Hedging Basket Options by Using a Subset of Underlying Assets

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Abstract

The purpose of this paper is to investigate the use of Principal Component Analysis in finding the efficient subset of underlying assets for hedging European basket options. This asset selection technique can be used together with other hedging strategies to enhance the hedging performance. Meanwhile, it become practical and essential when some of the underlying assets are illiquid or even not available to be traded. As an illustration, the optimal subset of assets is combined with a static hedging strategy that super-replicates a basket option with plain vanilla options on all the underlying assets with optimal strike prices. Through the combination of this super-hedging strategy and the newly-developed asset selection technique, we get a static hedging portfolio consisting of plain vanilla options only on the subset of dominant assets with optimal strikes. The strikes are chosen according to certain optimal criteria which depend on the risk attitude of investors while hedging basket options. The first hedging strategy could be a super-replication to eliminate all risks. Alternatively given a constraint on the investment into the hedge, optimal strikes are computed by minimizing a particular risk measure, e.g., variance of the hedging error or expected shortfall. Hence, the newly-developed static hedging portfolio by a subset of underlying assets is indeed to gain a tradeoff between the reduced hedging costs and the successful hedge. Through a numerical analysis, it is concluded that even without considering transaction costs hedging by using only a subset of assets works well particularly for in- and at-the-money basket options: a small hedging error is achieved with a relatively low hedging cost.

Key words: basket options, Principal Component Analysis, super-replication, expected shortfall.

1 Introduction

A basket option is an option whose payoff is linked to a portfolio or "basket" of underlying assets. The basket can be any weighted sum of underlyings as long as the weights are all positive. Various types of basket options have emerged in the market and become popular as a key tool for reducing risks since the early 1990s. They are either sold separately over-the-counter or sometimes issued as part of complex financial contracts, for instance, as "equity-kickers" in bond-like structures.

The typical underlying of a basket option is a basket consisting of several stocks, indices or currencies. Less frequently, interest rates are also possible. Several reasons to trade basket options are reported in the literature. The main advantage of basket options is that they tend to be cheaper than the corresponding portfolio of plain vanilla options. On one hand, this is due to the fact that usually the underlying assets in the basket are not perfectly correlated. On the other hand, a basket option minimizes transaction costs because an investor has to buy only one option instead of several ones. Thus, a basket option is considered as a cheaper alternative to hedge a risky position consisting of several assets. In addition, basket options are also ideal for clients who have a specific view of the market. They may be interested in diversified risk, or have a view on a particular sector, best expressed by a portfolio of individual stocks. So, the use of a basket of assets as an underlying allows products to be tailored to clients' needs. That's why the most widespread underlying of a basket option is a basket of stocks that represents a certain economy sector, industry or region.

The inherent challenge in pricing and hedging basket options stems primarily from the lack of availability of the distribution of a weighted average of correlated lognormals to find a closed-form pricing formula and then hedge ratios. An additional difficulty in evaluating basket options is due to the correlation structure involved in the basket, which is observed to be volatile over time as is the volatility. However, opposed to the volatility, correlations are not available in the market and must be estimated from sometimes scarce option data or from historical time series. Hence, the current common practice is to assume it to be constant.

So far, several methods have been proposed for hedging basket options. Basically, they could be classified into three categories. First, numerical methods such as Monte Carlo simulations are used by Engelmann and Schwendner (2001) [8] to compute Greeks with the assumption that the market is complete and basket options can be perfectly replicated. However, the lack of knowledge of the underlying distribution and related hedge parameters make it in general impossible to perfectly hedge basket options by buying or selling a portfolio of options on the individual assets. In this context, some researchers are endeavored to develop partial hedging strategies. For example, in the second category, some static hedging strategies are found to minimize the variance of the discrepancy between the final payoffs of the target basket option and the hedging portfolio: Pellizzari (2004) [23] achieves this objective directly with the help of Monte Carlo simulation and Ashraff, Tarczon and Wu (1995) [1] develop a variance-minimizing hedging strategy based on gamma hedging which additionally considers the cross-gamma effect. In the absence of a perfect hedge, in incomplete markets, the next best thing is the least expensive super-replicating strategy. For this purpose, many different methods are tried: First, both d'Aspremont and El-Ghaoui (2003) [2] and Laurence and Wang (2003) [18] treat the bound derivation as an optimization problem, in more detail a semi-definite problem, and solve it through the corresponding dual problem. The obtained bounds in their independent work are shown to be equivalent although with completely different approaches. Cherubini and Luciano (2002) [4] derive the upper and lower bounds for basket options by means of Fréchet bounds in the Copula framework. Furthermore, Hobson, Laurence and Wang (2004) [13] suggest a super-replicating portfolio based on Jensen's inequality.

In this paper, we develop the idea of using only a subset of constituent assets in hedging basket options. This is motivated by the fact that most of the new contracts are often related to a large number of underlying assets. In this case, the usual idea of using all the underlying assets to hedge basket options becomes a huge task. This is not only computationally expensive, but also creates high transaction costs which greatly reduce the hedging efficiency. Thus, it is desirable to find a strategy to hedge a basket option at a reasonable cost. Furthermore, hedging with subset assets becomes more practical and essential when some of the underlying assets are illiquid or even not available to be traded[1].

The same idea is first introduced in Lamberton and Lapeyre (1992) [17]. They assume that the market is complete such that all the assets including basket options can be perfectly hedged by a self-financing portfolio. Then the optimal subset of assets is achieved by minimizing the price difference between the self-financing portfolio and the hedging portfolio with only a subset of assets. The minimization is in essence a regression procedure. In turn, the selection of the subset of assets is equivalent to the selection of variables of a multiple regression. Accordingly, the numerical methods, such as forward, backward selection algorithms and stepwise regression methods, are recommended. Then, they design a dynamic approximate hedging portfolio which consists of the plain vanilla options on the optimal hedging assets. Alternatively, according to Nelken (1999) [15], the selection of hedging assets is in practice simply due to the liquidity or exposure of the assets in the basket.

This article is to introduce another approach, Principal Component Analysis (PCA) to figure out the optimal assets for hedging basket options. PCA is one of the classical data mining tools to reduce dimensions in multivariate data by choosing the most effective orthogonal factors to explain the original multivariate variables. This objective can be easily realized by decomposing the covariance matrix. Thus, this method is quite easy to implement with almost instant calculation as well as a reasonable accuracy. So far, this method is applied in finance mainly to identify the multiple risk factors in portfolio management and to figure out the dominant factor components driving the term structure movements of at-the-money (ATM) implied volatilities (cf. Fengler, Härdle and Schmidt (2002) [9]). Furthermore, it is also applied to find a low-rank correlation matrix nearest to a given correlation matrix. Particularly, Dahl and Benth (2002) [6] develop a method combining PCA and Quasi Monte Carlo simulations for a fast evaluation of Asian basket options. The idea is to capture the main or most of the information of the

[1] This is possible when the underlying is a mutual fund.

noise term (the covariance structure), which is complicated with a rather large number of dimensions in both time and asset, by considerably reduced dimensions. They call the dimension reduction technique as Singular Value Decomposition, which is equivalent to PCA when the covariance structure is studied. Similarly, PCA is adopted in the present paper to find the most effective underlying factors that capture the main information of the basket. Thereafter, one step further is taken to obtain the subset of the underlying assets that are highly related to these selected factors.

The selection of a subset of assets can be combined with other hedging strategies. In this paper, as an illustration, it is used together with a static super-hedging strategy. The basic idea of this static super-hedging strategy is to find a portfolio of plain vanilla options on the constituent assets of the basket with optimal strikes to super-replicate the basket option at the lowest cost. Through the combination of this super-hedging strategy and the newly-designed asset selection technique, we get a static hedging portfolio consisting of the plain vanilla options on the dominant assets in the basket in two steps: first find the appropriate set of hedging assets by means of PCA; and then figure out the optimal strikes of the options on the chosen subset of underlying assets.

Surely, a subset could not perfectly tract the original underlying basket and could leave some risk exposure uncovered. Hence, any hedging strategy with subset assets in the basket is actually a hedging portfolio in an incomplete market. The incompleteness is due to impracticability or impossibility of trading in all the underlying assets. In this context, the above two-step hedge strategy is modified to satisfy different optimality criteria by choosing strikes. The criterion depends on the risk attitude of investors while hedging basket options. They may favor a super-replication hedging portfolio to eliminate all risks. Alternatively with a constraint on the hedging cost at the initial date, optimal strikes are computed by minimizing a particular risk measure, e.g., variance of the hedging error or expected shortfall. In any case, one has an optimization problem to solve. Due to the lack of the distribution of the underlying basket, all the hedging strategies are numerically obtained through running Monte Carlo simulation. As shown by the numerical results, hedging error (measured by expected shortfall) at the maturity date decreases with the optimal strikes and hence the hedging cost. As a result, the new-proposed static hedging portfolio by the subset of underlying assets is to gain a tradeoff between the reduced hedging costs and the overall super-replication. In general, the hedging performance is better for in- and at-the-money basket options such that a small hedging error is achieved by investing a relatively lower hedging capital.

The remainder of the paper is organized as follows: Section 2 defines the assumptions and notations. In Section 3 the multivariate statistical method, PCA, is applied to select the optimal subset of assets for hedging basket options after a brief introduction of the required knowledge on this method. Then a two-step static hedging strategy is developed in Section 4 by combining a static super-hedging strategy with the optimal subset of assets. Numerical results are reported in Section 5. Finally, Section 6 concludes the paper.

2 Assumptions and Notations

We consider a financial market with continuous trading where all trading takes place in the finite time period $[0, T]$. The market consists of N risky assets S_i, $i = 1, \cdots, N$ and a risk free asset denoted by B which is usually called bank account. The dynamics of the bank account, which is continuously compounded with a constant positive risk free interest rate $r \geq 0$, are given by

$$dB(t) = rB(t)dt.$$

To model the N risky assets, we define the standard N-dimensional Wiener process $W = (W_1(t), \cdots, W_N(t))$ on the filtered probability space $(\Omega, \mathcal{F}_t, \mathcal{Q})$, where $\mathcal{Q}$ is the risk-neutral probability measure. As often assumed in the literature, the price process of each risky asset S_i, $i = 1, \cdots, N$ follows a geometric Brownian motion and one-dimensional Brownian motions, W_i, $i = 1, \cdots, N$, are correlated with each other according to a constant parameter. More explicitly, under the risk-neutral probability measure $\mathcal{Q}$ we have

$$\begin{aligned} dS_i(t) &= (r - q_i)S_i(t)dt + \sigma_i S_i(t)dW_i(t) \qquad & i = 1, \cdots, N \\ \rho_{ij}dt &= dW_i(t)dW_j(t) & i \neq j, \end{aligned}$$

where σ_i and q_i are the volatility and continuously compounded dividend yield of asset i, respectively and $\rho_{i,j} \in [-1, 1]$ denotes the constant correlation between assets i and j. Additionally, the determinant of the corresponding correlation structure is assumed to be unequal to 0 to ensure the completeness of the market.

In addition to the above-mentioned primary assets, there are also T-contingent claims, such as plain vanilla calls on each risky asset S_i with strike price $k \in \mathcal{K}^{(i)}$, the set of all strike prices traded in the market, and maturity date T

$$C_T^{(i)}(k) = (S_i(T) - k)^+, \qquad i = 1, \cdots, N,$$

as well as a basket call option on all the N risky assets with the same maturity date T and strike price K

$$BC_T(K) = \left(\sum_{i=1}^{N} \omega_i S_i(T) - K \right)^+,$$

where each risky asset is weighted by a positive constant ω_i, $i = 1, \cdots, N$. That is, if $\sum_{i=1}^{N} \omega_i S_i(T)$, the sum of asset prices S_i weighted by positive constants ω_i at date T, is more than K, the payoff equals the difference; otherwise, the payoff is zero.

Assuming that the Black-Scholes (BS) model is valid, the price of each plain vanilla call option on asset i is given by

$$C_0^{(i)}(k) = e^{-q_i T} S_i(0)\Phi(d_1) - e^{-rT} k\Phi(d_2), \qquad i = 1, \cdots, N$$

where $d_1 = \dfrac{\ln \frac{S_i(0)}{k} + (r - q_i + \frac{1}{2}\sigma_i^2)T}{\sigma_i \sqrt{T}}$, $d_2 = d_1 - \sigma_i \sqrt{T}$ and $\Phi(y) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{y} e^{-\frac{z^2}{2}} dz$ cumulative distribution function of the standard normal distribution.

In the present paper, only the hedging method on a basket call option is concerned. However, as pointed out by Laurence and Wang (2003) [18] and Deelstra, Liinev and Vanmaele (2004) [7], there is a put-call parity result for basket options, which is given by

$$\left(K-\sum_{i=1}^{N}\omega_iS_i(T)\right)^+=\left(\sum_{i=1}^{N}\omega_iS_i(T)-K\right)^+ +\left(K-\sum_{i=1}^{N}\omega_iS_i(T)\right).$$

Hence, a hedging strategy on a basket call option can be translated directly into one on the corresponding basket put option.

To measure the effectiveness of a hedging portfolio (HP), the hedging cost (HC) is defined as the price of the hedging portfolio at the initial date 0. Meanwhile, the hedging error at the maturity date T is simply denoted as HE, giving the difference between the payoffs of the basket option and the hedging portfolio, i.e., $BC_T(K)-HP_T$.

3 Optimal Asset Selection

In this section we are dealing with the selection of a subset of underlying assets to hedge basket options. Given the multi-dimensional nature of basket options, the derived hedging strategy is often composed of all the underlying assets. In practice, the underlyings in the contract are differently weighted and sometimes some with pretty small weights. Thus, one can simply hedge such basket options by neglecting those assets. However, it is rather arbitrary and lacks a theoretical foundation for the general case. This paper is to offer a criterion for assets selection. For this purpose, a method, Principal Component Analysis (PCA), is introduced.

3.1 Principal Component Analysis and Application to Basket Options Hedging

PCA is a popular method for dimensionality reduction in multivariate data analysis. Thus, it is useful in visualizing multidimensional data, and most importantly, identifying the underlying principal factors of the original variables. PCA is originated by Pearson [21] and proposed later by Hotelling [14] for the specific adaptations to correlation structure analysis. Its idea has been well described, among others, in Harman (1967) [10], Härdle and Simar (2003) [11] and Srirastava and Khatri (1979) [25]. We follow here the lines of Härdle and Simar (2003).

The main objective of PCA is to reduce the dimensionality of a data set without a significant loss of information. This is achieved by decomposing the covariance matrix into a vector of eigenvalues ordered by importance and eigenvectors. To be precise, consider the asset prices vector $\mathcal{S}=(S_1,\cdots,S_N)^T$ with

$$E(\mathcal{S})=\mu \qquad \text{and} \qquad Var(\mathcal{S})=\Sigma=E\left[(\mathcal{S}-\mu)(\mathcal{S}-\mu)^T\right].$$

PCA is to decompose the covariance matrix into its eigenvalues and eigenvectors as

$$\Sigma=\Gamma\Lambda\Gamma^T, \tag{1}$$

where $\Lambda = diag(\lambda_1, \cdots, \lambda_N)$ is the diagonal eigenvalue matrix with $\lambda_1 > \cdots > \lambda_N$ and Γ the matrix of corresponding eigenvectors

$$\Gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1N} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{N1} & \gamma_{N2} & \cdots & \gamma_{NN} \end{pmatrix}$$

or simply $(\gamma_1, \cdots, \gamma_N)$ given by the columns of the matrix. Principal Components transformation is then defined as the product of the eigenvectors and the original matrix less mean vector

$$P = \Gamma^T(\mathcal{S} - \mu). \tag{2}$$

That is, the PC transformation is a linear transformation of the underlying assets. Its elements $P_1, \cdots, P_N$ are named as i-th Principal Components (PCs) since they can be considered as the underlying factors that influence the underlying assets with decreasing significance as measured by the size of the corresponding eigenvalues.

The ability of the first N_1 $(N_1 < N)$ PCs to explain the variation in data is measured by the relative proportion of the cumulated sum of eigenvalues

$$\pi_{N_1} = \frac{\sum_{j=1}^{N_1} \lambda_j}{\sum_{j=1}^{N} \lambda_j}.$$

If a satisfactory percentage of the total variance has been accounted for the first few components, the remaining PCs can be ignored as the assets are already well represented without significant loss of information. The usual practice is to choose the first N_1 PCs that account for over 75% of the variance or simply identify $N_1 = 3$ for the convenience of visualizing the data.

The weighting of the PCs, or simply the element of each eigenvector, describes how the original variables are interpreted by the factors. This could be validated by considering the covariance between the PC vector P and the original vector $\mathcal{S}$

$$\begin{aligned} Cov(\mathcal{S}, P) &= E(\mathcal{S}P^T) - E\mathcal{S}EP^T \\ &= E(\mathcal{S}\mathcal{S}^T\Gamma) - \mu\mu^T\Gamma \\ &= \Sigma\Gamma \\ &= \Gamma\Lambda\Gamma^T\Gamma \\ &= \Gamma\Lambda. \end{aligned} \tag{3}$$

It implies that the correlation $r_{ij} = \rho_{S_i,P_j}$ between the variable S_i and the PC P_j is[2]

$$r_{ij} = \frac{\gamma_{ij}\lambda_j}{(\sigma_{S_i}^2\lambda_j)^{1/2}} = \gamma_{ij}\left(\frac{\lambda_j}{\sigma_i^2}\right)^{1/2}.$$

Clearly, γ_{ij} is proportional to the covariance of asset S_i and P_j. The higher it is, the more related is the i-th asset to the j-th PC. Hence, γ_{ij} are usually called as factor

[2] Note that $Var(P_j) = \lambda_j$. For the detailed derivation please check the referred books.

loadings, interpreting the relationship between the original variables S_i, $i = 1, \cdots, N$ and the derived factors, i.e., PCs P_j, $j = 1, \cdots, N_1$. The standard practice is to calculate r_{ij}^2 and then take the value as the proportion of variance of S_i explained by P_j. This is verified by first

$$\sum_{j=1}^{N} \lambda_j \gamma_{ij}^2 = \gamma_i^T \Lambda \gamma_i$$

the (i,i)-element of the matrix $\Gamma\Lambda\Gamma^T = \Sigma$ and indeed

$$\sum_{j=1}^{N} r_{ij}^2 = \frac{\sum_{j=1}^{N} \lambda_j \gamma_{ij}^2}{\sigma_i^2} = \frac{\sigma_i^2}{\sigma_i^2} = 1.$$

It should be noted that the PCs are not scale invariant, e.g., the PCs derived from the covariance matrix give different results when the variables take different scales. Consequently, instead of the covariance matrix, the correlation matrix is recommended to be decomposed.

Now based on the principle of PCA, asset selection could be completed in four steps as follows:

- Step I: Find the covariance matrix of the underlying assets.

  As assumed in Section 2, each underlying asset follows a geometric Brownian motion. Written in matrix form, we have

$$d\mathcal{S}_t = d\begin{pmatrix} S_1(t) \\ S_2(t) \\ \vdots \\ S_N(t) \end{pmatrix} = \begin{pmatrix} S_1(t) \\ S_2(t) \\ \vdots \\ S_N(t) \end{pmatrix} rdt + \begin{pmatrix} \sigma_1 S_1(t) dW_1(t) \\ \sigma_2 S_2(t) dW_2(t) \\ \vdots \\ \sigma_N S_N(t) dW_N(t) \end{pmatrix}$$

  by assuming the dividend is zero[3]. And the covariance of weighted assets is given by

$$Cov(d\omega\mathcal{S}) = \begin{pmatrix} \omega_1^2\sigma_1^2 S_1^2(t) & \omega_1\omega_2\sigma_1\sigma_2\rho_{12}S_1(t)S_2(t) & \cdots & \omega_1\omega_N\sigma_1\sigma_N\rho_{1N}S_1(t)S_N(t) \\ \omega_1\omega_2\sigma_1\sigma_2\rho_{12}S_1(t)S_2(t) & \omega_2^2\sigma_2^2 S_2^2(t) & \cdots & \omega_2\omega_N\sigma_2\sigma_N\rho_{2N}S_2(t)S_N(t) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1\omega_N\sigma_1\sigma_N\rho_{1N}S_1(t)S_N(t) & \omega_2\omega_N\sigma_2\sigma_N\rho_{2N}S_2(t)S_N(t) & \cdots & \omega_N^2\sigma_N^2 S_N^2(t) \end{pmatrix} dt.$$

  If this covariance is to be studied by PCA, that means we evaluate the covariance of the change of the basket, instead of the basket. Indeed, it serves as a trick since the covariance of the basket with multivariates of lognormal distribution is rather complicated. Because of the properties of the geometric Brownian motion, the procedure is simplified and the effects of the involved parameters on the basket option price are maintained. However, one problem to be fixed here is what non-anticipating value should be taken for $S_i(t)$.

[3]The calculation procedure is actually the same for the case with a constant continuous dividend rate since dividends have no any effect on the covariance structure.

The direct means is to consider the ratio of $\frac{d\omega\mathcal{S}}{\mathcal{S}}$ such that asset prices $S_i(t)$ are cancelled out in the covariance structure as below

$$Cov(\frac{d\omega\mathcal{S}}{\mathcal{S}}) = \begin{pmatrix} \omega_1^2\sigma_1^2 & \omega_1\omega_2\sigma_1\sigma_2\rho_{12} & \cdots & \omega_1\omega_N\sigma_1\sigma_N\rho_{1N} \\ \omega_1\omega_2\sigma_1\sigma_2\rho_{12} & \omega_2^2\sigma_2^2 & \cdots & \omega_2\omega_N\sigma_2\sigma_N\rho_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1\omega_N\sigma_1\sigma_N\rho_{1N} & \omega_2\omega_N\sigma_2\sigma_N\rho_{2N} & \cdots & \omega_N^2\sigma_N^2 \end{pmatrix} dt.$$

This covariance structure should work well except for the case in which the spot prices of the underlying assets differ significantly from one another, where the underlying asset with the extremely high price should be considered in any case (even with a relatively low volatility) due to its absolute dominant effect on the basket option price.

Contrary to the usual practice of decomposing the correlation matrix as recommended in PCA text books, the modified covariance matrix is used in this application. This is motivated by the fact that weights and individual asset volatilities do have a great impact on the price of the basket option.

In practice, this step is done by first studying the time series of the asset price to achieve the basic correlation structure and the (ATM) volatility of all the underlying assets as those given above[4]. Then combine these structures further with weights to obtain the modified covariance structure.

- Step II: Decompose the covariance matrix into the eigenvalue vector ordered by importance and the corresponding eigenvectors. This evaluation procedure could be easily done by many programs such as Matlab, Mathematica and C++ etc.
- Step III: Choose principal components according to the cumulative proportion of the explained variance.
- Step IV: Select the optimal subset of N_1 underlying assets by examining the cumulative r^2 for each asset with the principal components chosen in the previous step. The selection can be done in two ways: First, if the number of hedging assets, N_1, is beforehand determined, the list of least important assets is checked out after a comparison of cumulative r^2. If there is no prior requirement on the number of assets, a more careful study of the cumulative r^2 has to be done to find the most effective assets.

4 Combination of Asset Selection and a Static Super-Hedging Strategy

The selection of the optimal subset of assets can be used together with other hedging methods. Here in this paper, we are going to combine it with a static super-hedging portfolio that is going to be developed in the next subsection.

[4]Due to the complexity of volatility, traders and analysts have developed rules of thumb: using ATM volatility for each component asset to price basket options

4.1 A Static Super-Hedging Strategy for Basket Options

This super-hedging portfolio, i.e., the least expensive super-replicating portfolio, consists of plain vanilla call options on all the constituent assets traded in the market with optimal strike price. This method is first derived in an idealized situation where all the option prices on the constituent assets with a continuum of strikes are known. That is, $\mathcal{K}^{(i)}$, the set of all strike prices of options traded in the market on the underlying asset S_i, is a continuum interval. With this full information, the portfolio could be obtained by simply solving a Lagrangian problem in the BS framework. However, calls are traded only with a limited number of strikes in practice. The above obtained portfolio thus has to be calibrated accordingly. The calibration procedure could be named as "convexity correction", approximating the option's price with the optimal strike by two options with two neighboring strikes.

4.1.1 Hedging with a Continuum of Strikes

The objective of this method is to find a super-hedging portfolio whose final payoff is always larger than that of a basket call option. This idea is stimulated by Jensen's inequality for the final payoff of a basket call option:

$$\begin{aligned} BC_T(K) &= \left(\sum_{i=1}^{N} \omega_i S_i(T) - K\right)^+ = \left[\sum_{i=1}^{N} \omega_i \left(S_i(T) - \frac{b_i}{\omega_i}K\right)\right]^+ \\ &\leq \sum_{i=1}^{N} \omega_i \left(S_i(T) - \frac{b_i}{\omega_i}K\right)^+. \end{aligned}$$

The first transformation is to take ω_i out of the bracket and this is fulfilled if and only if $\sum_{i=1}^{N} b_i = 1$ and the second one is due to Jensen's inequality. That means, the payoff of any portfolio consisting of N plain vanilla call options is larger or at least equal to that of the corresponding basket call option. Moreover, as a consequence of the no-arbitrage assumption, the price of a financial product is given by the discounted expected final payoff under the risk-neutral measure. We could then find the corresponding relationship between the price of a basket call option and that of the hedging portfolio

$$e^{-rT} E\left[\left(\sum_{i=1}^{N} \omega_i S_i(T) - K\right)^+\right] \leq \sum_{i=1}^{N} \omega_i e^{-rT} E\left[\left(S_i(T) - \frac{b_i}{\omega_i}K\right)^+\right]. \tag{4}$$

For the purpose of hedging, we would like to look for a portfolio of plain vanilla call options with the optimal strike prices which depend on the choice of the b_i's such that it is the cheapest hedging strategy to dominate the final payoff of a basket call option.

Hence, we now have a minimization problem to solve: minimize the price of a weighted portfolio of standard options with respect to b_i's subject to the condition that the sum of b_i's is equal to 1.

$$\begin{aligned} \min_{b_i} \quad & \sum_{i=1}^{N} \omega_i e^{-rT} E\left[\left(S_i(T) - \frac{b_i}{\omega_i}K\right)^+\right] \\ \text{s.t.} \quad & \sum_{i=1}^{N} b_i = 1. \end{aligned} \tag{5}$$

Its solution, i.e., the optimal sequence of weights b_i^*, is given by the following proposition[5].

Proposition 4.1. *Suppose the underlying assets of a basket option follow geometric Brownian motions and the Black-Scholes model is valid, then the optimal* b_i^**'s satisfying*

$$BC_0(K) \leq \sum_{i=1}^{N} \omega_i e^{-rT} E\left[\left(S_i(T) - \frac{b_i^*}{\omega_i}K\right)^+\right]$$

are uniquely obtained by solving a set of equations:

$$\begin{aligned} b_i &= \frac{\omega_i S_i}{K}\left(\frac{b_1 K}{\omega_1 S_1}\right)^{\frac{\sigma_i}{\sigma_1}} \exp\left\{T\left[\left(1-\frac{\sigma_i}{\sigma_1}\right)\left(r+\frac{1}{2}\sigma_1\sigma_i\right)+\left(\frac{\sigma_i}{\sigma_1}q_1 - q_i\right)\right]\right\} \quad (6) \\ \sum_{i=1}^{N} b_i &= 1. \end{aligned}$$

4.1.2 Hedging with a Discrete Set of Strikes

In practice, $\mathcal{K}^{(i)}$, the set of all strike prices of options traded in the market on the underlying asset S_i, is often not a continuum range or interval, but a discrete set. Hence, a direct impact on the hedging portfolio is caused since the optimal hedging product may not exist.

Recall that one of the properties of a convex function is that for any $c \in (a, b)$ such that $c = \beta * a + (1-\beta) * b$ where $\beta \in [0, 1]$, the following holds

$$\varphi(c) \leq \beta\varphi(a) + (1-\beta)\varphi(b),$$

where $\varphi(.)$ is a convex function. That is, the value of the convex function at a particular point is bounded from above by a linear interpolation of the two neighboring values. This could be used to maintain the super-replication feature of the desired hedging portfolio since the Black-Scholes call option price is well-known to be convex with respect to the strike price.

To illustrate it in our case, we define $\mathcal{K}^{(i)} = (k_0^{(i)}, k_1^{(i)}, \cdots, k_m^{(i)})$ the set of the traded $m+1$ strikes in increasing order, i.e., $k_j^{(i)} < k_{j+1}^{(i)}$ for $j+1 \leq m$ and $k_0^{(i)} = 0$, i.e., the least possible strike is such that the call option is the asset itself. Assume the former achieved optimal strikes are not always traded in the market. For those assets whose call options with strike price $k_{optimal}^{(i)} = \frac{b_i^*}{\omega_i}K$ are not traded, one can replace them by a linear combination of two call option prices with the neighboring strikes $k_j^{(i)}$ and $k_{j+1}^{(i)}$ such that

$$C^{(i)}(k_{optimal}^{(i)}) \leq \beta^* C^{(i)}(k_j^{(i)}) + (1-\beta^*)C^{(i)}(k_{j+1}^{(i)}),$$

where $\beta^* = \frac{k_{optimal}^{(i)} - k_{j+1}^{(i)}}{k_{optimal}^{(i)} - k_j^{(i)}}$. In this way, the upper bound for a basket call option can be

[5]Its proof is provided in Appendix.

generally expressed as

$$\sum_{k^{(i)}_{optimal}\text{traded}} \omega_i e^{-rT} E\left[\left(S_i(T) - \frac{b_i^*}{\omega_i}K\right)^+\right] +$$
$$\sum_{k^{(i)}_{optimal}\text{non-traded}} \omega_i e^{-rT}\left(\beta^* E\left[\left(S_i(T) - k_j^{(i)}\right)^+\right] + (1-\beta^*)E\left[\left(S_i(T) - k_{j+1}^{(i)}\right)^+\right]\right).$$

Hence when only a limited number of strikes are traded on each asset, one can still find a super-hedging strategy that consists of one or two traded calls on each constituent asset.

In all, this hedging portfolio is an upper bound. Thus, all the risks are avoided, which is the second best for risk managers as the first best, perfect hedging, is almost impossible or complicated for basket options. Meanwhile, the hedging portfolio consists of a portfolio of plain vanilla options, thus is independent of the correlation structure between assets. As mentioned above, one of the difficulty of basket options hedging lies in controlling the correlation structure. Furthermore, the lack of reliable data on correlation worsens the problem. Consequently, it is favorable to achieve a hedging method independent of correlations. As easily observed, if the underlying assets are perfectly correlated, the upper bound is exactly equal to the price of the basket option. In this case, it is a perfect hedge. In the case of high correlation, for example when all the constituent stocks belong to the same industry, a high performance can be expected. However, in the case of low correlations, then it may perform not that well, which additionally serves as the underlying reason for introducing the idea of hedging by only a subset of assets to reduce the relatively large difference between the final payoffs of basket options and the hedging portfolio.

The similar idea is applied by Nielsen and Sandmann (2003) [20] to derive an upper bound for Asian options; and also by Hobson, Laurence and Wang (2004) [13] to super-hedge basket options. In the latter paper, this problem is analyzed in a general framework for obtaining a model-independent super-hedging portfolio, thus with a focus on proving the existence of such a super-hedging strategy, but without any hint on the form of the solution.

4.2 Combination of Asset Selection and the Static Super-hedging Strategy

In this subsection, some new two-step static hedging methods are proposed by following the idea of the above super-hedging strategy. First, the optimal subset of assets are picked out by PCA. Then, the hedging portfolio is proposed to be composed of the call options written on these N_1 most important underlying assets in the basket with the optimal strikes according to certain optimality criteria, which is defined through risk measures. The obtained hedging strategies are indeed to gain a trade-off between reduced hedging costs and the overall super-replication on basket options.

4.2.1 Problem Formulation

As the first step of the newly-designed static hedging method, PCA is utilized to find the subset of important assets in the basket. In this way, all the underlying assets are newly indexed and regrouped into two subsets: one subset of N_1 assets of high significance S_j, where $j = 1, \cdots, N_1$ and one with the other $N - N_1$ assets S_j, where $j = N_1 + 1, \cdots, N$. Then the final payoff of the basket option can be rewritten as

$$\begin{aligned}
\left(\sum_{j=1}^{N}\omega_j S_j(T) - K\right)^+ \quad &=_{(A)} \quad \left(\sum_{j=1}^{N_1}\omega_j S_j(T) - \lambda K + \sum_{j=N_1+1}^{N}\omega_j S_j(T) - (1-\lambda)K\right)^+ \\
&\leq_{(B)} \quad \underbrace{\left(\sum_{j=1}^{N_1}\omega_j S_j(T) - \lambda K\right)^+}_{I} + \underbrace{\left(\sum_{j=N_1+1}^{N}\omega_j S_j(T) - (1-\lambda)K\right)^+}_{II} \\
&\overset{\sum_{j=1}^{N_1}\alpha_j=1}{\underset{\sum_{j=N_1+1}^{N}\beta_j=1}{=}} \quad \left[\sum_{j=1}^{N_1}\omega_j\left(S_j(T) - \frac{\alpha_j}{\omega_j}\lambda K\right)\right]^+ + \left[\sum_{j=N_1+1}^{N}\omega_j\left(S_j(T) - \frac{\beta_j}{\omega_j}(1-\lambda)K\right)\right]^+ \\
&\leq_{(C)} \quad \underbrace{\sum_{j=1}^{N_1}\omega_j\left(S_j(T) - \frac{\alpha_j}{\omega_j}\lambda K\right)^+}_{I'} + \underbrace{\sum_{j=N_1+1}^{N}\omega_j\left(S_j(T) - \frac{\beta_j}{\omega_j}(1-\lambda)K\right)^+}_{II'}. \qquad (7)
\end{aligned}$$

That is, a basket call option's payoff is always dominated by two portfolios of plain vanilla call options denoted as I' and II' in (C). This result is achieved by applying two times Jensen's inequality in (B) and (C), respectively. Serving as a trick for the further derivation, the strike of the basket option K is in (A) split into λK and $(1-\lambda)K$ where $\lambda \in [0, 1]$ such that the final payoff of the basket option is first dominated by two basket call options on the two disjoint subsets of the original underlying assets as is expressed in (B). Then following the same idea as in the previous section, one could find portfolios of plain vanilla call options to further dominate the two basket options. Clearly, if $N_1 = N$ and $\lambda = 1$ (or $N_1 = 0$ and $\lambda = 0$), the obtained hedging portfolio consists of all the underlying assets. That is, hedging with all the assets discussed in Section 4.2 is one special case.

With the assumption of no arbitrage, we can get the same relationship for the price at the initial date, time 0, of the basket option and the portfolio of plain vanilla call options, after taking expectations and discounting their final payoffs:

$$BC_0(K) \leq \underbrace{\sum_{j=1}^{N_1}\omega_j e^{-rT} E\left[\left(S_j(T) - \frac{\alpha_j}{\omega_j}\lambda K\right)^+\right]}_{I'} + \underbrace{\sum_{j=N_1+1}^{N}\omega_j e^{-rT} E\left[\left(S_j(T) - \frac{\beta_j}{\omega_j}(1-\lambda)K\right)^+\right]}_{II'}. \qquad (8)$$

Since the new hedging portfolio is only related to the dominant assets, our concern here is simply on part I'. Obviously for each given value of $\lambda \in [0, 1]$, one can follow the calculating process given in Section 4.1 to find the optimal α_j^* to super-replicate the basket option on the dominant assets with strike λK. Thus, in the second step, we have to search for the optimal λ^* to cover as well as possible the risks that basket options are exposed to, and thus the corresponding optimal strike prices $\frac{\alpha_j^*}{\omega_j}\lambda^* K$ of the plain vanilla calls in the hedging portfolio.

As mentioned in the Introduction, hedging basket options by using subset underlying assets is indeed a hedge in the incomplete market where the incompleteness is resulted from impracticability and impossibility of trading in all the underlying assets. Consequently, a perfect replication would be impossible or quite difficult. In this context, hedges are derived through optimization to satisfy certain optimality criteria.

4.2.2 Static Hedging Strategies with Subset Underlying Assets

Basically, the optimality criteria depend on the risk attitude of investors and are defined by particular risk measures. For instance, the criteria considered in the paper are to achieve super-replication, minimum variance of the hedging error or minimum expected shortfall given a certain initial hedging cost.

Criterion 1: To Super-Replicate the Basket Option The first constraint imposed on part I' is to maintain the price at the maturity date of the hedging portfolio always higher than that of the basket option. Hence, this hedging portfolio is to achieve super-replication which eliminates all the risks of holding a basket option. This is obtained through an optimization with the constraint of no possible sub-replication. More explicitly,

$$\min_{\lambda,\, \alpha_j} \quad \underbrace{\sum_{j=1}^{N_1} \omega_j e^{-rT} E\left[\left(S_j(T) - \frac{\alpha_j}{\omega_j}\lambda K\right)^+\right]}_{I'} \tag{9}$$

$$s.t. \quad \mathbb{P}\left[\sum_{j=1}^{N_1} \omega_j e^{-rT}\left(S_j(T) - \frac{\alpha_j}{\omega_j}\lambda K\right)^+ > \left(\sum_{j=1}^{N} \omega_j S_j(T) - K\right)^+\right] \geq 1 \tag{10}$$

$$\sum_{j=1}^{N_1} \alpha_j = 1.$$

In this way, the obtained hedging portfolio by using subset assets is composed of the vanilla options on the subset hedging assets with the optimal strike prices such that the basket options are super-replicated. However from a practical point of view, this hedging portfolio may be not that effective and requires a high hedging cost. This is partly due to the property of super-hedging portfolio whose hedging costs have to be high for staying always on the safe side. In addition, since the hedging portfolio is composed of only those significant assets, more capital has to be input for the risks resulted from neglecting those insignificant assets. As a result, partial hedging strategies may be taken to gain the tradeoff of reduced hedging costs and overall super-replication.

Criterion 2: To Minimize the Variance of the Hedging Error Given $HC = BC_0(K)$ When investing less capital, the hedge is to minimize the remaining risks. Here in this case, the shortfall risk is measured by the variance of the hedging error. Namely, a hedging portfolio is obtained to minimize the variance of the hedging error when the hedging cost is constrained to be exactly the basket option price. Formally, it is expressed as

$$\min_{k_j} \quad E\left[\left(\Big(\sum_{j=1}^{N}\omega_j S_j(T) - K\Big)^+ - \sum_{j=1}^{N_1}\omega_j\left(S_j(T)-k_j\right)^+\right)^2\right] \tag{11}$$

$$s.t. \quad e^{-rT}E\Big[\Big(\sum_{j=1}^{N}\omega_j S_j(T) - K\Big)^+\Big] = \sum_{j=1}^{N_1}\omega_j e^{-rT}E\left[(S_j(T)-k_j)^+\right] \tag{12}$$

$$k_j \geq 0 \quad \forall \quad j = 1, \cdots, N_1$$

One may observe, the control variables here are not λ and α's. Instead, this optimization problem is reformulated by directly searching the optimal strikes k's of the hedging portfolio. This nevertheless gives the same result but considerably simplifies the computation.

Criterion 3: To Minimize the Expected Shortfall Given a Certain Hedging Cost One main drawback of this quadratic criterion is that it punishes both positive and negative difference between the hedging portfolio and the basket option. Actually for the purpose of hedging, only negative difference is not favored. To avoid such a problem, some other effective risk measures could be considered. The expected shortfall (ES) is in the context of hedging basket option defined as $E[(BC_T - HP_T)^+]$. Obviously, it accounts only the positive hedging error. Meanwhile as a risk measure, it takes into account not only the probability of exposed risks but also its size. Hence, it is often used recently in the literature as a risk indicator. In this case, the optimization problem becomes then as follows:

$$\min_{k_j} \quad E\left[\left(\Big(\sum_{j=1}^{N}\omega_j S_j(T) - K\Big)^+ - \sum_{j=1}^{N_1}\omega_j\left(S_j(T)-k_j\right)^+\right)^+\right] \tag{13}$$

$$s.t. \quad \sum_{j=1}^{N_1}\omega_j e^{-rT}E\left[(S_j(T)-k_j)^+\right] \leq V_0 \tag{14}$$

$$k_j \geq 0 \quad \forall \quad j = 1, \cdots, N_1,$$

where V_0 is the maximal capital that investors would like to input to hedge the basket option.

In summary, the newly-designed hedging portfolio is composed of the plain vanilla call options on only dominant underlying assets in the basket with optimal strikes. This hedging portfolio is achieved by first identifying the subset of hedging assets by means of PCA, and then figuring out the optimal strikes for the call options on these assets based on different optimality criteria, i.e. super-replication, minimum variance or minimum ES given a certain investment into the hedge. The criteria chosen depend on the risk attitude of investors when hedging basket options. The more risk averse he is, the tighter the criterion on the hedging error is, and the more probable the hedging portfolio with subset assets super-hedges basket options. In this context, all these static hedging strategy is to find the compromise between reduced hedging costs and the overall super-replication. It is worth mentioning that all the optimization problems above are solved numerically by running Monte Carlo simulations due to the lack of a distribution of the underlying basket.

5 Numerical Results

In this section we will give some numerical results of this new two-step static hedging strategy. Here we use the example that is first presented in Milevsky and Posner (1998) [19]. Basically, it is an index-linked guaranteed investment certificate offered by Canada Trust Co., fusing a zero coupon bond with a basket option that is stuck at the spot rate of the underlying indices. Here we are interested in hedging the embedded basket option of a weighted average of the renormalized G-7 indices as

$$BC(T) = \left(\sum_{i=1}^{7} \omega_i \frac{S_i(T)}{S_i(t)} - 1 \right)^{+} .$$

That is, effectively, a call option on the rate of return of a basket of indices. The necessary pricing parameters are given in Table 1 and 2. In addition to the data given above for the basket option, a flat and constant interest rate of 6.3% is assumed[6].

country	index	weight (in %)	volatility (in %)	dividend yield (in %)
Canada	TSE 100	10	11.55	1.69
France	CAC 40	15	20.68	2.39
Germany	DAX	15	14.53	1.36
U.K.	FTSE 100	10	14.62	3.62
Italy	MIB 30	5	17.99	1.92
Japan	Nikkei 225	20	15.59	0.81
U.S.	S&P 500	25	15.68	1.66

Table 1: G-7 Index-linked Guaranteed Investment Certificate

	Canada	France	Germany	U.K.	Italy	Japan	U.S.
Canada	1.00	0.35	0.10	0.27	0.04	0.17	0.71
France	0.35	1.00	0.39	0.27	0.50	-0.08	0.15
Germany	0.10	0.39	1.00	0.53	0.70	-0.23	0.09
U.K.	0.27	0.27	0.53	1.00	0.46	-0.22	0.32
Italy	0.04	0.50	0.70	0.46	1.00	-0.29	0.13
Japan	0.17	-0.08	-0.23	-0.22	-0.29	1.00	-0.03
U.S.	0.71	0.15	0.09	0.32	0.13	-0.03	1.00

Table 2: Correlation Structure of G-7 Index-linked Guaranteed Investment Certificate

5.1 Asset Selection Through PCA

Given the data above, the covariance structure of G-7 Index-linked guaranteed investment certificate is easily calculated as the product of weights, variance and correlation

[6] One important issue has to be mentioned for this illustrative example. Since the underlying assets are stock indices of different countries, exchange rate risks between the different currencies will be involved in the basket options pricing and hedging. Here, in order to focus fully on the hedging issue, we neglect this risk by simply assuming that all the indices are traded in the market denominated in the same currency.

matrix. An implementation of the decomposition on this modified covariance gives then the eigenvalue vector in the order of significance

$$\lambda = (0.0017453, 0.0011528, 0.00089155, 0.00038515, 0.00011805, 0.000054436, 0.000026893)^T,$$

and the eigenvectors γ_j by the columns of the matrix

$$\Gamma = \begin{pmatrix} 0.20343 & -0.081831 & 0.084609 & -0.043365 & 0.068215 & 0.86423 & 0.43742 \\ 0.34734 & 0.52281 & 0.64216 & -0.42132 & 0.016241 & -0.062038 & -0.10967 \\ 0.17536 & 0.36359 & 0.1157 & 0.78463 & -0.38794 & 0.14753 & -0.18911 \\ 0.16872 & 0.12967 & -0.0061681 & 0.34844 & 0.90951 & -0.051608 & -0.058339 \\ 0.080311 & 0.15239 & 0.046137 & 0.16156 & -0.067293 & -0.43374 & 0.8657 \\ -0.1257 & -0.62724 & 0.72511 & 0.23806 & 0.02119 & -0.088621 & -0.0037514 \\ 0.86977 & -0.39282 & -0.19783 & -0.027897 & -0.11138 & -0.16985 & -0.089545 \end{pmatrix}.$$

Now with the knowledge of eigenvalues and eigenvectors, one can determine the most significant factors according to the (cumulative) proportions of explained variance. As the result in Table 3 shows, the first PC already explains around 40% of the total variation. An additional 46% is captured by the second and the third PCs. The fourth PC explains a considerably smaller amount of total volatility. Thus, the three dominant PCs together account for more than 87% of the total variation associated with all 7 assets. This suggests that we can capture most of the variability in the data by choosing the first three principal components and neglecting the other four.

eigenvalue	proportion of variance	cumulated proportion
0.0017453	0.399	0.399
0.0011528	0.26355	0.66255
0.00089155	0.20382	0.86637
0.00038515	0.08805	0.95442
0.00011805	0.026987	0.98141
0.000054436	0.012445	0.99385
0.000026893	0.0061481	1

Table 3: Proportion of Variance Explained by PCs

Then the final step is to find the optimal subset of the underlying assets by checking the cumulative r^2 of each asset with the important three components given in Table 4. If two assets are planned to be used in the hedging portfolio, we have to find out the five least important assets from the basket. To achieve this result, the individual r^2 with the first three PCs and the cumulative r^2 are reported in Table 4. Obviously, assets S_1 (Canada TSE 100), S_2 (France CAC 40), S_3 (Germany DAX), S_4 (U.K. FTSE 100) and S_5 (Italy MIB 30) appear to be the least important ones. As a result, the subset of optimal hedging assets is composed of S_6 (Japan Nikkei 225) and S_7 (U.S. S&P 500). If the restriction of the number of assets is relaxed, one can order the assets in the list of significance: S_7, S_6, S_2, S_1, S_5, S_3 and S_4. Besides, an obvious division can be found between S_1 and S_2 as indicated by the large discrepancy of the cumulative r^2 (the difference between 92.84% and 64.71%). Therefore, we can finally determine the subset of assets for the purpose of hedging consisting of three assets of S_2 (France CAC 40), S_6

(Japan Nikkei 225) and S_7 (U.S. S&P 500).

	r_{i1}	r_{i2}	r_{i3}	$r_{i1}^2 + r_{i2}^2 + r_{i3}^2$
S_1	0.73580	-0.24055	0.21873	0.64711
S_2	0.46779	0.57224	0.61812	0.92836
S_3	0.33612	0.56641	0.15851	0.45893
S_4	0.48211	0.30115	-0.012597	0.32328
S_5	0.37300	0.57523	0.15315	0.49347
S_6	-0.16842	-0.68302	0.69438	0.97705
S_7	0.92694	-0.34024	-0.15069	0.99769

Table 4: Correlation Between the Original Variables and the PCs

5.2 Static Hedging with the Selected Three Dominant Assets

Now with the selected assets, the static hedging strategy could be achieved by figuring out the optimal strikes for the call options on these assets based on the calculating procedure given in the former section. In the following, only the numerical results of hedging portfolios with three assets are shown.[7] Generally, hedging with three assets works better than that with two assets due to the importance of S_2 in the basket as analyzed in Section 5.1. Moreover, as the weights in the basket are not changed after assets selection, the hedging subset surely better duplicates the original basket when more assets are included in the hedging portfolio. Nevertheless, the proper number of assets should be chosen in practice by comparing the additional hedging cost and the reduced hedging error.

To give a hint of the performance of this new static hedging method, the hedging cost is compared to the basket options price. All the basket option prices and the corresponding hedging portfolios are obtained numerically by Monte Carlo simulations with the number of simulated paths equal to $500,000$. Such a simulation procedure guarantees that the basket option price is relatively accurate to the fourth digits as shown in Table 5. In addition to the hedging cost, the expected value of the hedging errors at the maturity date is reported for each hedging portfolio to account for the hedging effect. Based on the definition in Section 2, negative hedging errors are favorable, suggesting that the basket option is well hedged with no risk exposure any more. Meanwhile, a special attention is given to the ES which plays a major role as a risk indicator to measure the hedging result. Especially, we vary the strike of this basket option with different values as $K \in \{0.90, 0.95, 1.00, 1.05, 1.10\}$ and the maturity date as $T \in \{1, 3, 5, 10\}$ years to gain an overall view of the hedging performance crossing maturities and strikes.

Table 6 presents the results of the static super-hedge portfolio with only three assets based on the first criterion. First, super-replication is not available for those options with long maturity of 5 and 10 years due to large volatility involved with long time. Otherwise, this hedging strategy well dominates basket options, as shown by negative

[7]The results for hedging with 2 assets are given if required.

	$T=1$	$T=3$	$T=5$	$T=10$
$K=1.10$	0.00014	0.00026	0.00038	0.00071
$K=1.05$	0.00013	0.00025	0.00037	0.00070
$K=1.00$	0.00011	0.00024	0.00036	0.00070
$K=0.95$	0.00009	0.00022	0.00034	0.00069
$K=0.90$	0.00006	0.00020	0.00033	0.00068

Table 5: Standard Error of MC Simulation for Basket Options with $500,000$ Simulations

Table 6: Super-Hedging Portfolio with Three Dominant Assets

K	T	BC	λ^*K	HC	$E[HE]$	k_2	k_5	k_7
0.90	1	0.1398	0.3033	0.3062	-0.1773	0.4217	0.5371	0.5305
	3	0.2080	0.0288	0.5490	-0.4119	0.0206	0.0585	0.0560
0.95	1	0.0978	0.3477	0.2644	-0.1775	0.4994	0.6102	0.6030
	3	0.1714	0.0845	0.5030	-0.4006	0.0810	0.1642	0.1582
1.00	1	0.0625	0.4060	0.2099	-0.1569	0.6048	0.7049	0.6972
	3	0.1374	0.0570	0.5254	-0.4687	0.0492	0.1127	0.1084
1.05	1	0.0358	0.4316	0.1861	-0.1601	0.6520	0.7460	0.7382
	3	0.1077	0.1848	0.4194	-0.3765	0.2166	0.3447	0.3335
1.10	1	0.0185	0.4961	0.1282	-0.1168	0.7741	0.8490	0.8407
	3	0.0826	0.1683	0.4336	-0.4240	0.1926	0.3155	0.3052

expected hedging errors and zero shortfall probability as required in the calculation procedure. However, super-replication requires rather low λ and hence pretty high hedging cost which amounts to even over 10 times the basket option price for the case $T = 1$ and $K = 1.10$. Especially, Figure 1 is designed to demonstrate how λ influences the hedging cost and the hedging error. Clearly, λ has two opposing effects on the hedging performance: an reduction in λ decreases the expected shortfall and meanwhile increases the hedging cost. Thus, higher hedging cost is unavoidable to achieve super-replication. Besides, it demonstrates that all hedging strategies proposed in this paper are exactly to gain a tradeoff between successful hedge and reduced hedging cost by varying strikes.

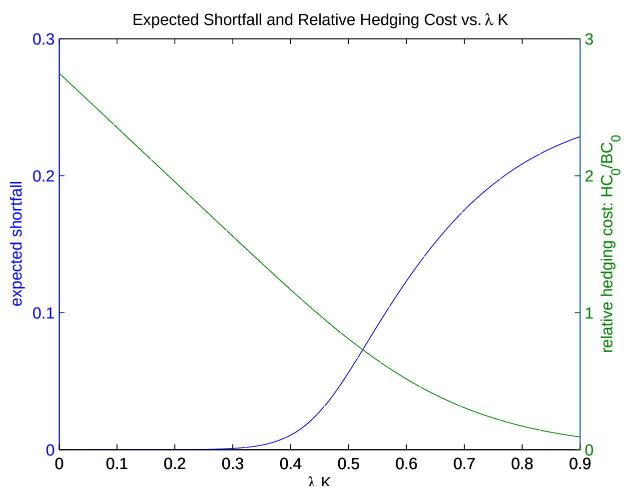

Figure 1: Expected Shortfall and Relative Hedging Cost vs. λK for the Basket Option with $T = 3$ and $K = 0.9$

When relaxing the strong requirement of super-replication, the hedging cost can be surely decreased, for instance, the hedging portfolio obtained by taking the second criterion. As set in the model, the variance of the hedging error is minimized while maintaining the hedging cost as the basket option price. It surely leads to some sub-replication, although the hedging error is in average zero. The relative ES shown in Table 7 is around 10% for in- and at-the-money options and comes to a relatively high level for options with out-of-the-moneyness and short maturity. Moreover, the ES differs insignificantly across the maturity for those in-the-money basket options. For all cases, the obtained optimal strikes of the hedging portfolio increases with maturity and decreases with strike of the basket option.

For minimum-expected-shortfall hedging portfolio, two constraints are provided on the hedging cost and the numerical results are given in Table 8, respectively. The first one is HP_7, the hedging cost of the static super-hedging portfolio on all 7 underlying assets. As shown in Figure 1, the ES decreases with the hedging cost. Hence, the minimal ES is achieved when the hedging cost of the hedging portfolio with subset assets comes to HP_7. In this case, the hedging error turns out to be negative in average and the ES decreases greatly to about 5% crossing all maturities and strikes of the underlying basket option. This result indicates that hedging with three assets gives a relatively satisfactory performance: only a reasonable low hedging error is aroused when investing the same

Table 7: Minimum-Variance Hedging Portfolio with Three Dominant Assets

K	T	BC	HC	$ES\%$	k_2	k_5	k_7
0.90	1	0.1398	0.1398	9.15	0.7500	0.8244	0.8219
	3	0.2080	0.2080	10.10	0.6502	0.7805	0.7667
	5	0.2628	0.2628	9.92	0.5468	0.7308	0.7044
	10	0.3631	0.3631	9.51	0.3040	0.5553	0.5116
0.95	1	0.0978	0.0978	12.53	0.8485	0.9044	0.9019
	3	0.1714	0.1714	11.94	0.7386	0.8609	0.8511
	5	0.2302	0.2302	11.16	0.6328	0.8134	0.7850
	10	0.3387	0.3387	10.05	0.3723	0.6441	0.5927
1.00	1	0.0625	0.0625	17.81	0.9508	0.9827	0.9849
	3	0.1374	0.1374	14.31	0.8376	0.9422	0.9312
	5	0.1989	0.1989	12.58	0.7184	0.8907	0.8736
	10	0.3135	0.3135	10.74	0.4436	0.7284	0.6805
1.05	1	0.0358	0.0358	24.84	1.0558	1.0617	1.0726
	3	0.1077	0.1077	17.10	0.9378	1.0212	1.0155
	5	0.1698	0.1698	14.27	0.8138	0.9713	0.9540
	10	0.2894	0.2894	11.49	0.5144	0.8125	0.7642
1.10	1	0.0185	0.0185	33.03	1.1678	1.1431	1.1581
	3	0.0826	0.0826	20.39	1.0398	1.1034	1.1014
	5	0.1435	0.1435	16.19	0.9114	1.0520	1.0411
	10	0.2667	0.2667	12.14	0.5938	0.8986	0.8470

Note: $ES\%$ denotes the relative ES, namely expected shortfall divided by the expected basket option payoff at the maturity date T measured in percentage.

capital as the hedging portfolio with all 7 underlying assets.

To achieve a smaller ES, we raise the initial hedging cost constraint to the Value at Risk at the level 10% of the basket option payoff. Due to the lack of the distribution of the underlying basket, this has to be obtained by running the simulation. Under this construction, the hedging cost of the hedging portfolio becomes surely higher (equal to $VaR_{0.10}$). It gives then a quite promising result that the ES is greatly reduced and turns out to be almost zero, except those basket options with long maturity of 10 years. As observed also in the results above, lower hedging costs are required for in- and at-the-money basket options to achieve almost the same relative ES compared with those of out-of-the-moneyness. Consequently, if aiming at gaining the tradeoff of the reduced hedging cost and successful replication, the hedging strategies in this paper performs better for in- and at-the-money basket options. To clearly show the regions of sub- and super-replication, the payoffs of the basket option ($T = 3$, $K = 0.9$) and its minimum-ES hedging portfolio given $HC_0 = VaR_{0.10}$ are simulated and plotted in Figure 2. It can be observed that the basket option is completely hedged if the value of the basket comes out to be under or around the strike. The possibility of sub-replication rises with the value of the basket over 1.06. Nevertheless, the hedging error is rather small compared to the basket option.

Table 8: Minimum-Expected-Shortfall Hedging Portfolios with Three Dominant Assets

K	T	BC	$V_0 = HP_7$						$V_0 = VaR_{0.10}$					
			HC	E[HE]	$ES\%$	k_2	k_5	k_7	HC	E[HE]	$ES\%$	k_2	k_5	k_7
0.90	1	0.1398	0.1505	-0.0114	5.87	0.7244	0.8058	0.8035	0.2618	-0.1300	0.01	0.4978	0.6239	0.6044
	3	0.2080	0.2272	-0.0231	6.36	0.6030	0.7390	0.7284	0.4144	-0.2493	0.02	0.2598	0.3498	0.3298
	5	0.2628	0.2843	-0.0295	6.65	0.4912	0.6775	0.6559	0.5225	-0.3559	0.03	0.0536	0.0714	0.8910
	10	0.3631	0.3810	-0.0337	7.55	0.2612	0.4913	0.4516	0.5141	-0.2836	1.20	0	0	0
0.95	1	0.0978	0.1150	-0.0183	5.60	0.8028	0.8718	0.8686	0.2131	-0.1228	0.01	0.5969	0.6994	0.6922
	3	0.1714	0.1966	-0.0304	6.23	0.6719	0.8052	0.7955	0.3731	-0.2437	0.02	0.3050	0.4443	0.4272
	5	0.2302	0.2568	-0.0364	6.71	0.5615	0.7446	0.7219	0.4872	-0.3521	0.03	0.1120	0.1494	0.1870
	10	0.3387	0.3599	-0.0398	7.62	0.3146	0.5672	0.5246	0.5147	-0.3305	0.88	0	0	0
1.00	1	0.0625	0.0848	-0.0237	5.29	0.8812	0.9325	0.9307	0.1676	-0.1119	0.01	0.6905	0.7780	0.7700
	3	0.1374	0.1687	-0.0378	5.97	0.7434	0.8681	0.8559	0.3308	-0.2336	0.03	0.3814	0.5295	0.5161
	5	0.1989	0.2311	-0.0441	6.52	0.6241	0.8069	0.7901	0.4509	-0.3453	0.03	0.1149	0.3154	0.2514
	10	0.3135	0.3395	-0.0489	7.60	0.3713	0.6344	0.5929	0.5145	-0.3774	0.64	0	0	0
1.05	1	0.0358	0.0604	-0.0262	4.56	0.9532	0.9901	0.9916	0.1197	-0.0893	0.02	0.7943	0.8641	0.8574
	3	0.1077	0.1436	-0.0433	5.70	0.8162	0.9253	0.9174	0.2899	-0.2201	0.03	0.4640	0.6148	0.5960
	5	0.1698	0.2073	-0.0514	6.39	0.6896	0.8680	0.8511	0.4157	-0.3370	0.05	0.2116	0.3735	0.3374
	10	0.2894	0.3201	-0.0577	7.56	0.4176	0.6974	0.6638	0.5140	-0.4217	0.50	0	0	0
1.10	1	0.0185	0.0416	-0.0246	3.75	1.0285	1.0475	1.0458	0.0729	-0.0580	0.12	0.9156	0.9618	0.9556
	3	0.0826	0.1214	-0.0469	5.38	0.8885	0.9846	0.9774	0.2498	-0.2019	0.04	0.5523	0.6976	0.6780
	5	0.1435	0.1853	-0.0572	6.20	0.7597	0.9298	0.9130	0.3765	-0.3192	0.05	0.2959	0.4588	0.4383
	10	0.2667	0.3015	-0.0653	7.42	0.4806	0.7681	0.7246	0.5145	-0.4653	0.37	0	0	0

Finally to elucidate the importance of hedging assets selection, we present in Table 9 the results of the hedging portfolio on arbitrary 3 underlying assets, namely, S_1, S_2 and S_3. In comparison with the 3 dominant assets, the hedging portfolio with the arbitrary selection has a considerably worse performance although it has the same number of hedging assets. Indeed, it performs even worse than the hedging portfolio with two assets.

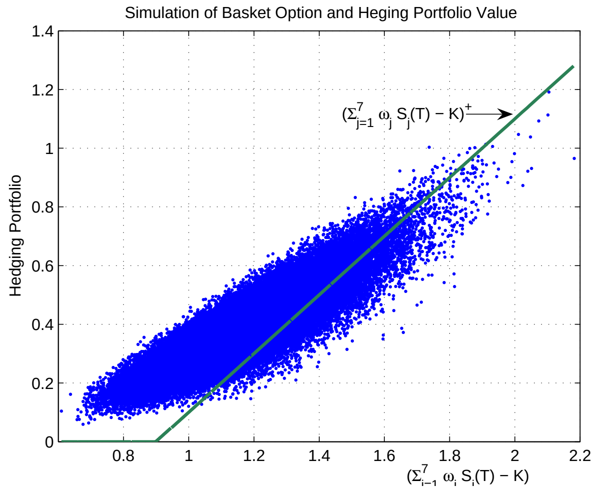

Figure 2: Simulations of the Basket Option (T=3, K=0.9) and the Minimum-Expected-Shortfall Hedge Portfolio with Constraint $V_0 = VaR_{0.10}$

These results suggest that the significance of the choice of the hedging assets in order to gain a good replication.

5.3 Remarks

Sometimes, the hedging performance is not that satisfactory especially for out-of-the-money options. This inefficiency is mainly due to the following two factors.

- First, the hedging sub-basket is composed of simply the selected dominant assets without reallocating weights. Therefore, the value of the subset is only part of the original basket. The only tool in the model to match the payoff of the basket option is to vary the strike prices of the hedging instruments. However, their power to match the distribution is fairly limited since they do not change the shape of the distribution of the hedging basket, but only shift the distribution to dominate the original basket. This can be easily observed in Figure 3. After neglecting those insignificant underlying assets, the sub-basket experiences less extreme cases. However, since it is part of the original basket, it is located on the left of the original basket. Therefore, the function of the strikes is to relocate the distribution of the hedging portfolio to the proper position near the basket option. As shown in the figure, the lower the hedging error is, the further the distribution shifted to the right. This weakness becomes more clear when the hedging instruments are the dominant hedging assets themselves for basket options of long maturity.

- In addition, all the hedging portfolios designed in this paper are static. Hence, it may require more capital to well hedge the basket option. However, the model is restricted to be static under the construction of hedging with plain vanilla options on the significant underlying assets on optimal strikes. As the control variables in this model are the strikes of these call options, frequent trading on options with different strikes would cause great loss and additional transaction costs.

As a result, other control variables have to be considered to improve the hedging effect. One possible instrument is to reallocate the weights of the hedging basket such that the

Table 9: Hedging Strategies with Three Arbitrary Assets

K	T	BC	Mean-Variance HP					Minimum-ES HP with $V_0 = HP_7$					
			HC	$ES\%$	k_2	k_5	k_7	HC	E[HE]	$ES\%$	k_2	k_5	k_7
0.90	1	0.1395	0.1395	15.42	0.7485	0.5970	0.7026	0.1505	-0.0117	12.0522	0.7367	0.5551	0.6739
	3	0.2083	0.2083	17.39	0.6262	0.3984	0.5578	0.2272	-0.0229	13.7017	0.5779	0.3359	0.5006
	5	0.2633	0.2633	17.51	0.4878	0.2172	0.3863	0.2843	-0.0288	14.3080	0.4317	0.1476	0.3012
	10	0.3638	0.3638*	21.01	0	0	0	0.3337	0.0566	21.0108	0	0	0
0.95	1	0.0976	0.0976	10.08	0.8361	0.7357	0.8090	0.1150	-0.0185	13.2384	0.8152	0.6640	0.7668
	3	0.1713	0.1713	19.96	0.7199	0.5532	0.6690	0.1966	-0.0305	14.3538	0.6738	0.4235	0.5956
	5	0.2297	0.2297	19.34	0.5931	0.3297	0.5101	0.2568	-0.0372	14.8146	0.5196	0.2380	0.4024
	10	0.3386	0.3386*	19.03	0	0	0	0.3334	0.0098	19.0328	0	0	0
1.00	1	0.0623	0.0623	25.79	0.9085	0.8711	0.9065	0.0848	-0.0239	13.8963	0.8786	0.7659	0.8473
	3	0.1373	0.1373	22.88	0.8016	0.6501	0.7755	0.1687	-0.0380	14.8961	0.7465	0.5167	0.6799
	5	0.1988	0.1988	21.46	0.6894	0.4395	0.6241	0.2311	-0.0442	15.5823	0.6093	0.3105	0.5084
	10	0.3133	0.3133	19.53	0.1205	0.0026	0.1693	0.3334	-0.0378	17.0348	0	0	0
1.05	1	0.0356	0.0356	33.16	0.9763	1.0030	0.9979	0.0604	-0.0264	14.0296	0.9369	0.8565	0.9163
	3	0.1078	0.1078	26.10	0.8730	0.7816	0.8670	0.1436	-0.0432	15.3744	0.8143	0.6058	0.7581
	5	0.1701	0.1701	23.56	0.7664	0.5615	0.7257	0.2073	-0.0509	16.1071	0.6843	0.3998	0.5907
	10	0.2889	0.2889	20.80	0.3386	0.0936	0.2321	0.3201	-0.0586	16.8005	0.0796	0.0540	0.0539
1.10	1	0.0183	0.0183	41.21	1.0439	1.1332	1.0845	0.0416	-0.0248	13.8981	0.9865	0.9446	0.9782
	3	0.0825	0.0825	29.56	0.9422	0.9052	0.9582	0.1214	-0.0469	15.7277	0.8739	0.6874	0.8309
	5	0.1431	0.1431	25.90	0.8411	0.6821	0.8238	0.1853	-0.0578	16.4462	0.7502	0.4735	0.6760
	10	0.2667	0.2667	22.03	0.4634	0.1696	0.3637	0.3015	-0.0653	17.3412	0.2037	0.0728	0.1979

Note: $*$ For these two cases, it is impossible to match the expectation values of the basket option and the hedging portfolio at the maturity date even when the hedging portfolios are composed of the hedging assets themselves. Hence, the variance is minimized without binding the constraint.

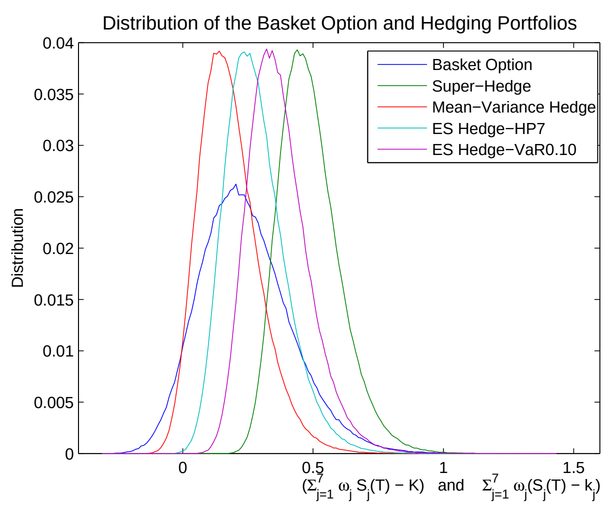

Figure 3: Distribution of the Underlying Basket (T=3, K=0.9) and the Hedging Portfolios

new hedging sub-basket can better match the distribution of the original basket. One this basis, dynamic hedging would also be possible by duplicating the basket option with the hedging assets. This would be the extension for the future works.

6 Conclusion

In summary, Principal Component Analysis, a popular multivariate statistical method for dimension reduction, is applied in basket options hedging, for selecting only a subset of assets. The selection procedure is completed mainly by decomposing the covariance structure of the underlying basket into eigenvalues and eigenvectors. Hedging basket options with only the selected assets can not only reduce transaction costs if combined with other hedging strategies, but also become practical and essential when some of the underlying assets are illiquid or even not available to be traded. Following this idea, a new two-step static hedging strategy is developed in this paper. It consists of the plain vanilla options on $N_1 < N$ dominant assets with optimal strike prices. The strikes are optimally chosen by numerically solving an optimization problem where the optimality criterion depends on the risk attitude of investors while hedging basket options. As given in the paper, the first objective is to eliminate all the risks that the basket option is exposed to. Alternatively, optimal strikes are obtained by minimizing a particular risk measure, e.g., the variance of the hedging error or the expected shortfall. As observed from the numerical results, the static hedging method here is indeed to achieve the tradeoff between reduced hedging costs and overall super-replication. Moreover even without considering reduced transaction costs, hedging with only subset assets works quite well particularly for in- and at-the-money options, generating a small hedging error with a relatively low hedging cost. Actually, its performance will become more satisfactory if the number of the assets in the underlying basket is large. Since the hedging performance is sensitive to the subset of the selected assets, it is recommended to examine the hedging cost and the involved transaction costs as well as the additional reduced hedging error of several subsets. To achieve an even better performance, hedging basket options with subset assets could be improved by reallocating weights of the hedging sub-basket to approximately match the distribution of the original basket. This could be the extension for the future research.

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Appendix: Proposition 1 Proof

Proof. First, the Lagrange function of the minimization is formed

$$\begin{aligned}\mathcal{L} &= \sum_{i=1}^{N}\omega_i e^{-rT}E\left[\left(S_i(T)-\frac{b_i}{\omega_i}K\right)^+\right]+\lambda\left(\sum_{i=1}^{N}b_i-1\right)\\ &= \sum_{i=1}^{N}\omega_i e^{-rT}\int_{max(\frac{b_i}{\omega_i}K,0)}^{\infty}\left(x_i-\frac{b_i}{\omega_i}K\right)f_i(x_i)dx_i+\lambda\left(\sum_{i=1}^{N}b_i-1\right)\end{aligned}$$

where $f_i(x_i)$ is the lognormal density function under the risk-neutral martingale measure for the stock S_i. A necessary and sufficient condition for the sequence b_i to minimize the Lagrange function is found through the first order conditions:

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial b_i} &= -\omega_i e^{-rT}\frac{\partial max\left(\frac{b_i}{\omega_i}K,0\right)}{\partial b_i}\left\{max\left(\frac{b_i}{\omega_i}K,0\right)-\frac{b_i}{\omega_i}K\right\}f_i\left[max\left(\frac{b_i}{\omega_i}K,0\right)\right]\\ &\quad -\omega_i e^{-rT}\int_{max(\frac{b_i}{\omega_i}K,0)}^{\infty}\frac{K}{\omega_i}f_i(x_i)dx_i+\lambda\\ &= 0\\ \frac{\partial\mathcal{L}}{\partial\lambda} &= \sum_{i=1}^{N}b_i-1=0.\end{aligned}$$

These conditions can be further simplified to

$$\frac{\partial\mathcal{L}}{\partial b_i} = -Ke^{-rT}\int_{max(\frac{b_i}{\omega_i}K,0)}^{\infty}f_i(x_i)dx_i+\lambda=0 \qquad \forall\ i=1,\cdots,N \tag{15}$$

$$\frac{\partial\mathcal{L}}{\partial\lambda} = \sum_{i=1}^{N}b_i-1=0 \tag{16}$$

since the term of the first condition $\omega_i e^{-rT}\frac{\partial max\left(\frac{b_i}{\omega_i}K,0\right)}{\partial b_i}\left\{max\left(\frac{b_i}{\omega_i}K,0\right)-\frac{b_i}{\omega_i}K\right\}f_i\left[max\left(\frac{b_i}{\omega_i}K,0\right)\right]$ is always equal to zero no matter which value $max\left(\frac{b_i}{\omega_i}K,0\right)$ is going to take.

With these conditions, one can first prove that $b_i\in[0,1]\ \forall\ i=1,\cdots,N$ is always satisfied. Assume any specific $\bar{i}$ we have $b_{\bar{i}}<0$. This implies that

$$\frac{\partial\mathcal{L}}{\partial b_i}|_{b_i=b_{\bar{i}}}=-Ke^{-rT}+\lambda=0.$$

In this case, the first order condition (12) can be reduced to

$$\int_{max(\frac{b_i}{\omega_i}K,0)}^{\infty}f_i(x_i)dx_i=1 \qquad \forall\ i=1,\cdots,N,\ ,$$

which implies the result that $b_i\le 0\quad\forall\ i=1,\cdots,N$. This contradicts however the second first order condition (13). Therefore, b_i's are always positive and lie in the interval

$[0,1]$.

Then, given $b_i \in [0,1]$, $\forall\, i = 1, \cdots, N$, the first order condition (12) can be stated as

$$\Phi(d_2(S_i, b_i)) = \Phi(d_2(S_j, b_j)) \qquad \forall \quad i, j$$

where $d_2(S_i, b_i) = \frac{\ln\left(\frac{S_i(0)}{\frac{b_i}{\omega_i}K}\right) + (r - q_i - \frac{1}{2}\sigma_i^2)T}{\sigma_i\sqrt{T}}$ as defined in the BS formula, and Φ again denotes the standard normal cumulative distribution function.

Furthermore, $\Phi(x)$ is bijective, the first condition (12) can be reduced to

$$d_2(S_i, b_i) = d_2(S_j, b_j) \quad \forall\, i = 1, \cdots, N.$$

Then b_i can be all expressed in b_1 as

$$b_i = \frac{\omega_i S_i}{K}\left(\frac{b_1 K}{\omega_1 S_1}\right)^{\frac{\sigma_i}{\sigma_1}} \exp\left\{T\left[\left(1 - \frac{\sigma_i}{\sigma_1}\right)\left(r + \frac{1}{2}\sigma_1\sigma_i\right) + \left(\frac{\sigma_i}{\sigma_1}q_1 - q_i\right)\right]\right\} \tag{17}$$

In summary, the optimal sequence b_i are all positive and determined by solving the equations system of

$$\begin{aligned} b_i &= \frac{\omega_i S_i}{K}\left(\frac{b_1 K}{\omega_1 S_1}\right)^{\frac{\sigma_i}{\sigma_1}} \exp\left\{T\left[\left(1 - \frac{\sigma_i}{\sigma_1}\right)\left(r + \frac{1}{2}\sigma_1\sigma_i\right) + \left(\frac{\sigma_i}{\sigma_1}q_1 - q_i\right)\right]\right\} \\ \sum_{i=1}^{N} b_i &= 1. \end{aligned}$$

The existing problem is if there is always a solution and if the solution is unique. This is shown in the following way:

First, b_i is a strictly increasing function of b_1 since the first derivative of b_i with respect to b_1

$$\begin{aligned} b_i' &= \frac{\omega_i S_i \sigma_i}{K}\left(\frac{b_1 K}{\omega_1 S_1}\right)^{\frac{\sigma_i}{\sigma_1}-1} \frac{K}{\omega_1 S_1 \sigma_1} \exp\left\{T\left[\left(1 - \frac{\sigma_i}{\sigma_1}\right)\left(r + \frac{1}{2}\sigma_1\sigma_i\right) + \left(\frac{\sigma_i}{\sigma_1}q_1 - q_i\right)\right]\right\} \\ &= \frac{\omega_i S_i \sigma_i}{\omega_1 S_1 \sigma_1}\left(\frac{b_1 K}{\omega_1 S_1}\right)^{\frac{\sigma_i}{\sigma_1}-1} \exp\left\{T\left[\left(1 - \frac{\sigma_i}{\sigma_1}\right)\left(r + \frac{1}{2}\sigma_1\sigma_i\right) + \left(\frac{\sigma_i}{\sigma_1}q_1 - q_i\right)\right]\right\} \end{aligned}$$

is always larger than zero.

Then the sum of b_i's as a function of b_1 given by

$$g(b_1) = \sum_{i=1}^{N} b_i = \sum_{i=1}^{N} \frac{\omega_i S_i}{K}\left(\frac{b_1 K}{\omega_1 S_1}\right)^{\frac{\sigma_i}{\sigma_1}} \exp\left\{T\left[\left(1 - \frac{\sigma_i}{\sigma_1}\right)\left(r + \frac{1}{2}\sigma_1\sigma_i\right) + \left(\frac{\sigma_i}{\sigma_1}q_1 - q_i\right)\right]\right\}$$

is also continuous and increasing in b_1, which could be proven again by checking its first derivative. Moreover,

$$g(b_1 = 0) = 0,$$

and

$$
\begin{aligned}
g(b_1 = 1) &= \sum_{1=1}^{n} \frac{\omega_i S_i}{K} \left(\frac{K}{\omega_1 S_1}\right)^{\frac{\sigma_i}{\sigma_1}} \exp\left\{T\left[\left(1 - \frac{\sigma_i}{\sigma_1}\right)\left(r + \frac{1}{2}\sigma_1\sigma_i\right) + \left(\frac{\sigma_i}{\sigma_1}q_1 - q_i\right)\right]\right\} \\
&= \frac{\omega_1 S_1}{K} \left(\frac{K}{\omega_1 S_1}\right)^{\frac{\sigma_1}{\sigma_1}} \exp\left\{T\left[\left(1 - \frac{\sigma_1}{\sigma_1}\right)\left(r + \frac{1}{2}\sigma_1\sigma_1\right) + \left(\frac{\sigma_1}{\sigma_1}q_1 - q_1\right)\right]\right\} \\
&\quad + \frac{\omega_2 S_2}{K} \left(\frac{K}{\omega_1 S_1}\right)^{\frac{\sigma_2}{\sigma_1}} \exp\left\{T\left[\left(1 - \frac{\sigma_2}{\sigma_1}\right)\left(r + \frac{1}{2}\sigma_1\sigma_2\right) + \left(\frac{\sigma_2}{\sigma_1}q_1 - q_2\right)\right]\right\} + \cdots \\
&= 1 + \frac{\omega_2 S_2}{K} \left(\frac{K}{\omega_1 S_1}\right)^{\frac{\sigma_2}{\sigma_1}} \exp\left\{T\left[\left(1 - \frac{\sigma_2}{\sigma_1}\right)\left(r + \frac{1}{2}\sigma_1\sigma_2\right) + \left(\frac{\sigma_2}{\sigma_1}q_1 - q_2\right)\right]\right\} + \cdots \\
&\geq 1.
\end{aligned}
$$

As a consequence, there is always a unique solution $b_i \in [0, 1]$. $\square$