in the bond only. In contrast, the income hedge and the mean-variance terms generally involve all risky assets. The remaining wealth, $W_t - \theta_t^{\top}\mathbf{1}_{n+1}$, is invested in the bank account. In Section 4 we consider the case where the income rate is spanned, which – given our other assumptions – allows the optimal strategies to be computed in closed form.

3 Calibration of parameters

In the calibration of income parameters, we apply modeling ideas from the discrete-time settings of Cocco, Gomes, and Maenhout (2005) and Campbell and Viceira (2002) and decompose log-income into a personal component and a common component, so that the income of individuals with characteristics i is given by

$$\log y_t^i = P^i(t) + \log y_t^c \tag{16}$$

where the personal component $P^i(t)$ reflects the expected life-cycle income of individuals with characteristics i. The dynamics of the common income component are described by an income process of the form in (10), but with constant parameters, $\xi_0(t) = \bar{\xi}_0$ and $\sigma_y(t) = \sigma_y$.

In order to adopt estimation results from Cocco, Gomes, and Maenhout (2005) and Campbell and Viceira (2002), we will in our calibration approach assume that the personal income component $P^i(t)$ is deterministic and described by a third-order polynomial in time (age),

$$P^i(t) = a^i + b^i t + c^i t^2 + d^i t^3$$

where a^i, b^i, c^i, and d^i are constant parameters. The polynomial form of $P^i(t)$ only applies for age 20 until 65 (years). In the discrete-time framework of Cocco, Gomes, and Maenhout (2005), it is thus assumed that the retirement income level for age 66 and higher is proportional to the income level at age 65 with a replacement rate, $\bar{P}^i$. We will adopt this form, and in our continuous-time setting we simply linearly interpolate the income level in the one-year period between age 65 and age 66. Cocco, Gomes, and Maenhout (2005) use Panel Study on Income Dynamics (PSID) data to estimate labor income as a function of age and other characteristics. In our numerical analysis of life-cycle allocations, we will adopt their estimated third-order polynomial structures and replacement rates for groups characterized by three different educational backgrounds: "No High school", "High school", and "College". The polynomial life-cycle income profiles are illustrated in Figure 1, and the polynomial coefficients used in the figure are reproduced from Cocco, Gomes, and Maenhout (2005) in Table 1.

[Figure 1 about here.]

[Table 1 about here.]

With the above decomposition and assumptions, it can be inferred (by an application of Ito's lemma) that the individual income process is of the form in (10) with,

$$\xi_0(t) = \bar{\xi}_0 + \frac{dP^i(t)}{dt} = \bar{\xi}_0 + \begin{cases} b^i + 2c^i t + 3d^i t^2 & \text{if } 20 \leq t \leq 65 \\ -(1-\bar{P}^i) & \text{if } 65 < t < 66 \\ 0 & \text{if } t \geq 66 \end{cases} \tag{17}$$

We take the income rate volatility to be $\sigma_y(t) = 0.10$ for $t \leq 65$ and $\sigma_y(t) = 0$ for $t \geq 65$.

In order to estimate parameters of the common component of labor income as well as correlations and parameters of real interest rates and stock prices, we consider the case with only a single stock. The relevant dynamics are described by the real interest rate dynamics in (1), the single-stock special case of (8), and an income process of the form in (10) with $\xi_t(t) = \bar{\xi}_0$ and constant income volatility σ_y. Let $Y_t = (\log y_t^c, \log S_t, r_t)'$, then the relevant dynamics can be summarized by the linear stochastic differential equation,

$$dY_t = (A + BY_t)dt + Vd\hat{z}_t \tag{18}$$

where

$$A = \begin{pmatrix} \bar{\xi}_0 - \frac{1}{2}\sigma_y^2 \\ \psi - \frac{1}{2}\sigma_S^2 \\ \kappa\bar{r} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & \xi_1 \\ 0 & 0 & 1 \\ 0 & 0 & -\kappa \end{pmatrix}, \quad VV' = \begin{pmatrix} \sigma_y^2 & \rho_{yS}\sigma_y\sigma_S & \rho_{yr}\sigma_y\sigma_r \\ \rho_{yS}\sigma_y\sigma_S & \sigma_S^2 & \rho_{Sr}\sigma_S\sigma_r \\ \rho_{yr}\sigma_y\sigma_r & \rho_{Sr}\sigma_S\sigma_r & \sigma_r^2 \end{pmatrix},$$

and $\hat{z}_t$ is a three-dimensional standard Brownian motion. The discrete-time solution to the linear stochastic differential equation in (18) is a discrete-time VAR(1)-model of the form

$$Y_{t+\Delta} = A(\Psi,\Delta) + B(\Psi,\Delta)Y_t + \tilde{\epsilon}_{t+\Delta}, \;\; \tilde{\epsilon}_{t+\Delta} \sim N(0, \Omega(\Psi,\Delta)) \tag{19}$$

where $\Psi = (\bar{\xi}_0, \xi_1, \psi, \kappa, \bar{r}, \sigma_y, \sigma_S, \sigma_r, \rho_{yS}, \rho_{yr}, \rho_{Sr})$ denotes the set of parameters to be estimated. The functions $A(\Psi,\Delta)$, $B(\Psi,\Delta)$, and $\Omega(\Psi,\Delta)$ can be obtained in closed-form using the general solution formula for linear stochastic differential equations (see, e.g., Karatzas and Shreve (1988, pp. 354-357)).

The VAR(1)-model in (19) is estimated by maximum likelihood using quarterly US data which span the period from 1951 until 2003.[5] The total number of observation time points is

[5] The numerical maximum likelihood estimation is carried out using the software program GAUSS. However, in evaluating the relevant first and second order moments we used the analytical evaluation tools in the software program Mathematica and pasted the relevant analytical results into the GAUSS program.

208. The per capital income data was obtained from the Personal Income and Its Disposition NIPA table of the National Economic Accounts. The income data is the per capita disposable personal income after personal current taxes and subtracted personal income from financial assets. The cum dividend stock returns are constructed using quarter-end values of the S&P 500 index over the period while the S&P 500 dividends and the CPI-index data are adopted from Shiller (2000); the updated data were downloaded from Robert Shiller's homepage. All income and stock prices are in real terms (using the CPI-index as deflator). Real interest rates are constructed by subtracting an estimate of the inflation rate from the 3-month nominal interest rate. The subtracted inflation rate is obtained as the average realized inflation rate in the last four quarters relative to the same quarters one year earlier. The applied data is illustrated in Figure 1 where the income index and the stock index are scaled so that they start out in one in 1951 (and zero for the logarithmic value).

[Figure 2 about here.]

The estimated parameters values are displayed in Table 2 together with our chosen benchmark parameters for the subsequent numerical analysis. Our benchmark parameters are chosen close to the estimated values (in most cases the benchmark parameters are just rounded off). Some parameters are chosen to be similar or comparable to those applied by, e.g., Campbell and Viceira (2002) who use $\psi = 4\%$ and a real interest rate level of 2%. The volatility in the per capita log-income process is estimated at 2.08%, but our benchmark parameter value is set significantly higher at 10%. This choice is motivated by results based on household level data that indicate that individual income volatility is at least at this level.[6]

Individual income may be more or less correlated with basic macro economic values such as real interest rates and stock returns, but we choose correlations close to the estimated values for per capita income as our benchmark parameter values. Other authors have previously estimated correlations between labor income and stock market returns. Davis and Willen (2000) find that – depending on the individuals sex, age, and educational level – the correlation between aggregate stock market returns and labor income shocks is between -0.25 and 0.3, while the correlation between industry-specific stock returns and labor income shocks is between -0.4 and 0.1. Campbell and Viceira (2002) report that the correlation between aggregate stock market returns and labor income shocks is between 0.328 and 0.516. Heaton and Lucas (2000) find that the labor income of entrepreneurs typically is more highly correlated with the overall stock market (0.14) than with the labor income of ordinary wage

[6]For example, the estimates presented in Cocco, Gomes, and Maenhout (2005) and Campbell and Viceira (2002), among others, suggest a permanent income volatility slightly above 10% (for all three educational groups) as well as transitory income volatility components of the magnitude 25%. Our model of income dynamics include basically only shocks of the permanent type.

earners (-0.07). In all our numerical experiments we will assume that individuals invest in only one stock, representing the stock market index. While a broad stock market index is clearly less than perfectly correlated with individual labor income, it is to be expected that a larger fraction of the income rate variations can be hedged using multiple risky assets, but we have no information about the typical correlations between a labor income stream and individual stocks–and hence we really do not know how large a fraction of the risk of a typical labor income stream that can be hedged in the financial markets.[7]

Our estimation approach does not involve the risk premium on real interest rate risk, but our benchmark parameter value is set to $\lambda_r = 0$. This value is not far from the about 1% historical excess return on longer-term *nominal* bonds which is usually reported in empirical studies; see, e.g., Dimson, Marsh, and Staunton (2002). The long-term yield is then 1.92%, the yield curve is increasing if the current short rate is below 1.88%, decreasing if the current short rate exceeds 2%, and humped for intermediary values. The steady state distribution of r has a standard deviation of 0.02 (and a mean equal to $\bar{r}$, of course). The interest rate risk in our model can be hedged with a single bond, which in our numerical examples is taken to be a 10-year (real) zero-coupon bond. With the benchmark parameters, this bond will have an expected rate of return equal to the short-term interest rate and a volatility of $\sigma_B = 0.03973$.

In many of our numerical experiments we will focus on issues that do not require life-cycle income considerations. In these numerical examples, we take $\xi_1 = 0$ as the benchmark and choose the income parameter function $\xi_0(t)$ as a constant equal to 4%. This choice reflects a general increase in real income of about 2% due to common real income growth (as reflected in the estimated value of $\bar{\xi}_0$) and an about 2% expected increase in income for relatively younger individuals due to getting working experience and competence (as reflected in the slope of the life-cycle income profiles in Figure 1 for relatively young investors). However, it may be noted that ξ_1 is estimated with relatively large error and close to zero. ξ_1 describes how expected income growth is related to the real interest level, and there is thus no clear empirical relationship in our framework. Our estimations, however, also indicated that the parameter estimates of ξ_1 and ρ_{yr} are negatively correlated. If, for example, ρ_{yr} is fixed at zero, then ξ_1 is estimated significantly positive at around 0.25. We will also provide numerical examples with a non-zero ξ_1 so that the income drift varies with the interest rate

[7] Allowing for multiple stocks may "help" as indicated by a small example. Assume n stocks that are similar in the sense that they have identical expected rates of return ψ_i, identical volatilities σ_i, identical correlations ρ_{iB} with the bond, identical correlations ρ_{yi} with the labor income rate, and all pairwise stock-stock correlations are equal to ρ. Then the income process is spanned whenever

$$(\rho_{yi} - \rho_{iB}\rho_{yB})^2 = \left(1 - \rho_{yB}^2\right)\left(\frac{1-\rho}{n} + \rho - \rho_{iB}^2\right).$$

The value of ρ_{yi} for this equation to hold is decreasing in n, the number of stocks.

level. For a given ξ_1, we will then pick ξ_0 so that the income drift will be 0.04 whenever the interest rate is equal to its long-term level, i.e. we assume that $\xi_0 + \xi_1 \bar{r} = 0.04$. For a positive ξ_1, this implies that the expected income growth rate is above [below] 4% for higher [lower] than average interest rates. In Section 6 we study the effects of introducing a realistic life-cycle pattern in expected income growth by allowing for time-dependence in ξ_0.

[Table 2 about here.]

4 Spanned income and no investment constraints

In this section we assume that the income stream is fully spanned by the traded assets, i.e. that $\|\rho_{yP}\| = 1$, and that there are no other constraints on the optimal strategies of the consumer-investor. In our numerical examples, we take the benchmark parameter values as discussed in Section 3, except that we have to restrict ourselves to the case where the income process is spanned by the financial assets. For a fixed value of the stock-bond correlation ρ_{SB}, the combinations of the stock-income and the interest rate-income correlation which ensures spanning satisfy the relation

$$\rho_{yS} + \rho_{SB}\rho_{yr} = \pm\sqrt{(1-\rho_{yr}^2)(1-\rho_{SB}^2)}. \tag{20}$$

With the benchmark value $\rho_{SB} = 0$, the points (ρ_{yr}, ρ_{yS}) satisfying (20) trace out the circle shown in Figure 3. Clearly, any combination of correlations on the circle will be far from the correlation values reported in Section 3. We consider different combinations where the correlations have the same absolute values, which must then be $1/\sqrt{2} \approx 0.7071$, and either the same sign or opposite signs, i.e. $\rho_{yr}, \rho_{yS} \in \{-0.7071, +0.7071\}$.

[Figure 3 about here.]

4.1 Human wealth

Since the income process is spanned, it can be valued as the dividend stream from a traded asset. The market value at time t of the income stream over the time period $[t, T]$ is

$$H(y, r, t) = \mathrm{E}_t^{\mathbb{Q}}\left[\int_t^T y_s e^{-\int_t^s r_v\, dv}\, ds\right], \tag{21}$$

where $\mathbb{Q}$ denotes the unique risk-neutral probability measure. We can think of the individual selling the remaining income stream for the amount $H(y, r, t)$, her "human wealth." As described below the optimal strategies can in this case be derived from the optimal strategies for the case without income but with a financial wealth of $W_t + H(y_t, r_t, t)$ instead of just W_t. Under our assumptions on the dynamics of the labor income rate and the short-term interest rate, we are able to derive an explicit expression of the human wealth as shown in the following proposition. The proof is given in Appendix A.[8]

Proposition 1 *Under the assumptions above, the human wealth is given by*

$$H(y,r,t) = yM(r,t) \equiv y \int_t^T h(t,s) \left(B^s(r,t)\right)^{1-\xi_1} \, ds, \tag{22}$$

where

$$\begin{aligned} \ln h(t,s) = & \int_t^s \left(\xi_0(u) - \sigma_y(u)\rho_{yP}^\top \lambda - (\xi_1 - 1)\rho_{yB}\sigma_r\sigma_y(u)b(s-u)\right) du \\ & + \xi_1(\xi_1 - 1)\frac{\sigma_r^2}{2\kappa^2}\left(s - t - b(s-t) - \frac{\kappa}{2}b(s-t)^2\right) \end{aligned} \tag{23}$$

with the function b given by (3).

Due to the assumed income rate process, the human wealth is separated as the product of the current income rate, y, and a multiplier, $M(r,t)$, depending only on the interest rate and time.

Figure 4 shows the income multiplier, $M(r,0)$, as a function of the time horizon, T, for the four different combinations of the two income-asset correlation parameters.[9] Firstly, note the magnitude of the values of the multiplier. For young individuals, the current labor income may be relatively small, but the present value of all future labor income can be 100 times higher so that the human wealth may very well be much, much higher than the current financial wealth. This is certainly relevant to take into account when planning your life-cycle consumption and investment strategies. Secondly, it is obvious that the value of the income stream depends on the correlations with the traded financial assets. In particular, the human wealth is considerably higher for an income stream which is negatively correlated with the stock market, than for an income stream of a similar magnitude, but positively correlated with the stock market. If the correlation is positive [negative], the future expected income is discounted at a rate higher [lower] than the risk-free rate. With a positive correlation between the short-term interest rate and the income rate, the income tends to be high when the discount rate is high, which lowers the present value of the income.

[Figure 4 about here.]

Figure 5 illustrates the relation between the income multiplier, $M(r,0)$, and the current interest rate, r, for an individual with a 30-year horizon for different combinations of the income drift parameters (ξ_0, ξ_1) so that $\xi_0 + \xi_1\bar{r} = 0.04$. The graphs are generated using

[8]Given (2), the human wealth expression in (22) can also be written in the exponential-affine form $H(y,r,t) = \int_t^T \exp\{A_0(t,s) + A_1(t,s)r + A_2(t,s)\ln y\}\, ds$. This is the case in any setting where the risk-neutral dynamics of r_t and $\ln y_t$ are affine; c.f., e.g., Duffie, Pan, and Singleton (2000). We focus on a non-trivial case where the functions A_0, A_1, A_2 can be stated in closed form (involving some simple integrals).

[9]Integrals are computed numerically using Romberg's method of order 10.

$\rho_{yr} = \rho_{ys} = 0.7071$, but the picture is very similar for the other combinations of income-asset correlations. As long as $\xi_1 < 1$, human wealth is a decreasing, convex function of the current short-term interest rate. We see that the human wealth varies considerably with the interest rate except for the case where $\xi_1 = 1$ for which the positive relation between the income growth rate and the interest rate exactly offsets the discounting effect. We also see that the exact decomposition of the expected income growth rate into a constant part and an interest rate related part does affect the capitalized value of the income process. For the case where $\xi_1 = 0$ so that the income rate drift is independent of the interest rate level, we see from (22) that the human wealth equals the value of a bond paying continuous coupons of size $h(t, s)$ at time s. Note that this coupon can be interpreted as an appropriately risk-adjusted expected income rate at time s. For general ξ_1, we can similarly interpret the human wealth as the value of a sort of flexible-rate continuous coupon bond with the time s coupon given by $h(t, s)(B^s(r, t))^{-\xi_1}$, which is increasing in the interest rate as long as ξ_1 is positive. For interest rates around the long-term level of 2%, the human wealth is relatively insensitive to the decomposition of expected income. This is partly due to the high value of the mean reversion parameter κ, which implies that the interest rate is pulled quickly towards the long-term level.

[Figure 5 about here.]

In a model with constant interest rates and a single risky asset whose price is perfectly positively correlated with labor income, human wealth will be decreasing in the volatility of the income rate. As we can see from (22), this can be different in our model with two risky assets depending on the precise correlation parameters, the market prices of risk, and also the value of the income growth parameter ξ_1. Figure 6 illustrates the relation between the 30-year income multiplier, $M(r, 0)$, and the income rate volatility, σ_y, for the four different combinations of the values of the two income-asset correlations. In the two cases with a negative stock-income correlation, the human wealth is in fact positively related to the income uncertainty, but this can also happen in cases with a positive stock-income correlation. The figure also illustrates that the precise value of the income rate volatility can be quite important for the human wealth.

[Figure 6 about here.]

4.2 Optimal strategies

4.2.1 Derivation

With time-additive CRRA utility it is well-known that indirect utility function with wealth W_t and no income is then of the form

$$V(W,r,t) = \frac{1}{1-\gamma} g(r,t)^{\gamma} W^{1-\gamma},$$

where $g(r,t)$ is a function that depends on the remaining investment horizon (and hence on time) and the risk aversion parameter γ; see, e.g., Ingersoll (1987, Ch. 13). With Vasicek interest rate dynamics, the function $g(r,t)$ can be computed explicitly, cf. Liu (1999) and Sørensen (1999). With a spanned income rate and no portfolio constraints, we can think of the individual having an initial financial wealth of $W_t + H(y_t, r_t, t)$ and no labor income instead of having initial wealth W_t and the income stream.[10] Under the assumptions of this section, we therefore have that the indirect utility function *with* labor income is given by

$$J(W,r,y,t) = V(W + H(y,r,t), r, t). \tag{24}$$

From the value function the optimal consumption and investment strategies can be derived from (14) and (15). We summarize the solution in the following proposition.

Proposition 2 *Under the assumptions stated above, the indirect utility function is given by*

$$J(W,r,y,t) = \frac{1}{1-\gamma} g(r,t)^{\gamma} \left(W + H(y,r,t)\right)^{1-\gamma}, \tag{25}$$

where the function $g(r,t)$ is defined by

$$g(r,t) = \int_t^T f(s-t) \left(B^s(r,t)\right)^{\frac{\gamma-1}{\gamma}} \, ds + \varepsilon f(T-t) \left(B^T(r,t)\right)^{\frac{\gamma-1}{\gamma}}$$

with $f(\tau)$ defined by

$$\ln f(\tau) = \left(-\frac{\delta}{\gamma} + \frac{1-\gamma}{2\gamma^2}\|\lambda\|^2\right)\tau + \frac{1-\gamma}{\gamma^2}\left((\bar{r} - y_\infty)(\tau - b(\tau)) - \frac{\sigma_r^2}{4\kappa} b(\tau)^2\right).$$

The optimal consumption rate is

$$c_t = \frac{W_t + H(y_t, r_t, t)}{g(r_t, t)}, \tag{26}$$

while the optimal allocation of financial wealth is determined by

$$\begin{aligned}\theta_t = {} & \frac{1}{\gamma}\left(W_t + H(y_t, r_t, t)\right)\left(\Sigma(r_t,t)^{\top}\right)^{-1}\lambda - y_t H_y(y_t, r_t, t)\sigma_y(t)\left(\Sigma(r_t,t)^{\top}\right)^{-1}\rho_{yP} \\ & + \left(H_r(y_t, r_t, t) - \frac{g_r(r_t,t)}{g(r_t,t)}\left(W_t + H(y_t, r_t, t)\right)\right)\frac{\sigma_r}{\sigma_B(r_t,t)}\mathbf{e}_1.\end{aligned} \tag{27}$$

[10] Bodie, Merton, and Samuelson (1992) apply this idea in the case of constant investment opportunities.

Note that there are two reasons to hedge changes in the interest rate, as reflected by the two parts in the parenthesis in the last term of θ_t in (27). Firstly, interest rates determines the future investment opportunity set leading to a hedge demand determined by the ratio g_r/g. Secondly, interest rates affect the capitalized value of future income, both through discounting effects and through the possible dependence of the income rate process on the interest rate. This is captured by the H_r part. The terms in the expression for θ_t that involve H compensate exactly for the dynamics of the human wealth, since by Itô's Lemma

$$dH(y_t, r_t, t) = \dots\, dt + H_y(y_t, r_t, t) y_t \sigma_y(t) \rho_{yP}^{\top}\, dz_t - H_r(y_t, r_t, t)\sigma_r\, dz_{rt}.$$

The percentage volatility vector of the total wealth $W_t + H(y_t, r_t, r)$ is therefore

$$\frac{1}{\gamma}\lambda - \frac{g_r(r_t, t)}{g(r_t, t)}\sigma_r \mathbf{e}_1,$$

just as for the case without income. The intuition is that a CRRA investor will determine her investment strategy in order to obtain a given percentage volatility vector of total wealth. This desired volatility vector is independent of how the total wealth is comprised by financial wealth and human wealth. In other words, the consumer-investor computes her optimal investment of total wealth and then corrects the investment strategy for the implicit investment that the income stream represents.

In the problem without labor income, the optimal strategies are such that wealth stays positive with probability one. By analogy, the optimal strategies for the problem with labor income ensure that *total* wealth stays positive with probability one. However, financial wealth in itself may very well go negative in some situations. For positive values of financial wealth it makes sense to talk of portfolio weights, i.e. the fractions of financial wealth invested in the different assets. Investment strategies are usually stated in such portfolio weights. We denote portfolio weights by $\pi = \theta/W$. Using (22), we get from (27) that the optimal portfolio weights are

$$\begin{aligned}
\pi_t &= \frac{1}{\gamma}\left(1 + \frac{y_t}{W_t}M(r_t, t)\right)\left(\Sigma(r_t, t)^{\top}\right)^{-1}\lambda - \frac{y_t}{W_t}M(r_t, t)\sigma_y(t)\left(\Sigma(r_t, t)^{\top}\right)^{-1}\rho_{yP} \\
&\quad + \left(\frac{y_t}{W_t}M_r(r_t, t) - \frac{g_r(r_t, t)}{g(r_t, t)}\left(1 + \frac{y_y}{W_t}M(r_t, t)\right)\right)\frac{\sigma_r}{\sigma_B(r_t, t)}\mathbf{e}_1 \\
&= \frac{1}{\gamma}\left(1 + \frac{y_t}{W_t}M(r_t, t)\right)\left(\Sigma(r_t, t)^{\top}\right)^{-1}\left(\lambda - \gamma\sigma_y(t)\rho_{yP}\right) + \sigma_y(t)\left(\Sigma(r_t, t)\right)^{-1}\rho_{yP} \\
&\quad + \left(\frac{y_t}{W_t}M_r(r_t, t) - \frac{g_r(r_t, t)}{g(r_t, t)}\left(1 + \frac{y_y}{W_t}M(r_t, t)\right)\right)\frac{\sigma_r}{\sigma_B(r_t, t)}\mathbf{e}_1.
\end{aligned} \tag{28}$$

It is now clear that the optimal portfolio weights do not depend on current financial wealth and labor income separately but only through the wealth-to-income ratio.

Using (26), we see that the propensity to consume out of wealth and the propensity to consume out of current income, respectively, are given by

$$\frac{c_t}{W_t} = \frac{1 + \frac{y_t}{W_t}M(r_t, t)}{g(r_t, t)}, \qquad \frac{c_t}{y_t} = \frac{\frac{W_t}{y_t} + M(r_t, t)}{g(r_t, t)}, \tag{29}$$

which also depend on the wealth-income ratio. As expected, the propensity to consume out of income, c_t/y_t, is increasing in the wealth-income ratio and the expected income growth rate and decreasing in the current income rate, while the dependence on the income volatility, the risk aversion coefficient, the investment horizon, and the interest rate level is parameter specific.

Below we study the optimal investment strategy in more detail. For notational simplicity we assume that the individual has no utility from terminal wealth (i.e. $\varepsilon = 0$). After that we will look at numerical examples.

4.2.2 Optimal bond allocation

Obviously, the optimal demand for bonds depends on the ratio g_r/g. A straightforward computation shows that

$$\frac{g_r(r,t)}{g(r,t)} = \frac{1-\gamma}{\gamma} G(r,t), \tag{30}$$

where the function $G(r,t)$ is defined by

$$G(r,t) = \frac{\int_t^T b(s-t) f(s-t) \left(B^s(r,t)\right)^{\frac{\gamma-1}{\gamma}}}{\int_t^T f(s-t) \left(B^s(r,t)\right)^{\frac{\gamma-1}{\gamma}} \, ds}. \tag{31}$$

Note that $G(r,t)$ is positive so that the ratio g_r/g is positive [negative] if the risk aversion parameter γ is smaller [greater] than 1, the risk aversion parameter of a log utility investor. The following lemma gives two additional characteristics of the function G which will be important for the discussion of the optimal bond demand. Appendix B contains the proof.

Lemma 1 *The function $G(r,t)$ defined by* (31) *has the following properties:*

(a) $G(r,t)$ is increasing in T,

(b) $G(r,t)$ is decreasing in r if $\gamma > 1$ and increasing in r if $\gamma < 1$.

The optimal bond demand at time t can be decomposed as

$$\theta_{Bt} = \theta_{Bt}^{\text{spec}} + \theta_{Bt}^{(1)} + \theta_{Bt}^{(2)} + \theta_{Bt}^{(3)}, \tag{32}$$

where

$$\theta_{Bt}^{\text{spec}} = \frac{1}{\gamma \sigma_B} (W_t + H_t) \left[\left(\lambda_r - \rho_{SB}^{\top} K^{-1} \lambda_S \right) \right] \tag{33}$$

$$\theta_{Bt}^{(1)} = \left(1 - \frac{1}{\gamma}\right) \frac{\sigma_r}{\sigma_B} (W_t + H_t)\, G(r_t, t), \tag{34}$$

$$\theta_{Bt}^{(2)} = -H \frac{\sigma_y(t)}{\sigma_B} \left(\rho_{yB} - \rho_{SB}^{\top} K^{-1} \hat{\rho}_{yS} \right) \tag{35}$$

$$\theta_{Bt}^{(3)} = (\xi_1 - 1) \frac{\sigma_r}{\sigma_B} y \int_t^T b(s-t) h(t,s) \left(B^s\right)^{1-\xi_1} \, ds. \tag{36}$$

The bond demand consists of a speculative demand $\theta_{Bt}^{\text{spec}}$ and three hedge demands $\theta_{Bt}^{(1)}$, $\theta_{Bt}^{(2)}$, and $\theta_{Bt}^{(3)}$. In the case with no income, the last two hedge demand terms disappear and $H = 0$ in the speculative term and the first hedge term. The *first* hedge term represents a hedge against the changes in the investment opportunity set generated by the varying interest rates. This hedge demand is positive for a "conservative" investor ($\gamma > 1$), but negative for an "aggressive" investor ($0 < \gamma < 1$). This effect is well-studied in the literature cited in the introduction. The *second* hedge term shows the contribution of the bond in the hedge of current shocks to the income rate. The sign of this term depends on the pairwise correlations between income, bond, and stocks. For example, with only one stock we get $\theta_{Bt}^{(2)} = -H\frac{\sigma_y(t)}{\sigma_B}\frac{\rho_{yB}-\rho_{yS}\rho_{SB}}{1-\rho_{SB}^2}$. If $\rho_{yB} = -\rho_{yr}$ is sufficiently negative, the bond is an effective hedge instrument against shocks to the income rate, inducing a larger investment in the bond. The *third* hedge term is due to the interest rate sensitivity of the capitalized labor income. Whenever $\xi_1 < 1$, this term is negative implying a reduction in the bond demand. If $\xi_1 < 1$, the capitalized labor income is decreasing in the interest rate level, just as the bond, so the capitalized labor income can partly substitute the bond investment. Hence, the last term reinforces the negative hedge demand for the bond of an aggressive investor, while for a conservative investor it will reduce the positive hedge demand for the bond.

Next, we investigate the relations between the optimal bond allocation and the interest rate level, the time horizon, and the risk aversion of the individual. The derivative of θ_{Bt} with respect to T is

$$\begin{aligned}\frac{\partial \theta_{Bt}}{\partial T} = -\frac{1-\gamma}{\gamma}(W+H)\frac{\sigma_r}{\sigma_B}\frac{\partial G}{\partial T} + \frac{yh(t,T)}{\sigma_B}\left(B^T\right)^{1-\xi_1}\Bigg[&\frac{1}{\gamma}\left(\lambda_r - \rho_{SB}^{\top}K^{-1}\lambda_S\right) \\ &- \frac{1-\gamma}{\gamma}\sigma_r G - \sigma_y(t)\left(\rho_{yB} - \rho_{SB}^{\top}K^{-1}\hat{\rho}_{yS}\right) + (\xi_1 - 1)\sigma_r b(T-t)\Bigg]\end{aligned}$$

Applying Lemma 1, we see from the first term that the optimal bond demand of a conservative investor with no labor income is increasing in the investment horizon. This is inconsistent with the traditional advice of investing more in bonds as the horizon shrinks. As discussed by Munk and Sørensen (2004), the hedge position in the traded bond is combined with a short-term deposit or loan to mimic an investment in a specific coupon bond reflecting the expected consumption stream of the investor. Other things equal, a longer horizon will increase the duration and volatility of this desired coupon bond, which requires a larger weight on the traded bond in the mimicking strategy. From the second term we see that the presence of labor income can either reinforce, dampen, or reverse the horizon effect, depending on the sign and magnitude of the term in the square brackets. Increasing the investment horizon implies a larger human wealth and hence a larger wealth effect on the optimal bond allocation. This is reflected by the first term in the square brackets in the expression above. In addition, the total human wealth risk to be hedged increases as reflected by the remaining terms in the brackets.

In our Vasicek interest rate model, the volatilities of zero-coupon bonds are independent of the interest rate level, but for coupon bonds and other fixed-income securities this is not true. Nevertheless, in our investigation of the effects of interest rate changes on the optimal bond allocation, we choose to fix the volatility of the traded bond. Then we obtain

$$\begin{aligned}\frac{\partial \theta_{Bt}}{\partial r} = &-\frac{1-\gamma}{\gamma}\frac{\sigma_r}{\sigma_B}\frac{\partial G}{\partial r}(W+H) + (\xi_1-1)^2\frac{\sigma_r}{\sigma_B} y \int_t^T b(s-t)^2 h(t,s)\left(B^s(r,t)\right)^{1-\xi_1}\,ds \\ &+ \frac{(\xi_1-1)y}{\sigma_B}\left(\int_t^T b(s-t)h(t,s)\left(B^s\right)^{1-\xi_1}\,ds\right)\left[\frac{1}{\gamma}\left(\lambda_r - \rho_{SB}^\top K^{-1}\lambda_S\right)\right. \\ &\left. - \frac{1-\gamma}{\gamma}\sigma_r G - \sigma_y(t)\left(\rho_{yB} - \rho_{SB}^\top K^{-1}\hat{\rho}_{yS}\right)\right].\end{aligned}$$

From Lemma 1, $-(1-\gamma)\frac{\partial G}{\partial r}$ is negative so in absence of labor income, the optimal bond allocation is a decreasing function of the interest rate level. Clearly, the second term is always positive, while the last term may be either positive or negative, depending on the risk aversion, risk premia, correlation coefficients, and the sign of $\xi_1 - 1$. With labor income, the optimal bond allocation may therefore respond very differently to interest rate changes.

The dependence of the optimal bond allocation on the risk aversion level is not qualitatively different with labor income than without since the last two terms in (32) do not involve γ. With utility of terminal wealth only, G is also independent of γ and it can be shown that the bond allocation increases with γ, at least in the range $\gamma \in (1,\infty)$. With utility of consumption, G depends on γ in a complicated way, and the bond allocation may then depend on γ in a non-monotonic way.

4.2.3 Optimal stock allocation

The optimal investment in the stocks is the sum of a speculative demand and a hedge demand, $\theta_{St} = \theta_{St}^{\text{spec}} + \theta_{St}^{\text{hdg}}$, where

$$\theta_{St}^{\text{spec}} = \frac{1}{\gamma}(W+H)\left[\operatorname{diag}(\sigma_S)\right]^{-1} K^{-1}\lambda_S, \tag{37}$$

$$\theta_{St}^{\text{hdg}} = -H\sigma_y(t)\left[\operatorname{diag}(\sigma_S)\right]^{-1} K^{-1}\hat{\rho}_{yS}. \tag{38}$$

The presence of labor income magnifies the optimal investment in the different stocks due to a wealth effect, i.e. increases both long and short positions. However, the hedge effect may change this conclusion. The sign of the hedge demand depends on the correlation structure. With a single stock, which is uncorrelated with the bond, the hedge demand is positive [negative] if the income-stock correlation is negative [positive].

We can combine the income-related terms in the speculative and the hedge demand and rewrite the total stock demand as

$$\theta_{St} = \frac{1}{\gamma}W\left[\operatorname{diag}(\sigma_S)\right]^{-1} K^{-1}\lambda_S + H\left[\operatorname{diag}(\sigma_S)\right]^{-1} K^{-1}\left[\frac{1}{\gamma}\lambda_S - \sigma_y(t)\hat{\rho}_{yS}\right]. \tag{39}$$

The total effect of income on the demand of the different stocks depends on the sign of the components in the vector $K^{-1}\hat{\rho}_{yS}$. Since H is increasing in T, this sign will also determine how the optimal stock demand varies with the investment horizon. For stocks in positive demand the popular advice to decrease the fraction of wealth invested in stocks over the life cycle so that θ_{St} increases with T is true for a riskless income stream, but with income uncertainty the validity of this advice is highly dependent on risk aversion, risk premia, and the correlation coefficients. For $\xi_1 < 1$, the human wealth is decreasing in the interest rate level, and hence the optimal stock demand will be decreasing [increasing] in r if the last term in the second expression above is positive [negative]. Intuitively, a higher interest rate lowers the present value of future labor income and hence total wealth today. This reduces the demand for stocks. Without labor income the optimal stock demand is independent of the interest rate level. Higher risk aversion has a dampening effect on stock demand with or without labor income.

4.2.4 Optimal cash position

The optimal position in the bank account, i.e. in "cash", is

$$\theta_{0t} = W_t - \theta_{Bt} - \theta_{St}^{\top}\mathbf{1}_n.$$

From the discussion above it is clear that the dependence of this ratio on risk aversion and investment horizon is ambiguous and determined by the precise parameter values as well as the current income and financial wealth. That the cash position is not uniformly increasing in risk aversion may seem surprising at first, but note that a cash position is only riskfree over the next instant. For an investor with utility from terminal wealth only, the true riskfree asset is the zero-coupon bond maturing at the end of the investment horizon. For an investor with utility from intermediate consumption, the true riskfree asset is more like a coupon bond, cf. Munk and Sørensen (2004).

4.3 Numerical illustrations of the optimal strategies

In the following illustrations we use the benchmark parameter values listed in Section 3 and assume that the current short-term interest rate is equal to the long-term level of 2% unless otherwise mentioned. We first assume that the expected growth rate of income is constant and equal to 4% ($\xi_0 = 0.04, \xi_1 = 0$). We consider an investor with a time preference rate δ of 0.03 and a relative risk aversion γ equal to 2. Given that both the stock-bond correlation and the risk premium on the bond are zero, there is no speculative demand for the bond. In fact, with the parameters given, the optimal speculative position in absence of labor income will be 100% in the stock. Due to the demand for hedging changes in the investment opportunity set, the optimal portfolio of the investor without labor income will be 50% in the stock, 39.4% in the (10-year) bond, and 10.6% in cash.

In Tables 3 and 4 we report information on the optimal strategies for various wealth-to-income ratios (all positive) and four different combinations of the two income-asset correlation coefficients. The results in Table 3 are for a consumer-investor with a 10-year horizon, while the results in Table 4 are for a 30-year horizon. We provide portfolio weights for the bond, the stock, and cash and the consumption-income ratio. The portfolio weights for the bond and the stock are decomposed into the different components. First, note the magnitude of the portfolio weights. For a consumer-investor with a 30-year horizon and a current wealth equal to 20% of current annual income, the total human wealth may be more than 250 times current financial wealth! It is thus clear that the optimal investments as a fraction of current financial wealth can be extreme. In the tables we see several portfolio weights of over 100, i.e. over 10,000 percent of financial wealth! Even with a current wealth-income ratio of 5, the human wealth of the consumer-investor with a 10-year horizon is in all four cases more than double the current wealth, and therefore the optimal strategies taken labor income to account are far from the optimal strategies ignoring the value of future labor income. We can also see that in all cases, the optimal consumption rate is higher than current income. For low current wealth the optimal consumption per year can even exceed the sum of current wealth and current income, i.e. consumption is financed in part by borrowing. Of course, this is due to the desire of the individual to smooth consumption over the life-cycle.

[Table 3 about here.]

[Table 4 about here.]

The combination of correlation coefficients closest to the estimates obtained in the calibration in Section 3 is in the top panel of the tables. For that case we see that the optimal strategies involve extensive borrowing, which is clearly unrealistic as discussed earlier in this paper. In Section 5, we look at the effects of imposing limits to borrowing. Also note, that in the top panel of the two tables, the bond position is far more extreme than the stock position.

Let us look at the different components of the optimal asset demands in Tables 3 and 4. The size of the speculative demands for stocks (and bonds, if there was any speculative demand) is only affected through the size of the human wealth which varies relatively little over the different combinations of correlations with a 10-year horizon, but significantly more with a 30-year horizon. The first hedge term in the bond demand basically hedges total wealth against changes in investment opportunities and is therefore also affected through the size of human wealth. These terms in the optimal portfolios are therefore of the same sign for all correlation pairs considered. The sign of the second hedge term in the bond demand and the (only) hedge term in the stock demand is determined by the sign of the correlation of the bond and the stock, respectively, with the income rate, since these terms reflect hedging against current income rate shocks. The magnitudes of these terms are determined by the human wealth, but also the volatilities of the assets relatively to the volatility of income. Since the bond price is less volatile than the stock price, it takes a larger bond position than stock position to "undo" the income rate risk. Therefore, the magnitude of this hedge term is larger for the bond than for the stock. Finally, the third hedge term in the bond demand represents a hedge of the present value of future income against interest rate risk. The human capital is like a bond investment, where the income rate plays the role of the coupon payments. As long as the parameter ξ_1 is smaller than one, the human capital is decreasing in the interest rate and thus is like an implicit investment in the bond. The explicit investment in the bond is therefore reduced as indicated by a quite substantial negative third hedge demand.

The results discussed above are for a fixed expected income growth rate of 4%. In Table 5, we study how the results with a 10-year horizon are affected if the expected growth rate is related to the interest rate. We look at four values of ξ_1, namely -0.25, 0.25, 0.75, and 1.25. In each case, the parameter ξ_0 is then fixed so that $\xi_0 + \xi_1 \bar{r} = 0.04$. The results in the table are for the case, where both the income-interest rate and the income-stock correlation are given by 0.7071, but the effects of varying (ξ_0, ξ_1) are very similar for the other correlation pairs. As illustrated in Section 4.1, the size of the human wealth is affected by the decomposition of expected income growth rate and this will influence the optimal strategies. With the 10-year horizon considered here, this effect is minor, but it will be more dramatic for longer-term horizons, i.e. for young consumer-investors. In addition, ξ_1 is directly influencing the third hedge term of the bond (and, consequently, also the demand for cash) as already indicated in the discussion above. As we can see from the table, this effect can be quite substantial and emphasizes the need to understand the behavior of labor income over the business cycle.

[Table 5 about here.]

Finally, we study in Table 6 the sensitivity of results to the level of interest rates. The current interest rate has some effect on the value of human wealth through the discounting of future income as discussed in Section 4.1, and this will affect all the components of the optimal portfolios. In addition, the interest rate level affects the hedge of wealth against changes in investment opportunities, i.e. the first hedge term for the bond, but this effect seems rather insignificant. In total, the optimal strategies are relatively insensitive to small variations in interest rates around the long-term level.

[Table 6 about here.]

5 Unspanned income uncertainty and liquidity constraints

5.1 Computational approach

We continue to use the set-up of Section 2, but now consider the case where $\|\rho_{yP}\| < 1$ implying that the income uncertainty is unspanned, i.e. not perfectly hedgeable. In addition we impose a liquidity constraint on the individual so that the financial wealth has to stay non-negative at all points in time. We will also consider the effects of imposing stricter constraints so that the investor is restricted to non-negative positions in both the bond, the stock, and the bank account. Due to the unspanned income and the investment constraints, it is no longer possible to derive a closed-form solution to the utility maximization problem, so we have to resort to numerical solution techniques.

Our numerical method is based on a finite difference backwards iteration solution of the HJB equation with an optimization over feasible consumption rates and portfolios at each (time,state) node in the lattice. The original formulation of the problem has three state variables (financial wealth, interest rate, and income rate). In order to simplify the implementation of the numerical solution algorithm, we will use a homogeneity property to reduce the number of state variable by one. It follows from the linearity of the wealth dynamics in (11) and the income dynamics in (10) that if the consumption and investment strategy (c, θ) is optimal with initial wealth and income (W, y), then $(kc, k\theta)$ is optimal with initial wealth and income (kW, ky). The assumed power utility function implies that the value function is homogeneous of degree $1-\gamma$ in (W, y), i.e.

$$J(kW, r, ky, t) = k^{1-\gamma} J(W, r, y, t),$$

from which it follows that

$$J(W, r, y, t) = y^{1-\gamma} F\left(\frac{W}{y}, r, t\right), \tag{40}$$

where we have defined $F(x, r, t) = J(x, r, 1, t)$. Of course, this is also true in the complete market case studied previously. There we have

$$F\left(\frac{W}{y}, r, t\right) = \frac{1}{1-\gamma} g(r, t)^{\gamma} \left(\frac{W}{y} + M(r, t)\right)^{1-\gamma}, \tag{41}$$

where $M(r, t)$ is given in Proposition 1. By substitution of (40) into the HJB equation (13) we get that $F = F(x, r, t)$ solves the nonlinear partial differential equation (PDE)

$$\begin{aligned}\hat{\delta}(r,t)F = \sup_{\hat{c},\pi}\Bigg\{ & \frac{\hat{c}^{1-\gamma}}{1-\gamma} + F_t + F_r\left(\kappa[\bar{r}-r] + (1-\gamma)\rho_{yr}\sigma_y(t)\sigma_r\right) \\ & + F_x\left(1 - \hat{c} + x\left[(1-\xi_1)r - \xi_0(t) + \gamma\sigma_y(t)^2 + \pi^\top\Sigma\left(\lambda - \gamma\sigma_y(t)\rho_{yP}\right)\right]\right) \\ & + \frac{1}{2}x^2F_{xx}\left(\pi^\top\Sigma\Sigma^\top\pi + \sigma_y(t)^2 - 2\sigma_y(t)\pi^\top\Sigma\rho_{yP}\right) \\ & + \frac{1}{2}\sigma_r^2F_{rr} - xF_{xr}\sigma_r\left(\pi^\top\Sigma\mathbf{e}_1 + \rho_{yr}\sigma_y(t)\right)\Bigg\},\end{aligned} \tag{42}$$

where we have introduced

$$\hat{\delta}(r,t) = \delta - (1-\gamma)(\xi_0(t) + \xi_1 r) + \frac{1}{2}\gamma(1-\gamma)\sigma_y(t)^2$$

and $\hat{c}_t = c_t/y_t$ is the consumption-to-income ratio and $\pi_t = \theta_t/W_t$ is the vector of portfolio weights. The terminal condition on F is $F(x, r, T) = \varepsilon x^{1-\gamma}/(1-\gamma)$.

We set up a lattice in (x, r, t) and solve the PDE (42) numerically using a backward iterative procedure starting from the terminal date T. At each time t_n in the lattice we first guess on the optimal controls $\hat{c}(x_i, r_j, t_n), \pi(x_i, r_j, t_n)$ and solve (42) using finite difference techniques for $F(x_i, r_j, t_n)$, which is then a guess on the value function at time t_n. Using that in the first-order conditions for the maximization in (42), we can derive a new guess on the optimal controls, which can again be used to find a new guess on the value function. We continue these iterations until the guess on the value function at t_n seems stable, and we can then move on to the previous time step t_{n-1}. Our solution technique is basically the same as that applied by Brennan, Schwartz, and Lagnado (1997) and is closely related to the well-documented Markov Chain Approximation Approach, which has previously been used to study various consumption/investment problems, cf. Fitzpatrick and Fleming (1991), Hindy, Huang, and Zhu (1997), and Munk (1999, 2000). Details on the numerical procedure can be found in Appendix C.

We ensure that financial wealth W, and hence the wealth-income ratio $x = W/y$, stay non-negative by restricting the individuals choice of consumption and portfolio whenever $x = 0$ to a zero investment in the risky assets and to a consumption level which is smaller than the current income, i.e. $\hat{c} \leq 1$. If we do not further restrict the consumption and portfolio choice, the optimal choice for a strictly positive values of x is given by the first-order conditions from (42), i.e.

$$\hat{c}_t = \left(F_x\right)^{-1/\gamma}, \tag{43}$$

$$\pi_t = -\frac{F_x}{xF_{xx}}\left(\Sigma^\top\right)^{-1}\left(\lambda - \gamma\sigma_y(t)\rho_{yP}\right) + \frac{F_{xr}}{xF_{xx}}\sigma_r\left(\Sigma^\top\right)^{-1}\mathbf{e}_1 + \sigma_y(t)\left(\Sigma^\top\right)^{-1}\rho_{yP}. \tag{44}$$

Note that if we substitute the expression (41) for F in the complete market case into (44), we get Equation (28). Imposing the liquidity constraint makes lower levels of financial wealth worse. Without the liquidity constraint, a consumer-investor with a large human wealth is not that concerned with a fall in financial wealth from a low level to zero, in fact the financial wealth can go negative. With the liquidity constraint, the consequences of losing financial wealth from a low level are more severe. If you end up at a zero financial wealth, you have to stay away from the risky assets and keep consumption below current labor income. You can only return to strictly positive wealth and risky positions if you consume strictly less than your income. We therefore expect significantly less risky positions at near-zero financial wealth levels relative to the case without the liquidity constraint. To enhance numerical stability we need to impose some, relatively wide constraints on the portfolio weights π_{Bt} and π_{St}. In the numerical results discussed below we have restricted the portfolio weights to the interval $[-5, 10]$. This constraints will be binding in some situations.

Imposing a strict no borrowing condition for all wealth levels, we must have $\pi_{Bt} + \pi_{St} \leq 1$. Of course, if the portfolio given by (44) satisfies this condition, it is still the optimal portfolio, but if the constraint is binding, we maximize in (42) over portfolios (π_{Bt}, π_{St}) with $\pi_{Bt} + \pi_{St} = 1$ and get

$$\begin{aligned}\pi_{Bt} = \frac{1}{\sigma_B^2 + \sigma_S^2 - 2\rho_{SB}\sigma_S\sigma_B} \Bigg\{ & -\frac{F_x}{xF_{xx}} \left(\sigma_B[\lambda_r - \gamma\sigma_y(t)\rho_{yB}] - \psi + \gamma\sigma_S\sigma_y(t)\rho_{yS}\right) \\ & + \frac{F_{xr}}{xF_{xx}} \left(\sigma_B - \sigma_S\rho_{SB}\right) \\ & + \sigma_S^2 - \rho_{SB}\sigma_S\sigma_B + \sigma_y(t)[\sigma_B\rho_{yB} - \sigma_S\rho_{yS}] \Bigg\}\end{aligned}$$

and, of course, $\pi_{St} = 1 - \pi_{Bt}$. In a similar manner we can impose non-negativity constraints on the portfolio weights.

Since the labor income stream is not spanned by the financial assets and the investor faces constraints on the use of future income, we can no longer value the income stream as the dividends from a financial assets. The value of the future income stream will now be investor-specific. We can assess the value of the income stream to the investor by the additional initial wealth, $\bar{H}(W, y, r, t)$, needed to exactly compensate for the utility loss from not receiving income:

$$y^{1-\gamma} F\left(\frac{W}{y}, r, t\right) = \frac{1}{1-\gamma} g(r,t)^{\gamma} \left(W + \bar{H}(W, y, r, t)\right)^{1-\gamma}, \tag{45}$$

implying that

$$\bar{H}(W, y, r, t) = (1-\gamma)^{1/(1-\gamma)} y F\left(\frac{W}{y}, r, t\right)^{1/(1-\gamma)} g(r,t)^{-\gamma/(1-\gamma)} - W. \tag{46}$$

As in the complete market case, we can write this as the product of the current income rate and a present value multiplier, $\bar{H}(W, y, r, t) = y\bar{M}(W/y, r, t)$, where the multiplier is given by

$$\bar{M}(x, r, t) = (1-\gamma)^{1/(1-\gamma)} F(x, r, t)^{1/(1-\gamma)} g(r,t)^{-\gamma/(1-\gamma)} - x. \tag{47}$$

We compare this with the value $M(r,t)$ computed using (22). Since we are now in an incomplete market, $M(r,t)$ is to be interpreted as the present value multiplier of future income assuming that the market price of the non-traded income risk (represented by z_y) is zero and that there are no portfolio constraints.

5.2 Numerical results

We use the benchmark parameter values of Section 3. The individuals are assumed to have a time preference rate of $\delta = 0.03$, a relative risk aversion of $\gamma = 2$, and a time horizon of 30 years, unless otherwise mentioned. Table 7 illustrate the optimal strategies for different values of the wealth-income ratio and different levels of the current short-term interest rate for the case with the very wide constraints on the portfolio weights in the bond and the stock, i.e. $\pi_{Bt}, \pi_{St} \in [-5, 10]$. We see that the optimal portfolio weights are dampened considerably relative to the complete market case (compare for example with Table 4), in particular for low wealth-income ratios where the liquidity constraint is “nearby.” Although less extreme, the optimal investment strategies still involve large positions in the bond and the stock financed by a large loan. We can see that the implicit present value multiplier on current income is in all cases 20-25% smaller in the constrained case than in the unconstrained case and that the multiplier increases with the wealth-income ratio since the liquidity constraint is less important for high values of current wealth-to-income. The optimal consumption-income ratio is much smaller for low wealth-income ratios in the constrained case than in the unconstrained case.

Table 8 shows the optimal strategies if short-sales constraints on all assets are imposed so that, in particular, borrowing is completely prohibited. For all the states considered in the table, the optimal strategy consists of investing 100% in the stock. Although the bond is desirable for different hedging motives, it does not enter the optimal constrained portfolio for the states and parameters considered in this table. The stock is too attractive for speculative motives. We see that the consumption-income ratio is higher in the constrained case for low wealth-income ratios, basically because the risk of the investment strategy is reduced considerably by the constraints and, hence, you do not need to save as larger a fraction of your current income as a buffer. On the other hand, for large wealth-income ratios, the consumption-income ratio is lower in the constrained case, which can be explained by the fact the constrained investment strategies can finance a lower average level of consumption. The implicit present value multiplier of income is naturally smaller in the case with short-sales constraints, but the reduction is relatively small.

We also note that the optimal strategies both with and without short-sales constraints are only little sensitive to the level of interest rates, exactly as we saw it in the complete market analysis. In the following results we will therefore assume that the current short rate is equal to the long-term level of 2%.

[Table 7 about here.]

[Table 8 about here.]

In Table 9 we report results for relative risk aversions of 5 and 10. With a high enough risk aversion, the motive for hedging total wealth against changes in investment opportunities is so strong that the bond dominates the optimal portfolio even when short-sales constraints are imposed. Also note that the implicit value of the future income decreases considerably with the risk aversion of the agent.

[Table 9 about here.]

In Tables 10 and 11 we look at how the results are affected by the correlation ρ_{yr} between shocks to labor income and shocks to the short-term interest rate. For the case with wide constraints on the portfolio weights considered in Table 10, we see that the portfolio weight in the stock is roughly independent of this correlation parameter. With a positive ρ_{yr}, the bond will be negatively correlated with the income rate and hence provides a good hedge against income shocks. Hence, other things equal, the bond weight will be increasing in ρ_{yr}. When short-sales constraints are imposed, we see from Table 11 that the optimal portfolio in all cases considered will be 100% in stocks, but for ρ_{yr} close to 1, the bond will start entering the optimal portfolio.

[Table 10 about here.]

[Table 11 about here.]

In Table 12 we see the effects of varying the correlation ρ_{yS} between the labor income rate and the stock returns. Again, the results are basically driven by the fact that the correlation parameter determines how well the stock can hedge shocks to labor income. With a negative ρ_{yS}, the optimal portfolio weight on the stock is thus significantly higher than with a positive ρ_{yS}. The optimal bond demand is also affected since the bond is an alternative hedging device. The benchmark value of ρ_{yr} is 0.25 so that the bond is a decent hedge against income risk. If the value of ρ_{yS} is positive, the bond is a better instrument for hedging income rate risk than the bond, which is reflected by a higher investment in the bond. Table 13 provide similar information for the case where short-sales constraints are imposed. The stock is so attractive for speculative motives that the optimal portfolio consists only of stocks, except for high wealth-income ratios and high positive stock-income correlations.

[Table 12 about here.]

[Table 13 about here.]

In Tables 14 and 15 we study how the optimal strategies and the income multiplier depend on the parameters ξ_0, ξ_1, i.e. on the variations in expected income growth rate over the business cycle. In Table 14 we have impose the wide bounds $[-5, 10]$ on the portfolio weights, while Table 15 covers the case with no borrowing allowed at all. We see that the portfolio weight of the bond increases with ξ_1, while the portfolio weight of the stock is almost independent of ξ_1. This is exactly as we saw it in the complete markets case, cf. Table 5. The implicit income multiplier $\bar{M}$ seems to be slightly decreasing in ξ_1. From Table 15 we see once again that when short-sales constraints are imposed, the optimal portfolio consists of 100% in stocks for all the ξ_1-values considered.

[Table 14 about here.]

[Table 15 about here.]

6 Life-cycle variations in labor income

We will now consider the case where the expected income growth rate, $\xi_0(t)$, depends on age as stated in (17). We will use $\bar{\xi}_0 = 0.02$ and the three sets of coefficients listed in Table 1 representing three different levels of education. We assume that the income rate volatility is $\sigma_y(t)$ is given by the benchmark value 0.10 up to age 65 and equal to zero after age 65. For simplicity we continue to assume that the agent has a fixed terminal date, which we set to age 80. We take the relative risk aversion to be 2 and the time preference rate to be 0.03 as in the previous numerical examples.

The Figures 7, 8, and 9 show how the optimal investment strategy for an individual with the income profile of a high school graduate varies over the life for wealth-income ratios of 0.2, 1, and 5, respectively. All results are for an interest rate of 2%. For a given wealth-income ratio the portfolio weights in the stock and the bond decrease up to the retirement, where the future income becomes fully certain and thus replaces a bond investment. However, over the life-cycle the wealth-income ratio of an individual will also vary. Typically, the wealth-income ratio is low for young individuals, then the ratio increases since the individual wants to save for retirement, and then it tends to decrease again towards and into retirement. Of course, the precise path of optimal portfolios over the life-cycle will also depend on the realized returns on the investments and the realized income.

[Figure 7 about here.]

[Figure 8 about here.]

[Figure 9 about here.]

In Figure 10 we have depicted the optimal portfolio weights in the bond and the stock of individuals in the three different educational groups for the case of a wealth-income ratio equal to 1. After retirement, the optimal portfolio weights are the same, but before retirement we see substantial differences between the individuals. However, if we impose a strict no borrowing constraint at all times, the optimal portfolios of all three individuals will consist of 100% in stocks. As discussed earlier, this will be different for other levels of risk aversion and other values of key parameters.

[Figure 10 about here.]

7 Concluding remarks

In this paper we have studied the optimal consumption and investment strategies of individuals receiving a non-deterministic stream of labor income in a setting with varying interest rates, which capture the business cycles of the underlying economy. We find the optimal allocation of investments to bonds, stocks, and cash. While it is well-known from the literature that the labor income increases the optimal speculative investments due to a wealth effect, we have argued and demonstrated in this paper that the relative allocation to bonds and stocks can be significantly affected by the presence of uncertain labor income for several reasons. First, bonds and stocks can be differently correlated with labor income shocks so that bonds may be better for hedging income rate shocks than stocks or vice versa. Second, risk-averse investors want to hedge total wealth against shifts in investment opportunities. In the set-up of our paper, investment opportunities are driven by the short-term interest rate so that the bond is the appropriate asset for this hedging motive. Third, since human wealth is defined as the discounted value of the future income stream, it will in general be sensitive to the interest rate level like a bond and, hence, the income stream represents an implicit investment in a bond, so that the explicit bond investment is reduced. However, the level of labor income may itself be sensitive to the level of interest rates, reflecting the intuition that wage increases are more frequent and larger in booming periods (high interest rates) than in recessions (low interest rates).

We have solved explicitly for the optimal strategies assuming that the labor income stream is perfectly spanned by traded assets and that no constraints are imposed on the strategies, in particular the investors are allowed to borrow using future labor income as implicit collateral. We have provided a detailed analytical and numerical analysis of this case, illustrating various hedging demands and parameter sensitivities. The optimal strategies will often involve extremely large positions in bonds and stocks and an extreme level of borrowing.

We have also provided a detailed study of the optimal strategies for the more realistic case, where the labor income is not fully spanned by the financial assets and the individual has to keep financial wealth non-negative at all times. As our results document, this reduces risk-taking considerably but will still involve a high degree of borrowing and large positions in the bond and stock. When we, in addition, prevent the individual from taking short positions (and hence from borrowing at all), we see that the optimal portfolio will often consist of 100% in the stock, but this result is sensitive to correlation parameters, the degree of risk aversion, and the investment horizon.

Finally, we have illustrated the optimal strategies for reasonable variations in the income process over the life-cycle. For more precise results on optimal strategies for different individuals, it will be important to identify the variations in the income rate volatility over the life-cycle and also the relevant income-asset correlations and the extent to which they change over the life-cycle.

A Proof of Proposition 1

The process $\hat{z} = (\hat{z}_r, \hat{z}_S)^\top$ defined by

$$\hat{z}_t = z_t + \lambda t$$

is a standard $(n+1)$-dimensional Brownian motion under the risk-neutral probability measure. Therefore, the risk-neutral dynamics of the short-term interest rate is

$$dr_t = \left(\hat{\phi} - \kappa r_t\right) dt - \sigma_r \, d\hat{z}_{rt},$$

where $\hat{\phi} = \kappa \bar{r} + \sigma_r \lambda_1$. This implies that

$$r_u = e^{-\kappa[u-t]} r_t + \frac{\hat{\phi}}{\kappa}\left(1 - e^{-\kappa[u-t]}\right) - \int_t^u \sigma_r e^{-\kappa[u-v]} \, d\hat{z}_{rv}.$$

Applying the Fubini rule for interchanging the order of integration, we get

$$\int_t^s r_u \, du = \left(\frac{r_t}{\kappa} - \frac{\hat{\phi}}{\kappa^2}\right)\left(1 - e^{-\kappa[s-t]}\right) + \frac{\hat{\phi}}{\kappa}(s-t) - \frac{\sigma_r}{\kappa}\int_t^s \left(1 - e^{-\kappa[s-u]}\right) d\hat{z}_{ru}. \tag{48}$$

The income dynamics under the risk-neutral probability measure is

$$dy_t = y_t \left[\left(\hat{\xi}_0(t) + \xi_1 r_t\right) dt + \sigma_y(t) \left\{ \rho_{yB} \, d\hat{z}_{rt} + \hat{\rho}_{yS}^\top \, d\hat{z}_{St} \right\} \right],$$

where $\hat{\xi}_0(t) = \xi_0(t) - \sigma_y(t)\rho_{yP}^\top \lambda$ and $\hat{z}_t = z_t + \lambda t$, and hence

$$\begin{aligned} y_s = y_t \exp\Bigg\{ & \int_t^s \left(\hat{\xi}_0(u) + \xi_1 r_u - \frac{1}{2}\sigma_y(u)^2\right) du \\ & + \int_t^s \sigma_y(u)\rho_{yB} \, d\hat{z}_{ru} + \int_t^s \sigma_y(u)\hat{\rho}_{yS}^\top \, d\hat{z}_{Su} \Bigg\}. \end{aligned} \tag{49}$$

Combining (48) and (49) we get

$$\begin{aligned} y_s e^{-\int_t^s r_u \, du} = y_t \exp\Bigg\{ & \int_t^s \left(\hat{\xi}_0(u) - \frac{1}{2}\sigma_y(u)^2\right) du + (\xi_1 - 1)\left(r_t - \frac{\hat{\phi}}{\kappa}\right) b(s-t) \\ & + \frac{\hat{\phi}}{\kappa}(\xi_1 - 1)(s-t) + \int_t^s \sigma_y(u)\hat{\rho}_{yS}^\top \, d\hat{z}_{Su} \\ & + \int_t^s \left(\sigma_y(u)\rho_{yB} - (\xi_1 - 1)\sigma_r b(s-u)\right) d\hat{z}_{ru} \Bigg\}. \end{aligned} \tag{50}$$

The exponent on the right hand side is normally distributed and applying the standard rule for expectations of exponentials of normal random variables we get

$$\mathrm{E}_t^{\mathbb{Q}}\left[y_s e^{-\int_t^s r_u \, du}\right] = y_t e^{F(t,s) + (\xi_1 - 1)b(s-t)r_t}$$

for some easily computed function $F(t,s)$. Applying (2), we get

$$\mathrm{E}_t^{\mathbb{Q}}\left[y_s e^{-\int_t^s r_u \, du}\right] = y_t e^{F(t,s) - (\xi_1 - 1)a(s-t)} \left(B^s(r_t, t)\right)^{1-\xi_1}.$$

Defining $\ln h(t,s) = F(t,s) - (\xi_1 - 1)a(s-t)$ and integrating over s, we arrive at (22).

B Proof of Lemma 1

In the proof we assume for notational simplicity that $\varepsilon = 0$ so that the individual has no utility from terminal wealth.

(a) Computing the derivative $\frac{\partial G}{\partial T}$, we can see that it will be positive whenever

$$b(T-t)\int_t^T f(s-t)\left(B^s(r,t)\right)^{\frac{\gamma-1}{\gamma}}\,ds > \int_t^T b(s-t)f(s-t)\left(B^s(r,t)\right)^{\frac{\gamma-1}{\gamma}}\,ds,$$

which is indeed the case since b is increasing.

(b) Computing the derivative $\frac{\partial G}{\partial r}$, we see that it will be positive if and only if

$$\begin{aligned}(\gamma-1)\Bigg[&\left(\int_t^T f(s-t)b(s-t)\left(B^s(r,t)\right)^{\frac{\gamma-1}{\gamma}}\,ds\right)^2\\ &-\left(\int_t^T b(s-t)^2 f(s-t)\left(B^s(r,t)\right)^{\frac{\gamma-1}{\gamma}}\,ds\right)\left(\int_t^T f(s-t)\left(B^s(r,t)\right)^{\frac{\gamma-1}{\gamma}}\,ds\right)\Bigg] > 0.\end{aligned}$$

The term in the square brackets is negative due to the Cauchy-Schwarz Inequality:

$$\begin{aligned}&\left(\int_t^T f(s-t)b(s-t)\left(B^s(r,t)\right)^{\frac{\gamma-1}{\gamma}}\,ds\right)^2\\ &\quad=\left(\int_t^T \left[f(s-t)^{\frac{1}{2}}\left(B^s(r,t)\right)^{\frac{\gamma-1}{2\gamma}}\right]\left[b(s-t)f(s-t)^{\frac{1}{2}}\left(B^s(r,t)\right)^{\frac{\gamma-1}{2\gamma}}\right]ds\right)^2\\ &\quad\leq\left(\int_t^T f(s-t)\left(B^s(r,t)\right)^{\frac{\gamma-1}{\gamma}}\,ds\right)\left(\int_t^T b(s-t)^2 f(s-t)\left(B^s(r,t)\right)^{\frac{\gamma-1}{\gamma}}\,ds\right).\end{aligned}$$

Consequently, $\frac{\partial G}{\partial r}$ is positive if $\gamma < 1$ and negative if $\gamma > 1$.

C Details on the numerical solution method

We solve the highly non-linear PDE (42) in the following way. We set up an equally spaced lattice in (x, r, t)-space defined by the grid points

$$\{(x_i, r_j, t_n) \mid i = 0, 1, \ldots, I;\ j = 0, 1, \ldots, J;\ n = 0, 1, \ldots, N\},$$

where $x_i = i\Delta x$, $r_j = r_0 + j\Delta r$, and $t_n = n\Delta t$ for some fixed positive spacing parameters Δx, Δr, and Δt. Since wealth and income are restricted to non-negative values $x_0 = 0$ is a natural lower bound for $x = W/y$, while the highest value x_I is an artificial upper bound. We place the long-term interest rate level $\bar{r}$ in the middle of the grid, $r_{J/2} = \bar{r}$, and since the model allows for interest rates of all real values, we introduce an artificial lower bound, r_0, and an artificial upper bound, r_J. Since the numerical approximation is likely to be relatively imprecise near the artificial bounds, we pick the values of these bounds so that it is highly unlikely that the state moves from the starting point (x, r), located near the center of the grid, to one of the imposed bounds. We denote the approximated value function in the grid point (x_i, r_j, t_n) by $F_{i,j,n}$ and use similar notation for the controls (portfolio weights and scaled consumption) and other state-dependent functions. We focus on the case with a single stock.

We apply a backward iterative procedure starting at the terminal date T, where we first set $F_{i,j,N} = \varepsilon x_i^{1-\gamma}/(1-\gamma)$. At any earlier time t_n, we know the approximated value function at time t_{n+1}, i.e. $F_{i,j,n+1}$ for all $i = 0, 1, \ldots, I$ and all $j = 0, 1, \ldots, J$, and the optimal controls at time t_{n+1}, i.e. $\hat{c}_{i,j,n+1}$ and $\pi_{i,j,n+1}$ for all $i = 0, 1, \ldots, I$ and all $j = 0, 1, \ldots, J$. To find the approximated value function and the optimal controls at time t_n, we first make an initial guess of the optimal controls $\hat{c}_{i,j,n}$ and $\pi_{i,j,n}$ for all (i, j). Since optimal controls do not tend to very dramatically over time, a good initial guess is $\hat{c}_{i,j,n} = \hat{c}_{i,j,n+1}$ and $\pi_{i,j,n} = \pi_{i,j,n+1}$. In the PDE (42), we can remove the sup-term if we substitute the optimal controls into the curly brackets, and then solve for the value function. Applying this idea we set up a finite difference approximation of the PDE without the sup-operator. In the finite difference approximation we use so-called “up-wind” approximations of the derivatives, which tends to stabilize the approach. For each (i, j) in the interior of the grid we thus obtain the equation

$$
\begin{aligned}
\hat{\delta}_{j,n} F_{i,j,n} = {} & \frac{\hat{c}_{i,j,n}^{1-\gamma}}{1-\gamma} + D_t^+ F_{i,j,n} + D_r^+ F_{i,j,n} \left(\kappa[\bar{r} - r_j]^+ - (1-\gamma)\rho_{yr}^- \sigma_{yn} \sigma_r \right) \\
& - D_r^- F_{i,j,n} \left(\kappa[\bar{r} - r_j]^- - (1-\gamma)\rho_{yr}^+ \sigma_{yn} \sigma_r \right) + \frac{1}{2} \sigma_r^2 D_r^2 F_{i,j,n} \\
& + D_x^+ F_{i,j,n} \Bigg\{ (1 - \hat{c}_{i,j,n})^+ + x_i \left[((1-\xi_1) r_j)^+ + \xi_{0n}^- + \gamma \sigma_{yn}^2 \right] \\
& \qquad + x_i \sigma_B \left(\pi_{i,j,n}^B (\lambda_r - \gamma \sigma_{yn} \rho_{yB}) \right)^+ + x_i \sigma_S \left(\pi_{i,j,n}^S \left(\frac{\psi}{\sigma_S} - \gamma \rho_{yS} \sigma_{yn} \right) \right)^+ \Bigg\} \\
& - D_x^- F_{i,j,n} \Bigg\{ (1 - \hat{c}_{i,j,n})^- + x_i \left[((1-\xi_1) r_j)^- + \xi_{0n}^+ \right] \\
& \qquad + x_i \sigma_B \left(\pi_{i,j,n}^B (\lambda_r - \gamma \sigma_{yn} \rho_{yB}) \right)^- + x_i \sigma_S \left(\pi_{i,j,n}^S \left(\frac{\psi}{\sigma_S} - \gamma \rho_{yS} \sigma_{yn} \right) \right)^- \Bigg\} \\
& + \frac{1}{2} x_i^2 D_x^2 F_{i,j,n} \left(\pi_{i,j,n}^\top \Sigma \Sigma^\top \pi_{i,j,n} + \sigma_{yn}^2 - 2\sigma_y(t) [\pi_{i,j,n}^B \sigma_B \rho_{yB} + \pi_{i,j,n}^S \sigma_S \rho_{yS}] \right) \\
& + x_i \sigma_r D_{xr}^+ F_{i,j,n} \left((\pi_{i,j,n}^B)^- \sigma_B + (\pi_{i,j,n}^S \rho_{SB})^- \sigma_S + \rho_{yr}^- \sigma_{yn} \right) \\
& - x_i \sigma_r D_{xr}^- F_{i,j,n} \left((\pi_{i,j,n}^B)^+ \sigma_B + (\pi_{i,j,n}^S \rho_{SB})^+ \sigma_S + \rho_{yr}^+ \sigma_{yn} \right),
\end{aligned}
\tag{51}
$$

where

$$D_t^+ F_{i,j,n} = \frac{F_{i,j,n+1} - F_{i,j,n}}{\Delta t},$$

$$D_x^2 F_{i,j,n} = \frac{F_{i+1,j,n} - 2F_{i,j,n} + F_{i-1,j,n}}{(\Delta x)^2}, \qquad D_r^2 F_{i,j,n} = \frac{F_{i,j+1,n} - 2F_{i,j,n} + F_{i,j-1,n}}{(\Delta r)^2},$$

$$D_x^+ F_{i,j,n} = \frac{F_{i+1,j,n} - F_{i,j,n}}{\Delta x}, \qquad D_x^- F_{i,j,n} = \frac{F_{i,j,n} - F_{i-1,j,n}}{\Delta x},$$

$$D_r^+ F_{i,j,n} = \frac{F_{i,j+1,n} - F_{i,j,n}}{\Delta r}, \qquad D_r^- F_{i,j,n} = \frac{F_{i,j,n} - F_{i,j-1,n}}{\Delta r},$$

$$D_{xr}^+ F_{i,j,n} = \frac{1}{2\Delta x \Delta r}\left(2F_{i,j,n} + F_{i+1,j+1,n} + F_{i-1,j-1,n} - F_{i+1,j,n} - F_{i-1,j,n} - F_{i,j+1,n} - F_{i,j-1,n}\right),$$

$$D_{xr}^- F_{i,j,n} = -\frac{1}{2\Delta x \Delta r}\left(2F_{i,j,n} + F_{i+1,j-1,n} + F_{i-1,j+1,n} - F_{i+1,j,n} - F_{i-1,j,n} - F_{i,j+1,n} - F_{i,j-1,n}\right).$$

Adding similar equations for all points of the boundary of the grid, we get a system of $(I+1)(J+1)$ equations (one for each grid point) that we can solve for the $(I+1)(J+1)$ values $F_{i,j,n}$ (more information on the solution of the equations is given below). Given that solution we compute a new guess on the optimal controls $\hat{c}_{i,j,n}$, $\pi_{i,j,n}$ from the first-order conditions from the maximization in the PDE (42), i.e. the Equations (43) and (44) in the unrestricted case, again using finite difference approximations of the derivatives based on the newly computed guess on the value function $F_{i,j,n}$. Given the new guess on the optimal controls, we solve again the system of equations and obtain a new guess on the value function at time t_n. We continue these iterations until the largest relative change in the value function over all (i,j) relative to the previous iteration is below some small threshold (we use 0.1%). Then we continue to time t_{n-1}. Typically, the solution requires 2-4 iterations at each time step.

We can write the equation system that we need to solve in the form

$$\mathbf{M}_n \mathbf{F}_n = \mathbf{d}_n,$$

where $\mathbf{F}_n$ is the $(I+1)(J+1)$-dimensional vector of values $F_{i,j,n}$, $\mathbf{M}_n$ is a matrix of dimension $(I+1)(J+1) \times (I+1)(J+1)$, and $\mathbf{d}_n$ is an $(I+1)(J+1)$-dimensional vector of known values. The matrix will be a band matrix so that the equation system can be solved relatively fast. The width of the band depends on the way in which order the points (i,j) are taken in the vector $\mathbf{F}_n$. The two natural candidates are

$$(0,0),(1,0),(2,0),\dots,(I,0),(0,1),(1,1),(2,1),\dots,(I,1),\dots,(0,J),(1,J),(2,J),\dots,(I,J)$$

and

$$(0,0),(0,1),(0,2),\dots,(0,J),(1,0),(1,1),(1,2),\dots(1,J),\dots,(I,0),(I,1),(I,2),\dots,(I,J).$$

The first results in a matrix with a band width of $2I+5$, whereas the band width is $2J+5$ using the second enumeration of points. In the complete market case we have observed that the value function and the optimal strategies are more sensitive to wealth and income than to the interest rate, and since we expect the same in the incomplete market case, we will use $I > J$. Therefore the second enumeration is more efficient, both in relation to the number of non-zero values to store in the computer and in relation to the speed with which the equation system can be solved.

The numerical results stated in Section 5 are based on an implementation of the above procedure with $I = 500$, $J = 40$, and four time steps per year. The imposed bounds are $x_I = 10$, $r_0 = -0.03$, and $r_J = 0.07$.

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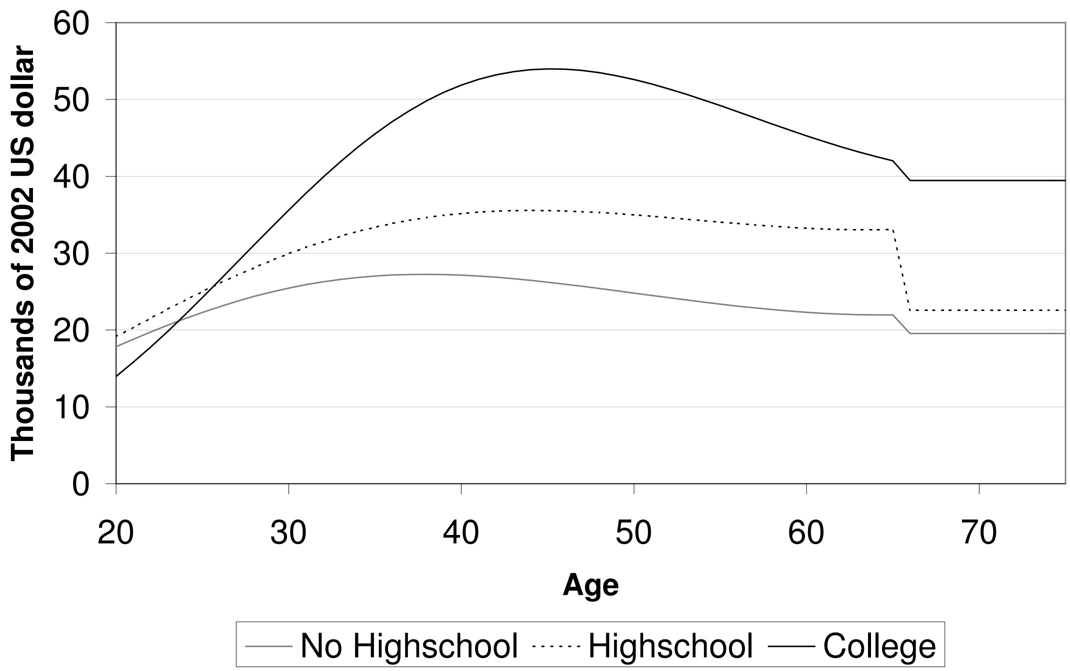

Figure 1: **Calibrated life-cycle income polynomials.**

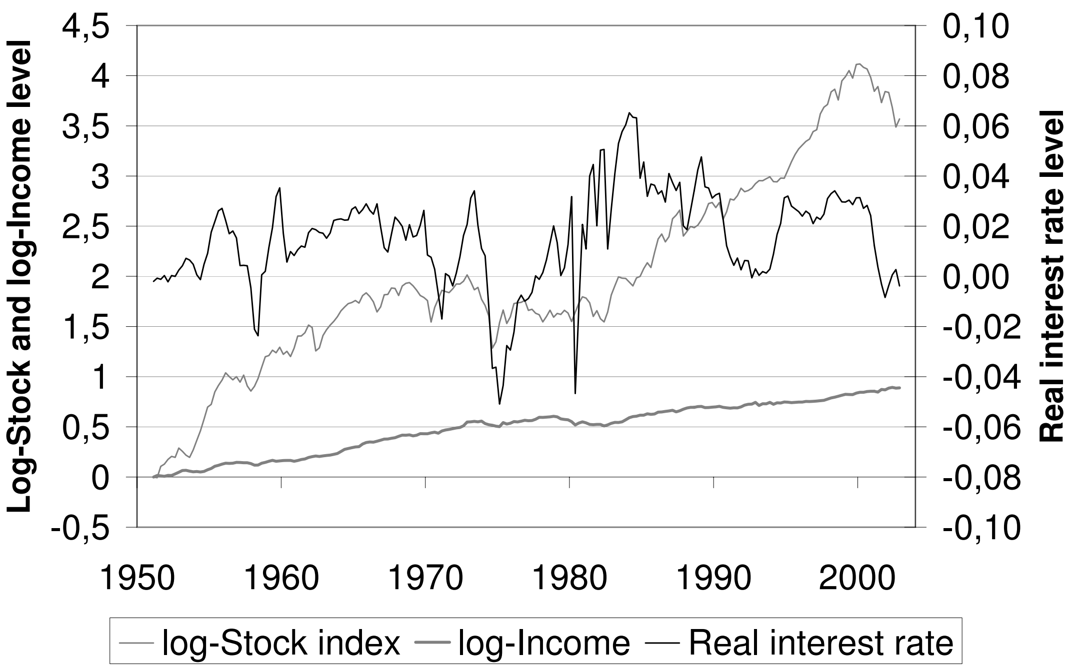

Figure 2: **Data series on income, stock index and real interest rates.**

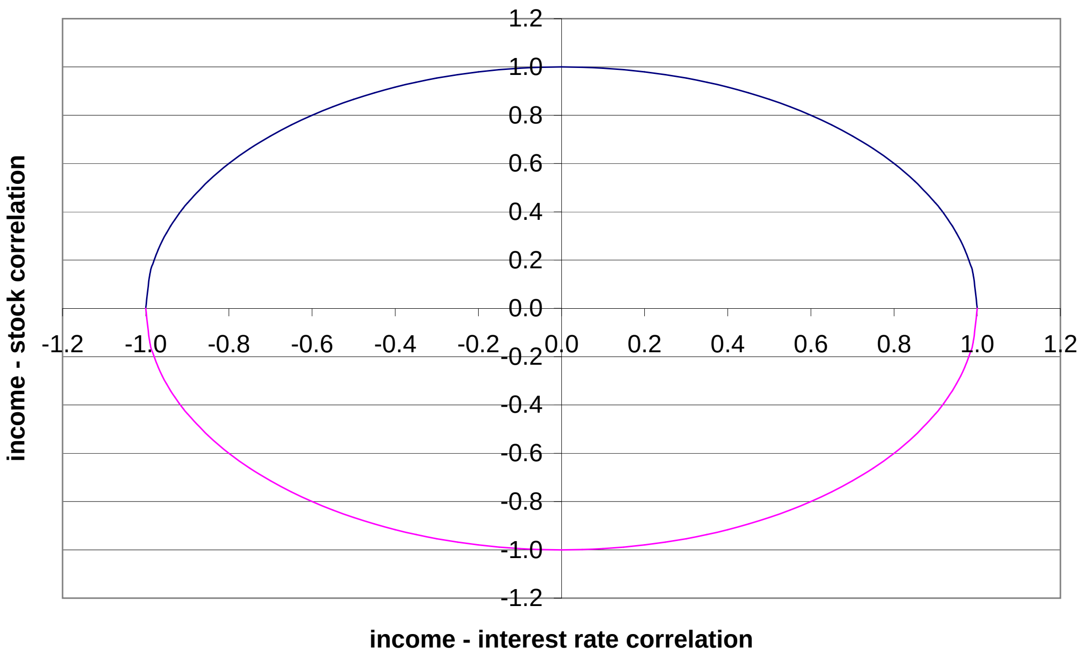

Figure 3: **Combinations of correlations with spanned labor income.** The figure shows the possible combinations of ρ_{yS} and ρ_{yr} in a complete market with $\rho_{SB} = 0$.

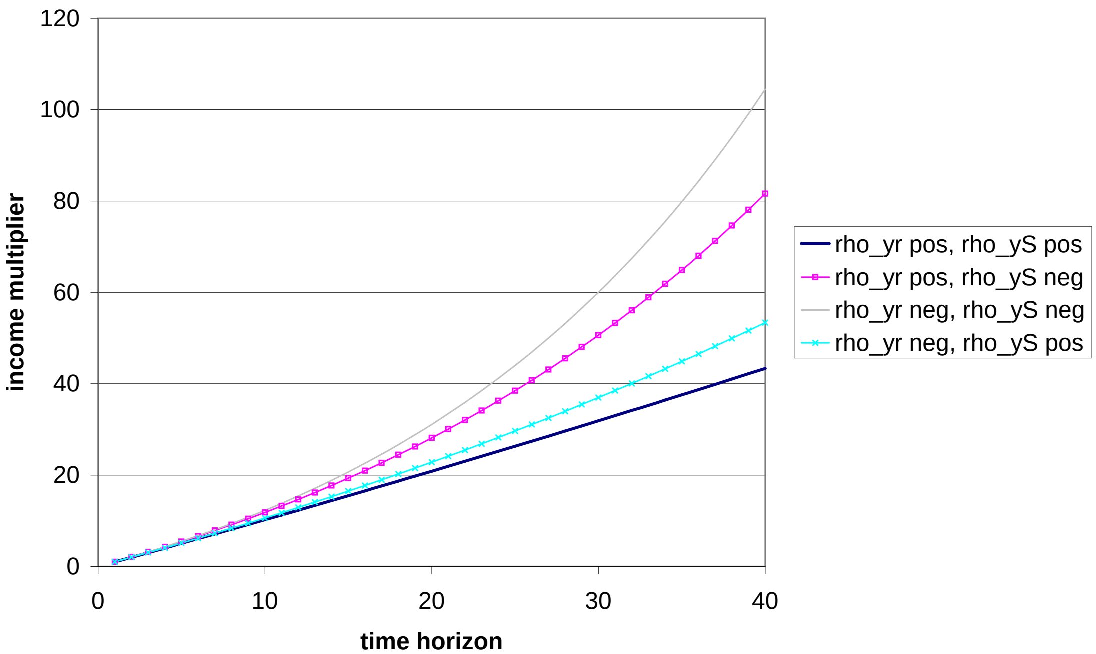

Figure 4: **The income multiplier and the time horizon.** The figure shows how the present value of future income pr. unit of current income, i.e. $M(r, t)$, depends on the time horizon. The four curves correspond to the different combinations of the income-asset correlations $\rho_{yr}, \rho_{yS} \in \{-0.7071, 0.7071\}$.

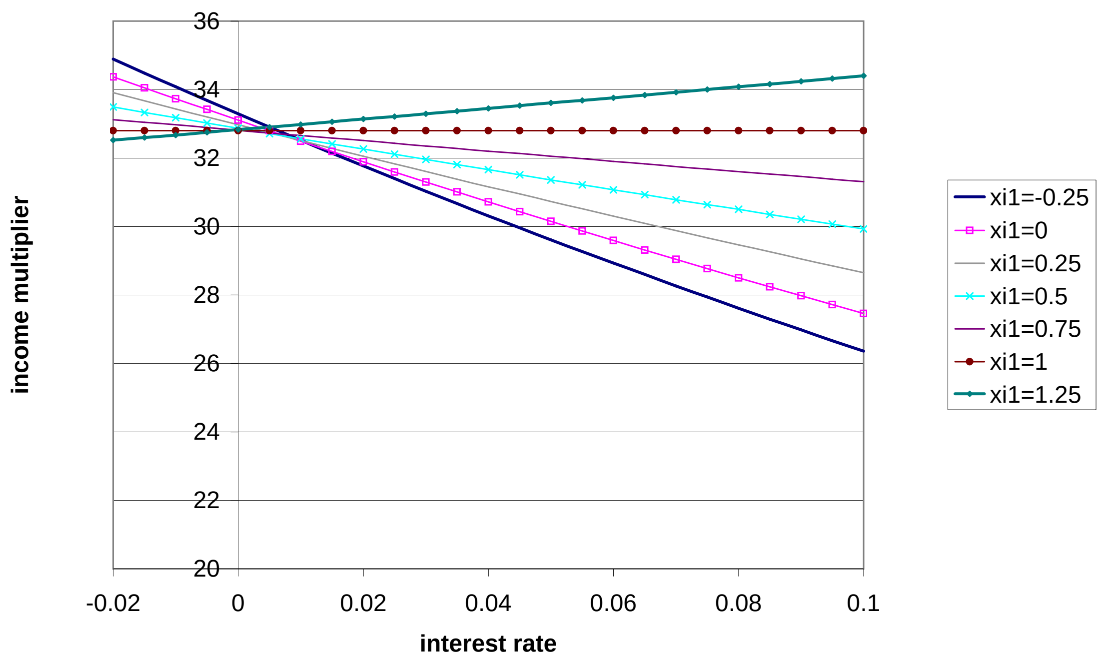

Figure 5: **The income multiplier and the interest rate.** The figure shows how the present value of future income pr. unit of current income, i.e. $M(r, t)$, depends on the interest rate level. The different curves correspond to different specifications of the expected growth of the income rate. Given the value of ξ_1, we determine ξ_0 such that $\xi_0 + \xi_1 \bar{r} = 0.04$. The time horizon is 30 years and $\rho_{yr} = \rho_{yS} = 0.7071$.

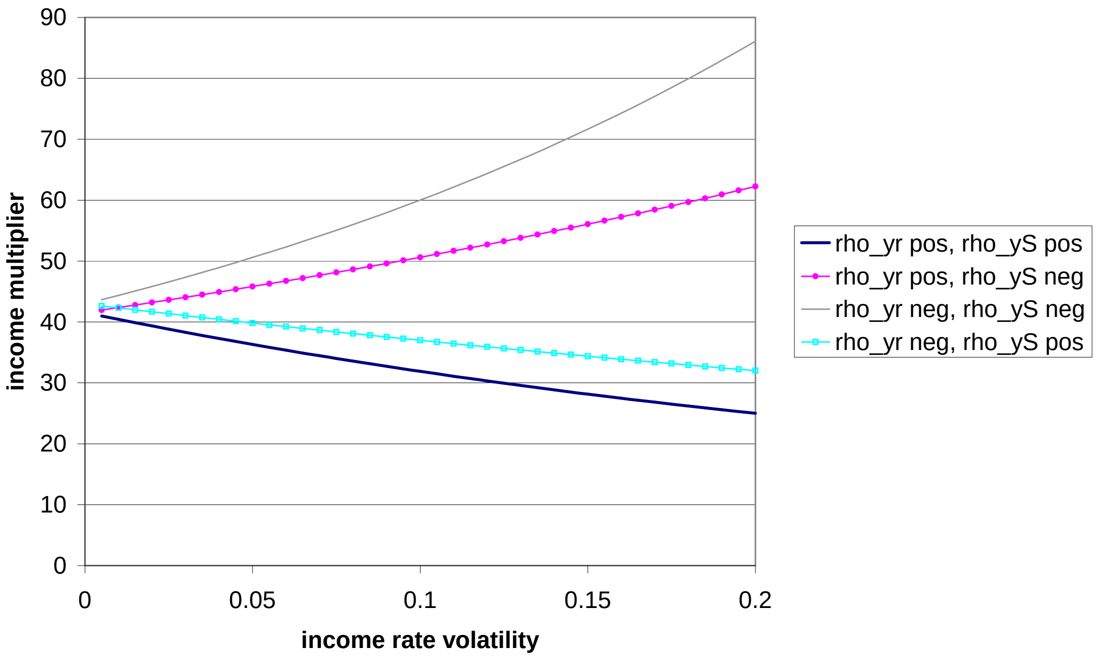

Figure 6: **The income multiplier and the income volatility.** The figure shows how the present value of future income pr. unit of current income, i.e. $M(r,t)$, depends on the income rate volatility, σ_y. The four curves correspond to the different combinations of the income-asset correlations $\rho_{yr}, \rho_{yS} \in \{-0.7071, 0.7071\}$. The time horizon is 30 years.

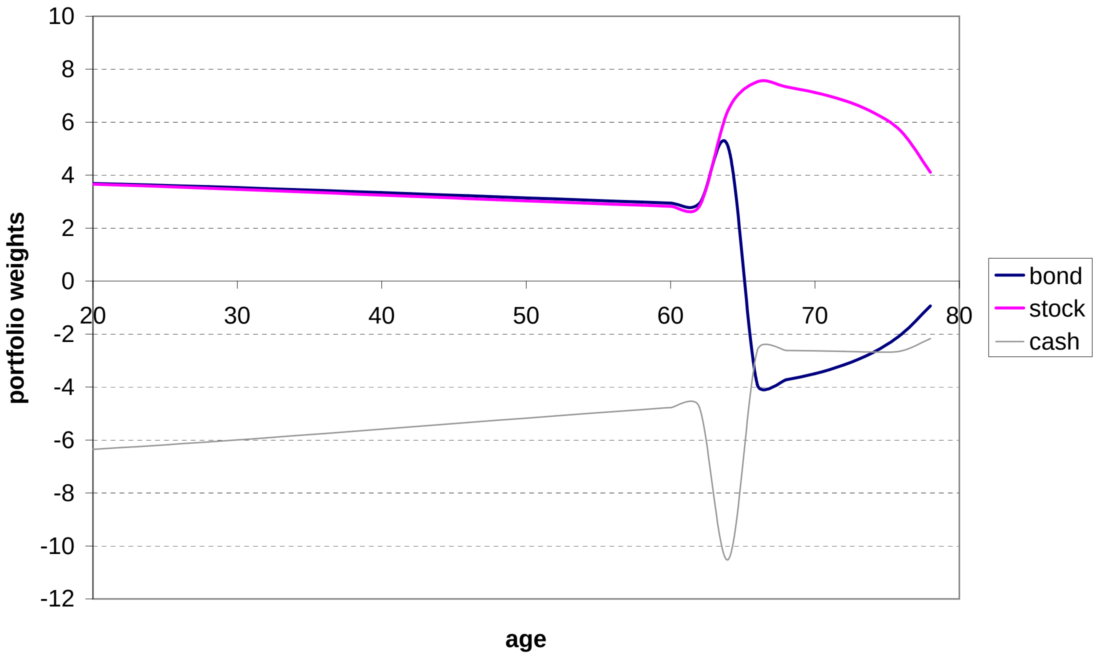

Figure 7: **The optimal investment strategies over the life of an individual with high school education for a wealth-income ratio of 0.2.**

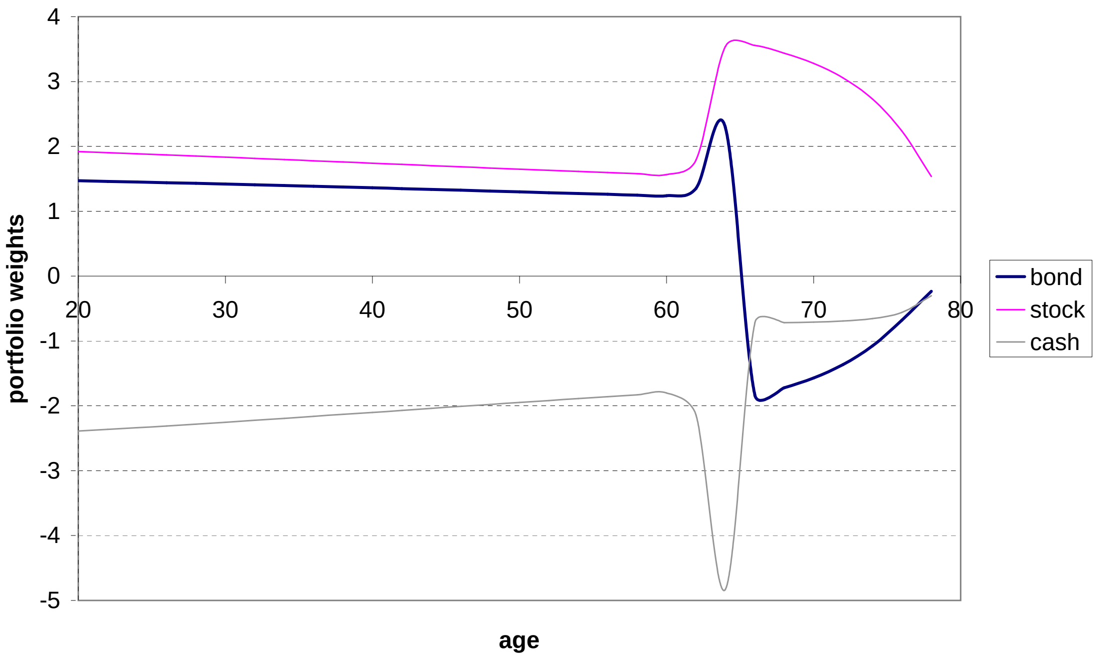

Figure 8: **The optimal investment strategies over the life of an individual with high school education for a wealth-income ratio of 1.**

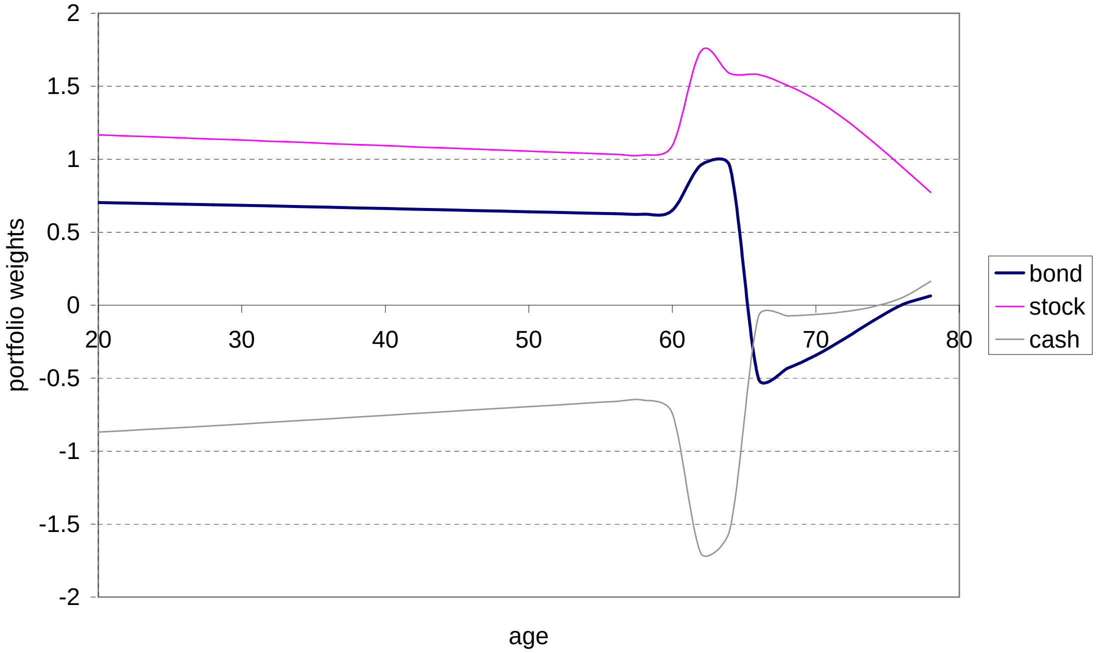

Figure 9: **The optimal investment strategies over the life of an individual with high school education for a wealth-income ratio of 5.**

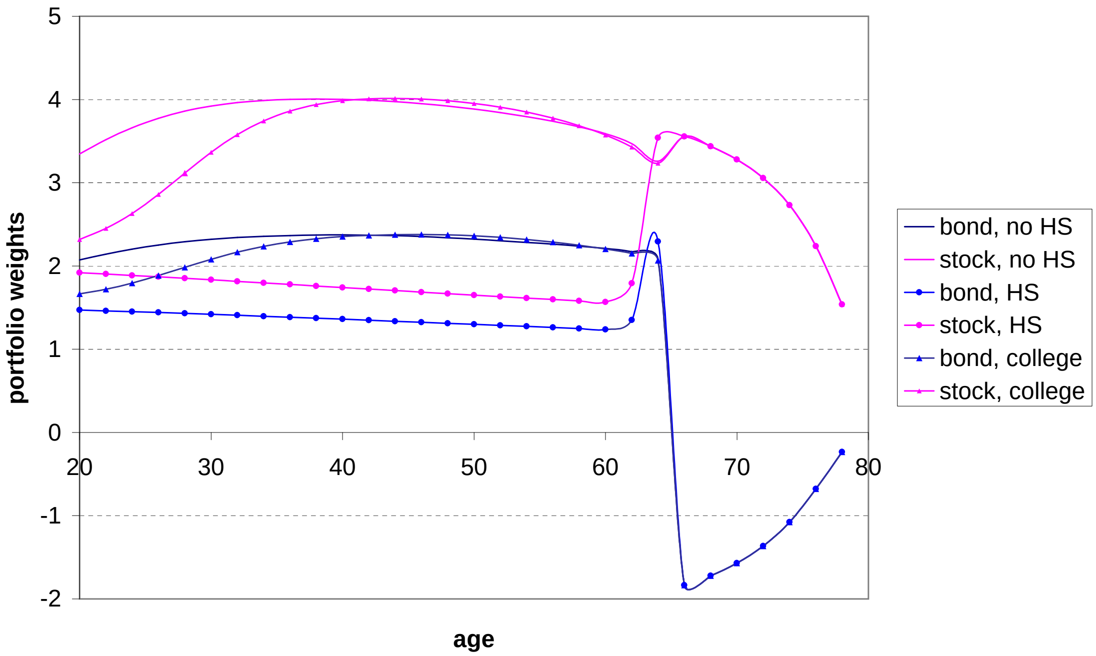

Figure 10: **The optimal investment strategies over the life of an individual for the three educational groups, all for a wealth-income ratio of 1.**

	No High school	High school	College
a^i	-2.1361	-2.1700	-4.3148
b^i	0.1684	0.1682	0.3194
c^i	-0.00353	-0.00323	-0.00577
d^i	0.000023	0.000020	0.000033
$\bar{P}^i$	0.88983	0.68212	0.93887
Initial age 20 level (in 2002 US dollar)	17,763	19,107	13,912

Table 1: **Coefficients in the life-cycle income polynomials.**

Parameter	Estimated parameter value	Benchmark parameter value
$\bar{\xi}_0$	0.0181 (0.0031)	–
ξ_0	–	0.04
ξ_1	-0.0436 (0.0634)	0.00
ψ	0.0665 (0.0224)	0.04
κ	0.6407 (0.1684)	0.50
$\bar{r}$	0.0155 (0.0047)	0.02
σ_y	0.0208 (0.0010)	0.10
σ_S	0.1613 (0.0079)	0.20
σ_r	0.0218 (0.0012)	0.02
ρ_{yS}	0.1674 (0.0728)	0.15
ρ_{yr}	0.2628 (0.0650)	0.25
ρ_{Sr}	0.0052 (0.0414)	0.00
λ_r	–	0.00

Notes: Standard errors in parentheses.

Table 2: **Estimated parameters and benchmark parameters.**

W/y	bond investment hedge1	hedge2	hedge3	total	stock investment spec	hedge	total	cash	c/y
$\rho_{yr} = 0.7071,\ \rho_{yS} = 0.7071,\ M = 10.22$									
0.2	20.53	90.96	-41.36	70.13	26.05	-18.07	7.984	-77.11	1.206
0.6	7.107	30.32	-13.79	23.64	9.018	-6.023	2.995	-25.63	1.252
1	4.422	18.19	-8.272	14.34	5.611	-3.614	1.997	-15.34	1.298
2	2.408	9.096	-4.136	7.368	3.055	-1.807	1.248	-7.616	1.414
5	1.2	3.638	-1.654	3.183	1.522	-0.7228	0.7994	-2.983	1.761
$\rho_{yr} = 0.7071,\ \rho_{yS} = -0.7071,\ M = 11.82$									
0.2	23.69	105.2	-48.85	80.07	30.06	20.9	50.97	-130	1.391
0.6	8.16	35.08	-16.28	26.95	10.35	6.968	17.32	-43.27	1.437
1	5.054	21.05	-9.769	16.33	6.412	4.181	10.59	-25.92	1.484
2	2.724	10.52	-4.885	8.362	3.456	2.09	5.547	-12.91	1.599
5	1.326	4.209	-1.954	3.581	1.682	0.8361	2.519	-5.1	1.946
$\rho_{yr} = -0.7071,\ \rho_{yS} = -0.7071,\ M = 12.07$									
0.2	24.18	-107.4	-50.02	-133.3	30.68	21.34	52.02	82.25	1.420
0.6	8.322	-35.81	-16.67	-44.16	10.56	7.113	17.67	27.49	1.466
1	5.151	-21.48	-10	-26.34	6.536	4.268	10.8	16.53	1.512
2	2.772	-10.74	-5.002	-12.97	3.518	2.134	5.652	8.32	1.628
5	1.345	-4.297	-2.001	-4.952	1.707	0.8536	2.561	3.392	1.975
$\rho_{yr} = -0.7071,\ \rho_{yS} = 0.7071,\ M = 10.42$									
0.2	20.93	-92.75	-42.32	-114.1	26.56	-18.42	8.132	107	1.229
0.6	7.239	-30.92	-14.11	-37.78	9.185	-6.142	3.044	35.74	1.275
1	4.501	-18.55	-8.464	-22.51	5.711	-3.685	2.026	21.49	1.321
2	2.448	-9.275	-4.232	-11.06	3.106	-1.842	1.263	10.8	1.437
5	1.215	-3.71	-1.693	-4.187	1.542	-0.737	0.8053	4.382	1.781

Table 3: **The sensitivity of the optimal strategies to income-asset correlations for a 10-year horizon.** The table shows how the components of optimal portfolios and the consumption-income ratio depend on the decomposition of the expected income growth rate. The investor has a 10-year horizon, a risk aversion of 2, and a time preference rate of 3%. The expected income growth rate is fixed at 4%.

W/y	bond investment				stock investment			cash	c/y
	hedge1	hedge2	hedge3	total	spec	hedge	total		
$\rho_{yr} = 0.7071,\ \rho_{yS} = 0.7071,\ M = 31.89$									
0.2	73.08	283.8	-150.4	206.5	80.22	-56.37	23.85	-229.3	1.619
0.6	24.66	94.59	-50.12	69.13	27.07	-18.79	8.283	-76.42	1.639
1	14.98	56.75	-30.07	41.66	16.44	-11.27	5.17	-45.83	1.659
2	7.718	28.38	-15.04	21.06	8.472	-5.637	2.835	-22.89	1.710
5	3.36	11.35	-6.015	8.697	3.689	-2.255	1.434	-9.131	1.861
$\rho_{yr} = 0.7071,\ \rho_{yS} = -0.7071,\ M = 50.62$									
0.2	115.7	450.5	-244.1	322.2	127.1	89.49	216.5	-537.7	2.564
0.6	38.88	150.2	-81.35	107.7	42.69	29.83	72.52	-179.2	2.584
1	23.51	90.1	-48.81	64.8	25.81	17.9	43.71	-107.5	2.604
2	11.98	45.05	-24.41	32.63	13.16	8.949	22.1	-53.73	2.655
5	5.067	18.02	-9.762	13.32	5.562	3.58	9.142	-21.47	2.800
$\rho_{yr} = -0.7071,\ \rho_{yS} = -0.7071,\ M = 55.31$									
0.2	126.4	-492.2	-267.6	-633.4	138.8	97.78	236.6	397.8	2.801
0.6	42.44	-164.1	-89.19	-210.8	46.59	32.59	79.18	132.6	2.821
1	25.65	-98.44	-53.52	-126.3	28.16	19.56	47.71	79.6	2.841
2	13.05	-49.22	-26.76	-62.93	14.33	9.778	24.11	39.82	2.891
5	5.494	-19.69	-10.7	-24.9	6.031	3.911	9.942	15.96	3.043
$\rho_{yr} = -0.7071,\ \rho_{yS} = 0.7071,\ M = 34.44$									
0.2	78.89	-306.5	-163.2	-390.8	86.61	-60.89	25.72	366	1.748
0.6	26.6	-102.2	-54.39	-130	29.2	-20.3	8.907	122	1.768
1	16.14	-61.3	-32.63	-77.79	17.72	-12.18	5.544	73.25	1.788
2	8.299	-30.65	-16.32	-38.67	9.111	-6.089	3.022	36.64	1.839
5	3.593	-12.26	-6.526	-15.19	3.944	-2.435	1.509	14.68	1.990

Table 4: **The sensitivity of the optimal strategies to income-asset correlations for a 30-year horizon.** The table shows how the components of optimal portfolios and the consumption-income ratio depend on the decomposition of the expected income growth rate. The investor has a 30-year horizon, a risk aversion of 2, and a time preference rate of 3%. The expected income growth rate is fixed at 4%.

W/y	bond investment				stock investment			cash	c/y
	hedge1	hedge2	hedge3	total	spec	hedge	total		
$\xi_0 = 0.045, \xi_1 = -0.25,\ M = 10.21$									
0.2	20.51	90.85	-51.63	59.73	26.02	-18.05	7.976	-66.7	1.204
0.6	7.099	30.28	-17.21	20.17	9.008	-6.016	2.992	-22.16	1.251
1	4.417	18.17	-10.33	12.26	5.605	-3.61	1.995	-13.26	1.297
2	2.405	9.085	-5.163	6.327	3.052	-1.805	1.248	-6.575	1.412
5	1.199	3.634	-2.065	2.767	1.521	-0.7219	0.799	-2.566	1.760
$\xi_0 = 0.035, \xi_1 = 0.25,\ M = 10.24$									
0.2	20.56	91.09	-31.07	80.58	26.09	-18.1	7.996	-87.58	1.207
0.6	7.117	30.36	-10.36	27.12	9.03	-6.032	2.999	-29.12	1.254
1	4.428	18.22	-6.215	16.43	5.618	-3.619	1.999	-17.43	1.300
2	2.411	9.109	-3.107	8.413	3.059	-1.81	1.25	-8.662	1.416
5	1.201	3.644	-1.243	3.601	1.524	-0.7238	0.7998	-3.401	1.763
$\xi_0 = 0.025, \xi_1 = 0.75,\ M = 10.28$									
0.2	20.64	91.44	-10.4	101.7	26.19	-18.16	8.024	-108.7	1.212
0.6	7.142	30.48	-3.468	34.15	9.063	-6.055	3.008	-36.16	1.258
1	4.443	18.29	-2.081	20.65	5.638	-3.633	2.005	-21.65	1.304
2	2.418	9.144	-1.04	10.52	3.069	-1.816	1.252	-10.77	1.420
5	1.204	3.657	-0.4162	4.445	1.528	-0.7266	0.801	-4.246	1.767
$\xi_0 = 0.015, \xi_1 = 1.25,\ M = 10.33$									
0.2	20.74	91.88	10.46	123.1	26.31	-18.25	8.06	-130.1	1.218
0.6	7.175	30.63	3.488	41.29	9.104	-6.084	3.02	-43.31	1.264
1	4.463	18.38	2.093	24.93	5.663	-3.651	2.012	-25.94	1.310
2	2.428	9.188	1.046	12.66	3.081	-1.825	1.256	-12.92	1.426
5	1.208	3.675	0.4185	5.302	1.533	-0.7301	0.8024	-5.104	1.773

Table 5: **The sensitivity of the optimal strategies to the decomposition of the expected income growth rate.** The table shows how the components of optimal portfolios and the consumption-income ratio depend on the decomposition of the expected income growth rate. The investor has a 10-year horizon, a risk aversion of 2, and a time preference rate of 3%. The income-asset correlations are $\rho_{yr} = \rho_{yS} = \sqrt{2}/2$.

W/y	bond investment				stock investment			cash	c/y
	hedge1	hedge2	hedge3	total	spec	hedge	total		
$r = 0.00,\ M = 10.56$									
0.2	21.23	93.94	-42.84	72.32	26.89	-18.66	8.229	-79.55	1.225
0.6	7.339	31.31	-14.28	24.37	9.297	-6.22	3.076	-26.45	1.271
1	4.561	18.79	-8.568	14.78	5.778	-3.732	2.046	-15.83	1.316
2	2.478	9.394	-4.284	7.587	3.139	-1.866	1.273	-7.86	1.43
5	1.228	3.757	-1.714	3.272	1.556	-0.7464	0.8092	-3.081	1.772
$r = 0.01,\ M = 10.39$									
0.2	20.88	92.43	-42.09	71.22	26.47	-18.36	8.106	-78.32	1.215
0.6	7.222	30.81	-14.03	24	9.156	-6.121	3.035	-26.04	1.261
1	4.491	18.49	-8.419	14.56	5.694	-3.672	2.021	-15.58	1.307
2	2.443	9.243	-4.209	7.476	3.097	-1.836	1.261	-7.737	1.422
5	1.214	3.697	-1.684	3.227	1.539	-0.7345	0.8042	-3.031	1.766
$r = 0.02,\ M = 10.22$									
0.2	20.53	90.96	-41.36	70.13	26.05	-18.07	7.984	-77.11	1.206
0.6	7.107	30.32	-13.79	23.64	9.018	-6.023	2.995	-25.63	1.252
1	4.422	18.19	-8.272	14.34	5.611	-3.614	1.997	-15.34	1.298
2	2.408	9.096	-4.136	7.368	3.055	-1.807	1.248	-7.616	1.414
5	1.2	3.638	-1.654	3.183	1.522	-0.7228	0.7994	-2.983	1.761
$r = 0.03,\ M = 10.06$									
0.2	20.19	89.51	-40.64	69.06	25.65	-17.78	7.865	-75.93	1.196
0.6	6.994	29.84	-13.55	23.28	8.882	-5.927	2.955	-25.24	1.243
1	4.354	17.9	-8.128	14.13	5.529	-3.556	1.973	-15.1	1.289
2	2.374	8.951	-4.064	7.261	3.015	-1.778	1.237	-7.497	1.406
5	1.186	3.58	-1.626	3.14	1.506	-0.7112	0.7946	-2.935	1.756
$r = 0.04,\ M = 9.898$									
0.2	19.86	88.08	-39.93	68.01	25.25	-17.5	7.748	-74.76	1.187
0.6	6.883	29.36	-13.31	22.93	8.749	-5.833	2.916	-24.85	1.234
1	4.287	17.62	-7.987	13.92	5.449	-3.5	1.95	-14.87	1.281
2	2.34	8.808	-3.993	7.155	2.975	-1.75	1.225	-7.38	1.398
5	1.172	3.523	-1.597	3.098	1.49	-0.6999	0.7899	-2.888	1.751

Table 6: **The sensitivity of the optimal strategies to the current interest rate.** The table shows how the components of optimal portfolios and the consumption-income ratio depend on the level of interest rates. The investor has a 10-year horizon, a risk aversion of 2, and a time preference rate of 3%. The income-asset correlations are $\rho_{yr} = \rho_{yS} = \sqrt{2}/2$. The expected growth rate of income is fixed at 4%.

W/y	π_B	π_S	π_0	c/y	$\bar{M}$
$r = 0,\ M = 40.50$					
0.2	5.067	6.746	-10.81	1.027	29.86
0.6	2.796	4.211	-6.007	1.121	30.25
1	2.102	3.362	-4.464	1.189	30.57
2	1.449	2.475	-2.924	1.32	31.2
5	0.9147	1.587	-1.502	1.618	32.46
$r = 0.02,\ M = 38.99$					
0.2	5.06	6.793	-10.85	1.016	28.81
0.6	2.788	4.215	-6.003	1.109	29.19
1	2.09	3.353	-4.443	1.176	29.5
2	1.437	2.455	-2.892	1.306	30.12
5	0.903	1.558	-1.461	1.606	31.34
$r = 0.04,\ M = 37.55$					
0.2	5.045	6.834	-10.88	1.005	27.79
0.6	2.779	4.217	-5.996	1.096	28.16
1	2.079	3.341	-4.42	1.163	28.46
2	1.425	2.434	-2.859	1.293	29.07
5	0.8914	1.529	-1.42	1.594	30.25

Table 7: **The optimal strategies with wide portfolio constraints.** The table shows the optimal strategies for various combinations of the wealth-income ratio and the current short-term interest rates. The investor has a time preference rate of 0.03, a relative risk aversion of 2, and a 30-year horizon. The portfolio weights are constrained to the interval $[-5, 10]$.

W/y	π_B	π_S	π_0	c/y	$\bar{M}$
$r = 0,\ M = 40.50$					
0.2	0	1	0	1.096	29.54
0.6	0	1	0	1.165	29.83
1	0	1	0	1.217	30.09
2	0	1	0	1.323	30.63
5	0	1	0	1.582	31.85
$r = 0.02,\ M = 38.99$					
0.2	0	1	0	1.083	28.49
0.6	0	1	0	1.15	28.78
1	0	1	0	1.201	29.03
2	0	1	0	1.307	29.56
5	0	1	0	1.569	30.75
$r = 0.04,\ M = 37.55$					
0.2	0	1	0	1.07	27.48
0.6	0	1	0	1.135	27.76
1	0	1	0	1.186	28
2	0	1	0	1.291	28.53
5	0	1	0	1.556	29.69

Table 8: **The optimal strategies with short-sales constraints.** The table shows the optimal strategies for various combinations of the wealth-income ratio and the current short-term interest rates. The investor has a time preference rate of 0.03, a relative risk aversion of 2, and a 30-year horizon. No borrowing or short sales are allowed.

W/y	π_B	π_S	π_0	c/y	$\bar{M}$
$\gamma = 5$, $M = 38.99$, wide constraints					
0.2	10	3.011	-12.01	1.05	25.61
0.6	6.604	1.783	-7.387	1.116	25.99
1	5.054	1.391	-5.446	1.165	26.29
2	3.531	0.9908	-3.522	1.26	26.85
5	2.209	0.6274	-1.836	1.482	27.95
$\gamma = 5$, $M = 38.99$, no short sales					
0.2	0	1	0	1.066	25.53
0.6	0	1	0	1.121	25.86
1	0	1	0	1.165	26.12
2	0.116	0.884	0	1.257	26.65
5	0.4333	0.5667	0	1.471	27.7
$\gamma = 10$, $M = 38.99$, wide constraints					
0.2	10	1.096	-10.1	0.8955	20.47
0.6	10	0.5416	-9.542	0.9331	20.72
1	7.529	0.4042	-6.933	0.9638	20.93
2	4.498	0.2679	-3.766	1.033	21.38
5	2.319	0.1693	-1.488	1.227	22.31
$\gamma = 10$, $M = 38.99$, no short sales					
0.2	0.7312	0.2688	0	0.886	20.08
0.6	0.8382	0.1618	0	0.9158	20.3
1	0.8598	0.1402	0	0.9448	20.51
2	0.8762	0.1238	0	1.014	20.98
5	0.8876	0.1124	0	1.209	22

Table 9: **The optimal strategies with different risk aversion coefficients.** The table shows the optimal strategies for investors with a relative risk aversion equal to 5 and 10, both for the case where the portfolio weights are constrained to the interval $[-5, 10]$ and the case where no short sales are allowed. The investor has a time preference rate of 0.03 and a 30-year horizon.

W/y	π_B	π_S	π_0	c/y	$\bar{M}$
$\rho_{yr} = -0.5,\ M = 40.72$					
0.2	-5	6.886	-0.8857	0.9886	28.96
0.6	-5	4.109	1.891	1.08	29.39
1	-5	3.195	2.805	1.148	29.75
2	-5	2.278	3.722	1.285	30.47
5	-3.333	1.444	2.889	1.607	31.87
$\rho_{yr} = 0,\ M = 39.56$					
0.2	-4.299	6.824	-1.525	1.013	28.81
0.6	-2.721	4.222	-0.5008	1.106	29.2
1	-2.132	3.351	-0.2191	1.173	29.51
2	-1.449	2.447	0.001899	1.304	30.14
5	-0.6514	1.548	0.1037	1.604	31.38
$\rho_{yr} = 0.5,\ M = 38.44$					
0.2	10	6.555	-15.56	1.008	28.87
0.6	8.046	4.092	-11.14	1.104	29.26
1	6.16	3.271	-8.431	1.172	29.58
2	4.247	2.409	-5.656	1.306	30.21
5	2.426	1.535	-2.961	1.611	31.44

Table 10: **The sensitivity of the optimal portfolio to the correlation between income and interest rate with wide portfolio constraints.** The table shows the optimal strategies for three values of the correlation coefficient between the labor income and the short-term interest rate. The investor has a time preference rate of 0.03, a relative risk aversion of 2, and a 30-year horizon. The portfolio weights are constrained to the interval $[-5, 10]$.

W/y	π_B	π_S	π_0	c/y	$\bar{M}$
$\rho_{yr} = -0.5$, $M = 40.72$					
0.2	0	1	0	1.078	28.49
0.6	0	1	0	1.14	28.79
1	0	1	0	1.189	29.05
2	0	1	0	1.292	29.63
5	0	1	0	1.559	30.93
$\rho_{yr} = 0$, $M = 39.56$					
0.2	0	1	0	1.082	28.49
0.6	0	1	0	1.147	28.78
1	0	1	0	1.197	29.03
2	0	1	0	1.302	29.58
5	0	1	0	1.566	30.81
$\rho_{yr} = 0.5$, $M = 38.44$					
0.2	0	1	0	1.085	28.49
0.6	0	1	0	1.153	28.77
1	0	1	0	1.205	29.02
2	0	1	0	1.312	29.54
5	0	1	0	1.573	30.7

Table 11: **The sensitivity of the optimal portfolio to the correlation between income and interest rate with short-sales constraints.** The table shows the optimal strategies for three values of the correlation coefficient between the labor income and the short-term interest rate. The investor has a time preference rate of 0.03, a relative risk aversion of 2, and a 30-year horizon. No borrowing or short sales are allowed.

W/y	π_B	π_S	π_0	c/y	$\bar{M}$
$\rho_{yS} = -0.5$, $M = 48.47$					
0.2	4.182	9.007	-12.19	0.937	29.9
0.6	2.407	5.6	-7.007	1.047	30.47
1	1.844	4.428	-5.272	1.126	30.93
2	1.287	3.17	-3.457	1.282	31.83
5	0.7941	1.855	-1.649	1.653	33.49
$\rho_{yS} = 0$, $M = 40.96$					
0.2	4.795	7.44	-11.24	0.9939	29
0.6	2.689	4.625	-6.314	1.09	29.42
1	2.028	3.668	-4.697	1.16	29.76
2	1.4	2.664	-3.063	1.296	30.45
5	0.8722	1.642	-1.515	1.611	31.79
$\rho_{yS} = 0.5$, $M = 34.86$					
0.2	5.865	4.728	-9.593	1.07	28.54
0.6	3.019	2.943	-4.962	1.157	28.83
1	2.223	2.374	-3.597	1.22	29.07
2	1.511	1.796	-2.307	1.343	29.54
5	0.9811	1.262	-1.243	1.614	30.45
$\rho_{yS} = 0.75$, $M = 32.26$					
0.2	6.429	2.745	-8.174	1.102	28.48
0.6	3.139	1.773	-3.912	1.188	28.74
1	2.279	1.476	-2.755	1.251	28.94
2	1.537	1.177	-1.713	1.372	29.33
5	1.01	0.9114	-0.9211	1.635	30.03

Table 12: **The sensitivity of the optimal portfolio to the correlation between income and stock with wide portfolio constraints.** The table shows the optimal strategies for four values of the correlation coefficient between the labor income and the stock return. The investor has a time preference rate of 0.03, a relative risk aversion of 2, and a 30-year horizon. The portfolio weights are constrained to the interval $[-5, 10]$.

W/y	π_B	π_S	π_0	c/y	$\bar{M}$
$\rho_{yS} = -0.5$, $M = 48.47$					
0.2	0	1	0	1.032	28.52
0.6	0	1	0	1.081	28.89
1	0	1	0	1.127	29.24
2	0	1	0	1.233	30.01
5	0	1	0	1.54	31.76
$\rho_{yS} = 0$, $M = 40.96$					
0.2	0	1	0	1.075	28.49
0.6	0	1	0	1.137	28.79
1	0	1	0	1.185	29.06
2	0	1	0	1.289	29.65
5	0	1	0	1.558	30.97
$\rho_{yS} = 0.5$, $M = 34.86$					
0.2	0	1	0	1.1	28.48
0.6	0	1	0	1.176	28.74
1	0	1	0	1.233	28.96
2	0	1	0	1.347	29.4
5	0	1	0	1.605	30.29
$\rho_{yS} = 0.75$, $M = 32.26$					
0.2	0	1	0	1.111	28.47
0.6	0	1	0	1.193	28.72
1	0	1	0	1.253	28.92
2	0	1	0	1.372	29.32
5	0.1223	0.8777	0	1.635	30

Table 13: **The sensitivity of the optimal portfolio to the correlation between income and stock with short-sales constraints.** The table shows the optimal strategies for four values of the correlation coefficient between the labor income and the stock return. The investor has a time preference rate of 0.03, a relative risk aversion of 2, and a 30-year horizon. No borrowing or short sales are allowed.

W/y	π_B	π_S	π_0	c/y	$\bar{M}$
$\xi_0 = 0.045, \xi_1 = -0.25$ ($M = 39.09$)					
0.2	2.727	6.808	-8.535	1.017	28.85
0.6	1.240	4.220	-4.460	1.110	29.23
1	0.8330	3.355	-3.188	1.177	29.55
2	0.5144	2.456	-1.971	1.308	30.17
5	0.3716	1.559	-0.931	1.607	31.39
$\xi_0 = 0.04, \xi_1 = 0$ ($M = 38.99$)					
0.2	5.060	6.793	-10.85	1.016	28.81
0.6	2.788	4.215	-6.003	1.109	29.19
1	2.090	3.353	-4.443	1.176	29.50
2	1.437	2.455	-2.892	1.306	30.12
5	0.903	1.558	-1.461	1.606	31.34
$\xi_0 = 0.035, \xi_1 = 0.25$ ($M = 38.95$)					
0.2	7.379	6.774	-13.15	1.013	28.74
0.6	4.330	4.207	-7.537	1.106	29.12
1	3.342	3.346	-5.688	1.173	29.43
2	2.354	2.449	-3.804	1.304	30.05
5	1.429	1.552	-1.981	1.604	31.27
$\xi_0 = 0.03, \xi_1 = 0.5$ ($M = 38.96$)					
0.2	9.680	6.743	-15.42	1.009	28.66
0.6	5.861	4.194	-9.055	1.102	29.05
1	4.583	3.334	-6.917	1.169	29.36
2	3.259	2.439	-4.698	1.300	29.98
5	1.941	1.541	-2.482	1.601	31.20
$\xi_0 = 0.02, \xi_1 = 1$ ($M = 39.13$)					
0.2	10	6.714	-15.71	0.997	28.46
0.6	8.851	4.143	-11.99	1.090	28.85
1	6.992	3.293	-9.285	1.157	29.17
2	4.997	2.402	-6.399	1.289	29.80
5	2.897	1.507	-3.404	1.594	31.03

Table 14: **The sensitivity of the optimal portfolio to the decomposition of the expected income growth rate with wide portfolio constraints.** Risk aversion 2, 30-year horizon...

W/y	π_B	π_S	π_0	c/y	$\bar{M}$
$\xi_0 = 0.045, \xi_1 = -0.25$ ($M = 39.09$)					
0.2	0	1	0	1.083	28.54
0.6	0	1	0	1.150	28.83
1	0	1	0	1.201	29.09
2	0	1	0	1.307	29.63
5	0	1	0	1.571	30.83
$\xi_0 = 0.04, \xi_1 = 0$ ($M = 38.99$)					
0.2	0	1	0	1.083	28.49
0.6	0	1	0	1.150	28.78
1	0	1	0	1.201	29.03
2	0	1	0	1.307	29.56
5	0	1	0	1.569	30.75
$\xi_0 = 0.035, \xi_1 = 0.25$ ($M = 38.95$)					
0.2	0	1	0	1.084	28.41
0.6	0	1	0	1.149	28.70
1	0	1	0	1.200	28.94
2	0	1	0	1.305	29.47
5	0	1	0	1.567	30.65
$\xi_0 = 0.03, \xi_1 = 0.5$ ($M = 38.96$)					
0.2	0	1	0	1.083	28.31
0.6	0	1	0	1.148	28.59
1	0	1	0	1.198	28.84
2	0	1	0	1.303	29.36
5	0	1	0	1.563	30.53
$\xi_0 = 0.02, \xi_1 = 1$ ($M = 39.13$)					
0.2	0	1	0	1.080	28.04
0.6	0	1	0	1.143	28.31
1	0	1	0	1.191	28.55
2	0	1	0	1.293	29.07
5	0	1	0	1.552	30.22

Table 15: **The sensitivity of the optimal portfolio to the decomposition of the expected income growth rate with no borrowing.** Risk aversion 2, 30-year horizon...