Pricing and Upper Price Bounds of Relax Certificates

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Abstract

Relax certificates are written on multiple underlying stocks. Their payoff depends on a barrier condition such that it is path–dependent. As long as none of the underlying assets crosses a lower barrier, the investor receives the payoff of a coupon bond. Otherwise, there is a cash settlement at maturity which depends on the lowest stock return. Thus, the products consist of a knock-out coupon bond and a knock-in minimum option. In a Black–Scholes model setup, the price of the knock–out part can be given in closed (or semi–closed) form in the case of one or two underlyings, but not for more than two. With the exception of the trivial case of one underlying, the price of the knock-in minimum option has to be calculated numerically. We thus also derive semi–closed form upper price bounds. These bounds are the lowest upper price bounds which can be calculated without the usage of numerical methods. In addition, the bounds are especially tight for the vast majority of relax certificates which are traded at a discount of the corresponding coupon bond. This is also illustrated with market data.

Keywords: Certificates, Barrier Option, Price Bounds

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1 Introduction

Recently, more and more structured products written on several instead of one underlying are issued. Amongst them are so–called relax certificates which can be interpreted as a generalized version of bonus certificates. While four issuers started to offer these products in 2006, more than 13 issuers are listed today.[1]

Normally, relax certificates[2] are written on three stocks belonging to a similar market segment like blue chips or primary products. They are also traded on indices. The payoff depends on whether and when any of the underlyings touches a lower barrier. As long as the barrier is not reached, the payments of the certificates correspond to those of a coupon bond where the coupon payments usually range from 10% to 17%.[3] However, if the lower barrier is hit, all future payments from the bond component are canceled. Instead, the investor receives a minimum option on the underlyings. Relax certificates thus combine a knock-out component (the bond) and a knock-in component (the minimum option). The time to maturity is usually smaller than that of ordinary bonus certificates. A typical choice are e.g. three years and three month with reference dates every 13 months or a maturity of about one year with a single reference date at maturity.

Relax certificates are advertised as follows: The bonus payments are appealing even in sideways moving and slightly bearish markets. The risk of loosing the bonus payments is low since this event is triggered by a significant loss in one of the underlying stocks. However, relax certificates are less attractive in highly bullish and highly bearish markets. In the first case, the investor would have been better off with a direct investment in the stocks. With relax certificates, she foregoes the participation in increasing stock prices.[4] In extremely bearish markets, the investor is also worse off. Here, she participates in the

[1]Cf. monthly reports of the EUWAX and the monthly statistics of the DDI.

[2]Similar products are also called Top-10-Anleihe, Easy Relax Express, Easy Relax Bonus, Multi-Capped Bonus or Aktienrelax. Furthermore, there are also relax certificates which bear some features of express certificates.

[3]Some examples for contracts which are traded in the market will be given in Section 4.

[4]There are also certificates where the investor can participate in the development of the underlying assets if the terminal value of the worst performing stock is larger than the face value of the coupon bond.

(highest) losses at the stock market. This contradicts the naming 'relax'.

In the following paper, we provide a detailed analysis of relax certificates. In particular, we analyze the pricing, upper price bounds and risk management. Related literature includes Wallmeier and Diethelm (2008) and Lindauer and Seiz (2008). They analyze (multi-) barrier reverse convertibles which are traded in Switzerland and are similar to the German relax certificates. Lindauer and Seiz (2008) rely on Monte Carlo simulations to price these contracts in a standard Black-Scholes framework with correlated assets. Wallmeier and Diethelm (2008) extend a multinomial tree introduced by Chen, Chung, and Yang (2002) to value barrier reverse convertibles on three underlyings. In contrast, our main focus is on closed-form or semi-closed form solutions.

Recall that relax certificates can be interpreted as a knock-out coupon bond and a knock-in minimum option. In the literature, there is an extensive analysis of barrier options. Without claiming completeness, closed-form solutions for standard barrier options are given by Rubinstein and Reiner (1991), Rich (1994) and Haug (1998). More exotic barrier options are, for example, considered in Kunitomo and Ikeda (1992) (two-sided barriers) and Heynen and Kat (1994a,b) (external barriers). For multi-asset barrier options, we refer to Wong and Kwok (2003) and Kwok, Wu, and Yu (1998). Closed-form solutions for pricing options on the minimum or maximum of two risky assets are firstly introduced in Stulz (1982). An extension to more than two risky assets is given by Johnson (1987).

The probability that at least one underlying reaches the barrier is important for pricing and risk management. In the simple case of one underlying asset, the distribution of the first hitting time is well known in a Black-Scholes setup, cf. for example Merton (1973). It can be calculated using the reflection principle as in Karatzas and Shreve (1999) or Harrison (1985). For two underlyings, a semi–closed form solution is given in He, Keirstead, and Rebholz (1998) where the distribution function is approximated by using an infinite Bessel function. Based on a more general work of Rebholz (1994), Zhou (2001) applies these results to credit risk modeling where similar problems occur. This is extended by Overbeck and Schmidt (2005) who use a deterministic time change for each Brownian motion. The first hitting time distribution of more than two underlyings, however, cannot be given in closed–form for a general correlation structure.

Our main findings are as follows. The decomposition into a knock-out coupon bond and a knock-in minimum option is useful to understand the structure of relax certificates. The contracts are designed such that relax certificates can be offered cheaper than the associated coupon bond. Formally, this gives a condition on admissible (or *attractive*) contract parameters in terms of the barrier and bonus payments. Basically, it implies that there is a lower bound on the bonus payments and/or an upper bound on the barrier level. In this case, a trivial upper price bound is indeed given by the corresponding coupon bond. This price bound can be tightened by subtracting the price of a put option on the minimum of the underlyings with a strike price equal to the barrier.

In addition, we show that price bounds can be determined by considering subsets of the underlyings. In the extreme case where the number of underlyings is reduced to one, the upper price bound can be calculated in closed–form in a Black-Scholes setup. This price bound is decreasing in the volatility of the underlying. This implies that the lowest upper price bound is given by using the stock with the highest volatility as underlying. Since the extreme case of one underlying obviously contradicts the basic idea of multiple underlyings, we also study higher dimensions. We show that tight but still tractable price bounds result from considering all subsets consisting of two underlyings.

In order to test the practical relevance of our theoretical results, we analyze relax certificates which are currently traded at the market. For typical contract specifications, the price of relax certificates on two or three underlyings is up to 10% lower than the price of the corresponding coupon bond. The risk that at least one of the underlying stocks hits the lower barrier can thus not be neglected and is highly economically significant. We also compare the market prices to the upper price bounds which are based on two underlyings only. It turns out that the market prices are well above these upper price bounds, which confirms that these contracts are overpriced and which also shows that the upper price bounds are rather tight.[5]

[5]The price bounds are calculated in a Black-Scholes model. For attractive relax certificates, however, the price bounds would be even lower if one takes into account the possibility of (downward) jumps.

The remainder of the paper is organized as follows. In Section 2, the payoff structure of relax certificates is defined and analyzed. In addition, we derive conditions on the contract parameters for which the certificates are attractive. This allows us to derive model independent upper price bounds. In Section 3, we assume a Black–Scholes model and give a representation of (exact) prices as well as (model-dependent) upper price bounds. In particular, we give a tight upper price bound in semi–closed form and discuss the dependence of the prices and price bounds on the characteristics of the underlyings. A comparison to market prices can be found in Section 4. Section 5 concludes.

2 Product Specification and Model Independent Price Bounds

2.1 Product Specification

In general, a relax certificate is written on n underlying stocks, where n is equal to 2 or 3 for currently traded relax certificates. Let $S_t^{(j)}$ be the price of stock j at time t. For ease of exposition, we set the initial value of all stocks equal to one, i.e. $S_0^{(j)} = 1$ $(j = 1, \ldots, n)$.[6]

The payoff of the relax certificate depends on whether at least one of the stocks has hit its lower barrier m $(m < 1)$, i.e. has lost the fraction $1 - m$ of its value. Usually, m is chosen to be quite low, e.g. $m = 0.5$, so that this event constitutes a significant loss in this stock. The first hitting time of stock j $(j = 1, \ldots, n)$ with respect to the barrier level m is denoted by $\tau_{m,j}$. The first hitting time of the portfolio of all underlying stocks is denoted $\tau_m^{(n)}$, i.e.

$$\tau_{m,j} := \inf\left\{t \geq 0, S_t^{(j)} \leq m\right\}, \tag{1}$$

$$\tau_m^{(n)} := \min\{\tau_{m,1}, \ldots \tau_{m,n}\}. \tag{2}$$

If none of the underlyings reaches the level m, $\tau_m^{(n)}$ is set to $\tau_m^{(n)} = \infty$.

[6]This is in line with currently traded relax certificates where the minimum option is written on the return of the underlying stocks in t_N.

The relax certificate can be decomposed into two parts, a knock-out (RO) and a knock-in (RI) component. Its total payoff at maturity t_N is $\mathrm{RC}_{t_N}^{(n)} = \mathrm{RO}_{t_N}^{(n)} + \mathrm{RI}_{t_N}^{(n)}$, where we assume that payments before maturity are accumulated at the continuously compounded risk-free rate r.[7] The set of all payment dates is denoted by $\underline{\underline{T}} = \{t_1, \ldots, t_N\}$, the current point in time is $t_0 = 0 < t_1$. If the barrier is not hit until $t_i \in \underline{\underline{T}}$ $(i = 1, ..., N)$, the investor receives a bonus payment which is given by δ times the nominal value and which can be interpreted as a coupon payment. At maturity t_N, she also receives the nominal value of the certificate which we normalize to one. This part of the payoff can be interpreted as a knock-out component $\mathrm{RO}_{t_N}^{(n)}$

$$\mathrm{RO}_{t_N}^{(n)} = \sum_{i=1}^{N} \delta e^{r(t_N - t_i)} \left(1 - 1_{\{\tau_m^{(n)} \leq t_i\}}\right) + \left(1 - 1_{\{\tau_m^{(n)} \leq t_N\}}\right) \tag{3}$$

where 1 denotes the indicator function. If the barrier is hit before t_N, the investor forgoes all future bonus payments as well as the repayment of the nominal value. Instead, she gets an option on the minimum of the n underlying stocks with maturity t_N. The payoff from this knock-in component $\mathrm{RI}_{t_N}^{(n)}$ at time t_N is given by

$$\mathrm{RI}_{t_N}^{(n)} = \min\{S_{t_N}^{(1)}, ..., S_{t_N}^{(n)}\} 1_{\{\tau_m^{(n)} \leq t_N\}}. \tag{4}$$

We summarize the payoff from the relax certificate in the following definition:

Definition 1 (Relax certificate) *The compounded payoff of a relax certificate with nominal value* 1*, bonus payments* δ*, lower boundary* m*, payment dates* $\underline{\underline{T}} = \{t_1, \ldots, t_N\}$*, and* n *underlying stocks* $S^{(1)}, ..., S^{(n)}$ *is*

$$RC_{t_N}^{(n)} = \sum_{i=1}^{N} \delta e^{r(t_N - t_i)} \left(1 - 1_{\{\tau_m^{(n)} \leq t_i\}}\right) + \left(1 - 1_{\{\tau_m^{(n)} \leq t_N\}}\right) + \min\left\{S_{t_N}^{(1)}, ..., S_{t_N}^{(n)}\right\} 1_{\{\tau_m^{(n)} \leq t_N\}}. \tag{5}$$

Note that we ignore any default risk of the issuer. This risk would reduce the payments and thus also the prices of the certificates as compared to the prices without default risk.

[7] Throughout the paper we assume that r $(r \geq 0)$ is a constant.

2.2 Model Independent Price Bounds

Relax certificates are advertised by rather high bonus payments and a price below the price of the corresponding coupon bond. We call these relax certificates *attractive*:

Definition 2 (Attractive relax certificate) *A relax certificate is called attractive iff*

$$RC_0^{(n)} < \sum_{i=1}^{N} \delta e^{-rt_i} + e^{-rt_N}. \tag{6}$$

The *discount* as compared to the price of a coupon bond is achieved by the knock-out feature of the bond component. However, note that in case of a knock-out, the payoff is not replaced by zero but by the payoff of a minimum option. For the relax certificate to be attractive, the investor has to switch from a "higher" to a "lower" payoff in this case, i.e. the foregone future bond payments must be worth more than the minimum option. A condition to ensure that this is indeed the case is given in the following lemma:

Lemma 1 (Attractive relax certificate: sufficient conditions) *A sufficient condition on the contract parameters* (δ, m) *to ensure* $RC_0^{(n)} < \sum_{i=1}^{N} \delta e^{-rt_i} + e^{-rt_N}$ *is given by*

$$m \quad \leq \quad \min_{\{j=0,\ldots,n\}} \delta \sum_{i:t_i>t_j} e^{-r(t_i-t_j)} + e^{-r(t_N-t_j)}. \tag{7}$$

In particular, a sufficient condition for Equation (7) to hold is given by

$$m \leq \frac{(1+\delta)e^{-rt_N}}{1+e^{-rt_N}}. \tag{8}$$

Proof: If the barrier is not hit, the payoffs of the relax certificate are equal to that of a coupon bond. If the barrier is hit at time τ, the investor foregoes the future payments from this bond and receives a minimum option instead. The value of this minimum option is bounded from above by the lowest stock price at time τ, which is equal to m.[8] Condition (7) ensures that immediately after a coupon payment, the value of the coupon bond is

[8] To be more precise, in the case of gap risk due to jump or liquidity risk the lowest stock price can be lower than m.

larger than the upper price bound on the minimum option. In between the coupon dates, the price of the coupon bond increases and is thus also larger than m. To derive condition (8), it is enough to notice:

$$\min_{\{j=0,\dots,n\}} \delta \sum_{i:t_i>t_j} e^{-r(t_i-t_j)} + e^{-r(t_N-t_j)} \quad \geq \quad \delta e^{-rt_N} + e^{-rt_N} \geq \frac{(1+\delta)e^{-rt_N}}{1+e^{-rt_N}}.$$

If m is smaller than the right-hand side, then Condition (7) holds for sure. □

Obviously, an upper price bound for an attractive relax certificate is given by the price of the corresponding coupon bond. This trivial superhedge can easily be tightened by selling some put options.

Proposition 1 (Semi-Static Superhedge) *Assume that the tupel (δ, m) satisfies Equation (8). Then, the following semi-static strategy is a superhedge for the relax certificate: At $t_0 = 0$, buy the corresponding coupon bond (with coupon payments δ and payment dates $\underline{\underline{T}}$) and sell a minimum put-option with underlyings $S = (S^{(1)}, ..., S^{(n)})$, maturity t_N and strike m. If $\tau_m^{(n)} < t_N$, liquidate the portfolio at $\tau_m^{(n)}$ and use the proceeds to buy the cheapest underlying asset.*

Proof: Consider the case $\tau_m^{(n)} < t_N$ first. At $\tau_m^{(n)}$, the value of the hedge portfolio is

$$CB_{\tau_m^{(n)}} - P^{Min}_{\tau_m^{(n)}} \geq e^{-r(t_N-\tau_m^{(n)})}(1+\delta) - P^{Min}_{\tau_m^{(n)}},$$

where CB denotes the value of the coupon bond and $P^{Min}_{\tau_m^{(n)}}$ the price of the minimum put-option at time $\tau_m^{(n)}$. The payoff of the minimum put-option at t_N is bounded by

$$P^{Min}_{t_N} = \left[m - \min\{S^{(1)}_{t_N}, ..., S^{(n)}_{t_N}\}\right]^+ \leq m$$

so that, at $\tau_m^{(n)} < t_N$, we have $P^{Min}_{\tau_m^{(n)}} \leq e^{-r(t_N-\tau_m^{(n)})}m$. With condition (8), it follows

$$\begin{aligned} CB_{\tau_m^{(n)}} - P^{Min}_{\tau_m^{(n)}} &\geq e^{-r(t_N-\tau_m^{(n)})}(1+\delta) - e^{-r(t_N-\tau_m^{(n)})}m \\ &= e^{-r(t_N-\tau_m^{(n)})}(1+\delta-m) \\ &\geq m. \end{aligned}$$

Therefore, the value of the hedge portfolio is large enough to buy the cheapest asset, which is worth m at $\tau_m^{(n)}$. Obviously, this asset superhedges the minimum option, which also holds true if it pays some dividends. Finally, for $\tau_m^{(n)} \geq t_N$ we have $CB_{t_N} - P^{Min}_{t_N} = CB_{t_N}$. □

Corollary 1 (Upper bound on $\mathbf{RC}_{t_0}^{(n)}$) *For an attractive relax certificate, it holds*

$$RC_{t_0}^{(n)} \leq \sum_{i=1}^{N} \delta e^{-rt_i} + e^{-rt_N} - P_{t_0}^{Min}. \tag{9}$$

Proof: The proof follows immediately from Proposition 1. □

The semi–static superhedge in Proposition 1 can be simplified by considering only a subset of underlyings, as will be shown in Section 3.2. Reducing the number of underlyings to one leads to a semi-static hedge where only one plain-vanilla put option instead of the more exotic minimum option is needed. The optimal choice which gives the lowest initial capital is then the most expensive put. The high price of the put can be due to a low stock price and/or a high volatility.

An issuer who sells the relax certificate as a substitute for selling a coupon bond might follow yet another hedging strategy. As long as the barrier is not hit, he might just refrain from hedging at all. If the barrier is hit, however, he is no longer short a coupon bond but a minimum option. Then, he can hedge by taking a long position in the worst performing stock. This implies paying back the bond before maturity at a rather low level m.

3 Pricing and upper price bounds

For the following analysis, we assume a Black–Scholes–type model setup with no dividends. Each stock price $S_t^{(j)}$ satisfies the stochastic differential equation

$$dS_t^{(j)} = \mu_j S_t^{(j)} dt + \sigma_j S_t^{(j)} dW_t^{(j)}, \tag{10}$$

where $\{W_t^{(j)}\}_{0 \leq t \leq T}$ is a standard Brownian motion under the real world measure P. The Wiener processes are in general correlated, i.e. for $i \neq j$ it holds that $\langle W^{(i)}, W^{(j)} \rangle_t = \rho_{ij} t$. In particular, we assume constant correlations. Note that Equation (10) implies that the dynamics of the stock prices under the risk neutral measure Q are

$$dS_t^{(j)} = r S_t^{(j)} dt + \sigma_j S_t^{(j)} dW_t^{Q,(j)} \tag{11}$$

where $\{W_t^{Q,(j)}\}_{0\leq t\leq T}$ is a standard Brownian motion under the equivalent martingale measure Q.

We could also allow for dividends. Basically, this would reduce both the prices of attractive relax certificates and their price bounds. To get the intuition, note that dividends reduce the prices of the stocks and thus increase the probability that the lower barrier is hit. Since the investor then goes from a "high" to a "low" payoff in case of an attractive relax certificate, and since dividend payments also reduce the value of the minimum option, the price of the relax certificate will decrease.

3.1 Prices of Relax Certificates

Let $\mathrm{RC}_{t_0}^{(n)}$ denote the price at t_0 of a relax certificate which is written on n underlying assets $S^{(1)}, \ldots, S^{(n)}$. Pricing by no arbitrage immediately gives:

Proposition 2 (Price of a relax certificate) *The t_0–price ($t_0 = 0 < t_1$) of a relax certificate with bonus payments δ, payment dates $\underline{\underline{T}} = \{t_1, \ldots, t_N\}$ and n underlying assets is given by $RC_{t_0}^{(n)} = RO_{t_0}^{(n)} + RI_{t_0}^{(n)}$. The prices of the components are*

$$RO_{t_0}^{(n)} = \delta \sum_{i=1}^{N} e^{-rt_i}\, Q\left(\tau_m^{(n)} > t_i\right) + e^{-rt_N}\, Q\left(\tau_m^{(n)} > t_N\right), \tag{12}$$

$$RI_{t_0}^{(n)} = E_Q\left[\int_{t_0}^{t_N} e^{-ru} C_u^{Min,n} dN_u\right] \tag{13}$$

where $N_t := 1_{\{\tau_m^{(n)} \leq t\}}$ and $C_t^{Min,n} := E_Q\left[e^{-r(t_N-t)} \min\left\{S_{t_N}^{(1)}, \ldots, S_{t_N}^{(n)}\right\} \mid \mathcal{F}_t\right]$.

Proof: Pricing by no arbitrage immediately gives

$$\begin{aligned}\mathrm{RC}_{t_0}^{(n)} &= \delta \sum_{i=1}^{N} e^{-rt_i}\, Q\left(\tau_m^{(n)} > t_i\right) + e^{-rt_N}\, Q\left(\tau_m^{(n)} > t_N\right) \\ &\quad + E_Q\left[e^{-rt_N} \min\left\{S_{t_N}^{(1)}, \ldots, S_{t_N}^{(n)}\right\} 1_{\left\{\tau_m^{(n)} < t_N\right\}}\right].\end{aligned}$$

Using iterated expectations yields

$$\begin{aligned}&E_Q\left[e^{-rt_N} \min\left\{S_{t_N}^{(1)}, \ldots, S_{t_N}^{(n)}\right\} 1_{\left\{\tau_m^{(n)} < t_N\right\}}\right] \\ &= E_Q\left[\int_{t_0}^{t_N} e^{-ru} E_Q\left[e^{-r(t_N-u)} \min\left\{S_{t_N}^{(1)}, \ldots, S_{t_N}^{(n)}\right\} |\mathcal{F}_u\right] 1_{\left\{\tau_m^{(n)} \in du\right\}}\right].\end{aligned}$$

With the definition of N_t, the pricing formula follows. The price $C^{\text{Min},n}$ of the minimum option on n assets is given in Appendix C for $n = 2$ and in Appendix D for $n \geq 2$. $\square$

The price of the knock-out bond component in Equation (12) depends on the distribution of the first hitting time $\tau_m^{(n)}$, i.e. the first time when one of the stocks hits the barrier. The price (13) of the knock-in minimum option depends on the joint distribution of the first hitting time and the stock prices at this first hitting time. In the case of one underlying, the first hitting time distribution is well known and can be derived using the reflection principle as in Karatzas and Shreve (1999) or Harrison (1985). The price of the relax certificate can then be calculated in closed–form:

Proposition 3 (Price of a relax certificate on one underlying) *For $n = 1$, the price $RC_{t_0}^{(n=1)}$ according to Equations (12) and (13) is given in closed–form where the survival probability $Q(\tau_m^{(n=1)} \geq t)$ needed in Equation (12) is given by:*

$$Q(\tau_m^{(n=1)} \geq t) = N\left(\frac{-\ln \frac{m}{S_0} + \left(r - \frac{1}{2}\sigma^2\right) t}{\sigma\sqrt{t}}\right) + e^{2\frac{r-\frac{1}{2}\sigma^2}{\sigma^2}\ln \frac{m}{S_0}} N\left(\frac{\ln \frac{m}{S_0} + \left(r - \frac{1}{2}\sigma^2\right) t}{\sigma\sqrt{t}}\right). \quad (14)$$

The minimum option in Equation (13) reduces to the underlying itself, and the price of the knock-in component is

$$RI_{t_0}^{(n=1)} = m \int_{t_0}^{t_N} e^{-r(u)} Q\left(\tau_m^{(n=1)} \in du\right) \quad (15)$$

where $Q\left(\tau_m^{(n=1)} \in du\right)$ is given in Corollary 2 of Appendix A.

Proof: Equation (14) is based on well known results which, for the sake of completeness, are given in Appendix A. Concerning Equation (15), first note that

$$\begin{aligned} C_t^{\text{Min},n=1} &= E_Q\left[e^{-r(t_N-t)} \min\left\{S_{t_N}^{(1)}\right\} \mid \mathcal{F}_t\right] \\ &= E_Q\left[e^{-r(t_N-t)} S_{t_N}^{(1)} \mid \mathcal{F}_t\right] = S_t^{(1)}. \end{aligned}$$

In addition, we know that for $\tau_m^{(1)} = u$ it holds that $S_u = m$. This gives

$$E\left[\int_{t_0}^{t_N} e^{-ru} C_u^{\text{Min},n=1} dN_u\right] = m\, E\left[\int_{t_0}^{t_N} e^{-ru} dN_u\right] = m \int_{t_0}^{t_N} e^{-ru} Q\left(\tau_m^{(n=1)} \in du\right).$$

□

For more than one underlying, closed–form solutions for Equations (12) and (13) do no longer exist in general. For the special cases of uncorrelated stock prices or perfectly positively correlated stock prices, the distribution of the first hitting time follows from the one-dimensional case. For $n = 2$, Zhou (2001) derives a semi–closed form solution for the first hitting time by approximating the distribution function using an infinite Bessel function. The price of the knock-out bond component can then be calculated in semi-closed form. The price of the knock-in minimum option, however, additionally depends on the distribution of the stock prices when the barrier is hit. For $n \geq 2$, an analytical pricing formula thus no longer exists in general.

Thus, even in the case of a simple Black–Scholes–type model setup, the prices of relax certificates have to be determined numerically. Possible methods are binomial or trinomial lattices – see e.g. Hull and White (1993) – or finite difference schemes – see e.g. Dewynne and Wilmott (1994) – which become rather time-consuming for more than one underlying. In this case, a Monte-Carlo simulation is usually preferred. However, the barrier feature causes some problems for the simulation. To get the intuition, consider a Monte Carlo simulation with a given refinement of the timeline and a simple Euler discretization of the stock prices. If one of the stocks breaches the barrier between two discretization dates, this event is not detected in the simulation. To control for this problem, a large number of sampling dates is needed in addition to the usual requirement of a large number of simulation paths. But still, the bias decreases very slowly, as shown by Boyle, Broadie, and Glassermann (1997), Boyle and Lau (1994) or Broadie, Glassermann, and Kou (1997). This problem is well known in the literature. It is analyzed by numerous authors suggesting different correction methods, like a continuity correction as proposed by Broadie, Glassermann, and Kou (1997), or the use of a (multi-dimensional) Brownian bridge as done by Beaglehole, Dybvig, and Zhou (1997) for one underlying and by Shevchenko (2003) for several underlyings.

3.2 Upper Price Bounds

Given that the pricing of relax certificates is subject to numerical problems, the question is whether we can find price bounds that are both easy to calculate and tight. The next proposition shows that the price of an attractive relax certificate is decreasing in the number of underlyings. Reducing the number of underlyings thus gives an upper price bound.

Proposition 4 (Upper price bound: relax certificates on some underlyings only) *Let $S = \left(S^{(1)}, \ldots, S^{(n)}\right)$ denote a set of underlyings. In addition, let $RC_{t_0}(\hat{S})$ denote the price of a relax certificate with bonus payments δ, payment dates $\underline{\underline{T}}$ and underlyings $\hat{S}$ where $\hat{S} \subseteq S$. If condition (7) on the bonus payments δ holds, then*

$$RC_{t_0}(S) \leq RC_{t_0}(S') \text{ for all } S' \subseteq S. \tag{16}$$

In particular, it holds

$$RC_{t_0}(S) \leq \min_{k,l \in \{1,\ldots,n\}} RC_{t_0}\left(S^{(k)}, S^{(l)}\right) \leq \min_{i \in \{1,\ldots,n\}} RC_{t_0}\left(S^{(i)}\right). \tag{17}$$

Proof: Notice that $\tau_m(S) \leq \tau_m(S')$, i.e. the 'big' certificate is knocked out no later than the 'small' one. Depending on whether and when the two certificates are knocked out, there are three cases. First, if both certificates survive until maturity, their payments coincide. Second, if both are knocked out at the same point in time, i.e. $\tau_m(S) = \tau_m(S') \leq t_N$, it holds

$$RC_{\tau_m(S)}(S') = C^{\min}_{\tau_m(S)}(S') \geq C^{\min}_{\tau_m(S)}(S) = RC_{\tau_m(S)}(S).$$

Third, if the 'big' certificate is knocked out while the small one still survives, i.e. if $\tau_m(S) \leq t_N$ and $\tau_m(S') > \tau_m(S)$, it follows

$$RC_{\tau_m(S)}(S') \geq C^{\min}_{\tau_m(S)}(S') \geq C^{\min}_{\tau_m(S)}(S) = RC_{\tau_m(S)}(S).$$

In all three cases, the value of the 'small' certificate is at least as high as the value of the 'big' certificate. This proves the first part of the proposition. The second part then follows as a special case. □

It is worth mentioning that the above result is model-independent. In the model of Black-Scholes this upper bound can be calculated in closed-form for $n = 1$. For $n = 2$, there is a semi–closed form solution for the knock-out component (12), and we now give an upper bound for the price of the knock-in component (13).

Proposition 5 (Upper price bound for knock-in part) *For $n \geq 2$, an upper price bound on the knock-in component is given by*

$$m \int_{t_0}^{t_N} e^{-ru} Q\left(\tau_m^{(n)} \in du\right) \leq m\, Q(\tau_m^{(n)} \leq t_N). \tag{18}$$

Proof: Using the law of iterated expectations gives

$$\begin{aligned} \mathrm{RI}_{t_0}^{(n)} &= E_Q\left[e^{-rt_N} \min\left\{S_{t_N}^{(1)}, ...S_{t_N}^{(n)}\right\} 1_{\{\tau_m^{(n)} \leq t_N\}}\right] \\ &= E_Q\left[E_Q\left[e^{-rt_N} \min\left\{S_{t_N}^{(1)}, ...S_{t_N}^{(n)}\right\} \mid \mathcal{F}_{\tau_m^{(n)}}\right] 1_{\{\tau_m^{(n)} \leq t_N\}}\right] \\ &= E_Q\left[E_Q\left[\min\left\{\hat{S}_{t_N}^{(1)}, ...\hat{S}_{t_N}^{(n)}\right\} \mid \mathcal{F}_{\tau_m^{(n)}}\right] 1_{\{\tau_m^{(n)} \leq t_N\}}\right] \end{aligned}$$

where $\hat{S}_t := e^{-rt} S_t$. $\hat{S}$ is a Q-martingale, so that $\min\{\hat{S}_{t_N}^{(1)}, ...\hat{S}_{t_N}^{(n)}\}$ is a Q-supermartingale. Together with the Optimal Stopping Theorem it follows

$$E_Q\left[\min\left\{\hat{S}_{t_N}^{(1)}, ...\hat{S}_{t_N}^{(n)}\right\} \mid \mathcal{F}_{\tau_m^{(n)}}\right] \leq \min\left\{\hat{S}_{\tau_m^{(n)}}^{(1)}, ...\hat{S}_{\tau_m^{(n)}}^{(n)}\right\} = m\, e^{-r\tau_m^{(n)}}.$$

This implies

$$\mathrm{RI}_{t_0}^{(n)} \leq m \int_{t_0}^{t_N} e^{-ru} Q\left(\tau_m^{(n)} \in du\right).$$

The second bound then follows. □

As a consequence we can state the following theorem.

Theorem 1 (Semi closed-form upper price bound for $n \geq 2$) *For $n \geq 2$, an upper price bound on the relax certificate on the underlyings $S = (S^{(1)}, ..., S^{(n)})$ is given by*

$$\min_{k,l \in \{1,..,n\}} \Big\{ m\, Q\left(\min\{\tau_{m,k}, \tau_{m,l}\} \leq t_N\right) \tag{19}$$
$$+ \delta \sum_{i=1}^{N} e^{-rt_i} Q\left(\min\{\tau_{m,k}, \tau_{m,l}\} > t_i\right) + e^{-rt_N} Q\left(\min\{\tau_{m,k}, \tau_{m,l}\} > t_N\right) \Big\}.$$

where

$$Q\left(\min\{\tau_{m,k},\tau_{m,l}\}>t\right)=\frac{2}{\alpha t}e^{a_k\ln\left(\frac{S_0^{(k)}}{m}\right)+a_l\ln\left(\frac{S_0^{(l)}}{m}\right)+bt}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi\theta_0}{\alpha}\right)\cdot e^{-\frac{r_0^2}{2t}}\int_0^{\alpha}\sin\left(\frac{n\pi\theta}{\alpha}\right)g_n(\theta)d\theta.$$

The parameters and the function g_n are defined in Corollary 3 in Appendix B.

PROOF: According to Proposition 4, it holds that

$$\mathrm{RC}_{t_0}^{(n)}\leq\min_{k,l\in\{1,..,n\}}\left\{\mathrm{RC}_{t_0}(S^{(k)},S^{(l)})\right\}=\min_{k,l\in\{1,..,n\}}\left\{\mathrm{RO}_{t_0}(S^{(k)},S^{(l)})+\mathrm{RI}_{t_0}(S^{(k)},S^{(l)})\right\}.$$

The value of the knock-out component follows from Proposition 2, while Proposition 5 gives an upper bound on the value of the knock-in minimum option. Putting the results together gives Equation (19).

The survival probability $Q\left(\min\{\tau_{m,k},\tau_{m,l}\}>t\right)$ follows from the results of He, Keirstead, and Rebholz (1998) and Zhou (2001). Details are given in Appendix B. □

The upper price bound given in Theorem 1 results from subsets of two underlyings. If the relax certificate itself is written on two underlyings only, the knock-out component is priced exactly, while only the knock-in part is approximated from above. Since the "main part" of the product is explained by the knock-out part, the price bound is especially tight in this case. This is illustrated by a numerical example which refers to an attractive relax certificate written on two underlyings $S^{(1)}$ and $S^{(2)}$ with initial values $S_0^{(1)}=S_0^{(2)}=1$. The time to maturity is 3 years, intermediate payment dates are $t_1=1$ and $t_2=2$ (years), the bonus payment is $\delta=0.11$ and the barrier is $m=0.5$. The model setup is given by the following scenarios: interest rate is equal to $r=0.05$, volatilities $\sigma_1=\sigma_2=\sigma$ vary between 0 and 0.5 and correlation is set to $-1+0.25i$ $(i=0,\ldots,8)$. Recall that exact prices, in particular for the knock–in component, have to be determined numerically.[9] Figure 1 shows that the knock-out part explains the "main part" of the price of the relax certificate for nearly all correlations and volatilities. The price contribution of the

[9] To be more precise, the prices are calculated using a Monte-Carlo simulation with 10.000 simulation runs and a step size of 100 steps per day. To control the accurateness of the approximation, the simulation results for the survival probabilities and the prices of the knock–out component are compared to the exact closed-form solutions.

knock-out part ranges from nearly 100% for a volatility of 0.1 and all correlations to at least 50% for all positive correlations and all volatilities. For negative correlations and very high volatilities ($\sigma \geq 0.4$) the knock-in part dominates because of the low survival probabilities.

The difference between the upper price bound and the exact price of the knock–in part is illustrated in Figure 2. The difference is increasing in volatility and decreasing in correlation. Observe that the overestimation of the true price by the upper price bounds given in Proposition 5 and Theorem 1 respectively is rather low. It ranges from approximately 0% (for $\sigma \leq 0.1$ and all correlations) to 2.8% (for $\sigma = 0.35$ and $\rho = -0.25$) of the price of the relax certificate.

For three or more underlyings, however, the upper price bounds will be worse, and the question is whether we can derive tighter price bounds. One possibility is to derive upper and lower bounds for the first-hitting time probabilities which can be calculated in (semi-) closed form, and then plug these bounds into the pricing equation (12) for the knock-out coupon bond and into the upper bound in Proposition 5 for the value of the knock-in minimum option. Recall that it is not possible to determine the hitting time probabilities for $n \geq 3$ in (semi-)closed form. Therefore the tightest bounds for $n = 3$ which are not based on numerical approximations are achieved by using:

Lemma 2 (Semi closed–form bounds on survival probabilities for n=3) *It holds*

$$\underline{Q}(\tau_m^{(n=3)} \leq t) \leq Q\left(\tau_m^{(n=3)} \leq t\right) \leq \overline{Q}(\tau_m^{(n=3)} \leq t)$$

where

$$\begin{aligned}
\overline{Q}(\tau_m^{(n=3)} \leq t) &= \min\big\{Q(\min\{\tau_{m,1}, \tau_{m,2}\} \leq t) + Q(\tau_{m,3} \leq t),\\
&\qquad Q(\min\{\tau_{m,1}, \tau_{m,3}\} \leq t) + Q(\tau_{m,2} \leq t),\\
&\qquad Q(\min\{\tau_{m,2}, \tau_{m,3}\} \leq t) + Q(\tau_{m,1} \leq t)\big\}\\
\underline{Q}(\tau_m^{(n=3)} \leq t) &= \max\big\{Q(\min\{\tau_{m,1}, \tau_{m,2}\} \leq t),\\
&\qquad Q(\min\{\tau_{m,1}, \tau_{m,3}\} \leq t), Q(\min\{\tau_{m,2}, \tau_{m,3}\} \leq t)\big\}.
\end{aligned}$$

Proof: It holds that

$$Q(\tau_m^{(n=3)} \leq t) = Q(\min\{\tau_{m,1}, \tau_{m,2}, \tau_{m,3}\} \leq t)$$

Notice that

$$\{\min\{\tau_{m,1}, \tau_{m,2}, \tau_{m,3}\} \leq t\} = \{\min\{\tau_{m,1}, \tau_{m,2}\} \leq t\} \cup \{\tau_{m,3} \leq t\}$$

Using

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$$

immediately gives the Lemma. □

To derive an upper price bound on a relax certificate, $Q(\tau_m^{(n=3)} > t)$ is replaced by $(1 - \underline{Q}(\tau_m^{(n=3)} \leq t))$ while $Q(\tau_m^{(n=3)} \leq t)$ is replaced by $\overline{\mathrm{Q}}(\tau_m^{(n=3)} \leq t)$. It is straightforward to show that the resulting upper price bound is higher than the one given in Theorem 1.

4 Market Comparison

4.1 Contract Specifications

We now analyze some relax certificates issued in 2007 and 2008 and compare their issue prices to our price bounds. Table 1 gives the contract specifications of six typical certificates. The underlyings and their implied volatilities can be found in Table 2. All barriers are set to a rather low value (50% or 60%), so that at least one of the underlying stocks has to loose a high fraction of its initial value for the coupon bond to be replaced by the minimum option. Furthermore, the bonus payments are large enough for all certificates to be attractive in the sense of Definition 2.

The relax certificate C1, issued by Commerzbank, is written on two stocks, namely Siemens and Daimler. The time to maturity is 14 month, and there are no intermediate payment dates. If both stocks never fall below 50% of their initial value, the payoff from the bonus certificate is 111 Euros. In principle, the high bonus payment (as compared to

C	Issue Price incl. load	n	t_N	δ	m	Bonus payments t_i	r_i
C1	101.00	2	14 months	0.11	50%	at maturity	0.0505
C2	101.00	3	17 month 4 days	0.16	50%	at maturity	0.0505
C3	101.00	3	3 years 3 months	0.10	50%	every 13 months	0.0505, 0.0484, 0.0472
C4	101.00	3	15 month 2 days	0.19	60%	at maturity	0.0505
C4	101.00	3	3 years	0.30	50%	at maturity	0.0472
C6	1000.00	3	20 months	0.20	60%	at maturity	0.0484

Table 1: Summary of traded product specifications and interest rates.

the current risk-free rate) should just compensate the investor for the risk that the lower barrier is hit, in which case she receives the worst of the two stocks at maturity.

The relax certificate C2, issued by HSBC Trinkaus, is additionally written on EON. It has both a longer time to maturity and a higher bonus payment, but the same initial price as C1. Theoretically, the higher bonus payment is set in such a way as to exactly offset the lower value resulting from the longer time to maturity. The third relax certificate, C3, is issued by HVB. It is written on three stocks, namely Allianz, BASF and Deutsche Post. Different from before, there are now two intermediate payment dates after 13 and 26 months.

In contrast to the certificates presented so far, C4 – C6 contain an additional component. The contracts also include a knock-out minimum call option on the underlyings with a strike price equal to the terminal payoff from the bond component. The investor thus participates in the stock market if all stocks perform well. Both C4 and C5 are issued by Société General. C4 is written on three banks, namely Deutsche Bank, Commerzbank and Postbank, whereas C5 is written on Allianz, Deutsche Telekom and DaimlerChrysler. C6 is launched by WestLB with underlyings Allianz, Bayer and RWE.

C	Underlyings	Implied Volatilitiy	C	Underlyings	Implied Volatility
C1	Daimler AG	0.33	C4	Deutsche Bank	0.34
	Siemens AG	0.35		Commerzbank	0.42
				Postbank	0.46
C2	Daimler AG	0.33	C5	Allianz	0.32
	Siemens AG	0.35		Deutsche Telekom	0.25
	EON	0.27		Daimler AG	0.33
C3	AllianC	0.32	C6	Allianz	0.32
	BASF	0.25		Bayer	0.32
	Deutsche Post	0.30		RWE	0.24

Table 2: Implied volatilities of the underlying stocks of the traded relax certificates.

4.2 Survival Probabilities and Price Bounds

Proposition 1 states that the coupon bond is a trivial upper price bound for an attractive relax certificate. The interest rates are inferred from the corresponding zero coupon bonds (swaps) via bootstrapping and are given in the last column of Table 1. The resulting prices of the coupon bonds are given in Table 3. For all certificates, the issue price is significantly lower than the price of the corresponding coupon bond. The risk that at least one of the stocks looses more than 50% respectively 40% of the initial value should thus not be neglected, and it reduces the price by 4% to 11%.

To assess the risk inherent in the relax certificate, we calculate the (risk-neutral) probability that the barrier will not be hit by one or two underlyings, where we set $\rho_{k,l} = 0.3$ and $\sigma_k = \sigma_l = 0.3$.[10] The results show that adding a further underlying increases the risk that the bond will be knocked out significantly. They also confirm that the risk of a knock-out is rather high even if we only consider two underlyings and thus calculate a lower bound for the knock-out probability for certificates with $n = 3$.

[10]For all certificates, the implied volatilities of at least two underlyings as given in Table 2 are above 30%, so that a volatility of $\sigma = 0.3$ yields an upper bound for the survival probability.

C	n	Issue Price incl. load	corresp. coupon bond	Survival probability		Upper price bound		
				one underlying	two underlyings	Knock-out component	Knock-in component	Price
C1	2	101.00	104.65	96.88%	93.60%	97.32	3.20	100.52
C2	3	101.00	107.99	94.97%	89.90%	96.11	5.25	101.36
C3	3	101.00	112.83	80.76%	65.99%	75.19	17.05	92.24
C4	3	101.00	111.72	87.58%	77.19%	86.24	13.69	99.93
C5	3	101.00	112.83	82.47%	68.67%	77.48	15.69	93.17
C6	3	1000.00	1107.37	81.18%	67.85%	751.10	192.90	944.00

Table 3: Relax certificates traded at the market

The table gives the price of the corresponding coupon bond, the survival probabilities based on one and two underlyings, and the upper price bounds based on two underlyings. The calculations are based on a volatility of $\sigma = 0.3$ and a correlation of $\rho = 0.3$.

In the next step, we consider the upper price bounds given in Theorem 1. Table 3 gives the upper price bounds, again based on $\rho_{k,l} = 0.3$ and $\sigma_k = \sigma_l = 0.3$. For all certificates, the price of the knock-out coupon bond exceeds the upper bound on the value of the knock-in minimum option to a large extent. Furthermore, the resulting upper price bound is below the issue price for all but one certificate. If we account for dividend payments of the stocks and for credit risk of the issuer, the upper price bound would even decrease further.

For C1 and C2, we also calculate the upper price bounds using the implied volatilities of the underlyings and a correlation which ranges from -1 to 1. Figure 3 shows the price bounds which result from reducing the number of underlyings to $n = 1$ and $n = 2$. For $n = 1$, the upper price bound follows from Proposition 3, while we rely on Theorem 1 for $n = 2$. For C1, the issue price exceeds the lowest upper price bound for all correlation levels. For C2, the issue price is below the upper bound only if we assume a correlation larger than 0.85.

There are two possible conclusions. First, relax certificates may be overpriced in the market. This is in line with the empirical results of Wallmeier and Diethelm (2008) for the swiss certificate market. Furthermore, the mispricing is the higher the higher the bonus payments (and thus the higher the discount due to the knock-out feature of the bond). We conjecture that the investors do not correctly estimate the risk associated with the barrier feature, but overweight the sure coupon. Second, the model of Black-Scholes may not be the appropriate choice. If we include (on average downward) jumps as in Merton (1976), however, the knock-out probability increases. The resulting price bounds will then even be lower than in the model of Black-Scholes such that the overpricing is even higher under a more realistic model setup.

5 Conclusion

Relax certificates can be decomposed into a knock-out coupon bond and a knock-in minimum option on all underlying stocks. The contracts are designed such that relax certificates can be offered at a discount compared to the associated coupon bond. Formally, this gives a condition on admissible (or *attractive*) contract parameters in terms of the barrier and the bonus payments.

The knock-out/knock-in event takes place when the worst-performing of the n underlying stocks hits a lower barrier, which is usually quite low. Nevertheless, our analysis shows that the probability of a knock-out can not be neglected and induces a significant price discount of the relax certificate as compared to the corresponding coupon bond. The risk is the larger the higher the volatility of the underlyings, the lower their correlation, and the larger the number of stocks the certificate is written on.

In general, numerical methods are needed to price relax certificates, and even in the Black-Scholes model, closed form solutions only exist for one underlying. However, closed-form or semi-closed form solutions are available for upper price bounds. A trivial upper price bound is given by the corresponding coupon bond. Furthermore, the price of a relax certificate on several underlyings is bounded from above by the price of the (cheapest) relax certificate on a subset of underlyings. We show that two underlyings allow to achieve meaningful and tractable price bounds. The most likely candidates to give this lowest upper price bound are the relax certificates on the most risky assets and/or the assets with the lowest correlation between the underlyings.

Finally, we test the practical relevance of our theoretical results by comparing the price bounds with market prices. The upper price bounds are calculated based on the implied volatilities of call options on the respective underlyings. It turns out that the relax certificates which are currently traded at the market are significantly overpriced. This result is true for nearly all correlation scenarios.

A First Hitting Time - One-Dimensional Case

To derive the distribution of the first hitting time in the one-dimensional case, we use some results given in He, Keirstead, and Rebholz (1998). They consider the probability density and distribution function of the maximum or minimum of a one-dimensional Brownian motion with drift. Along the lines of He, Keirstead, and Rebholz (1998), we define

$$\underline{X}_t := \min_{0\leq s\leq t} X_s \qquad \overline{X}_t := \max_{0\leq s\leq t} X_s$$

where $X_t = \alpha t + \sigma W_t$, $t \geq 0$ and α, σ are constants. W is a Brownian motion defined on some probability space.

Proposition 6 *Let $G(x,t;\alpha)$ and $g(y,x,t;\alpha_1)$ be defined as*

$$G(x,t;\alpha) := N\left(\frac{x-\alpha t}{\sigma\sqrt{t}}\right) - e^{\frac{2\alpha x}{\sigma^2}} N\left(\frac{-x-\alpha t}{\sigma\sqrt{t}}\right),$$

$$g(y,x,t;\alpha_1) := \frac{1}{\sigma\sqrt{t}}\phi\Big(\frac{x-\alpha_1 t}{\sigma\sqrt{t}}\Big)\left(1-e^{-\frac{4x^2-4x4y}{2\sigma^2 t}}\right)$$

where N denotes the cumulative distribution function of the standard normal distribution and $\phi(z)$ the density of the standard normal distribution.
For $x \geq 0$, it holds

$$P(\overline{X}_t \leq x) = G(x,t;\alpha), \qquad P(X_1(t)\in dy, \overline{X}_1(t) \leq x) = g(y,x,t;\alpha_1)dy.$$

For $x < 0$, it holds

$$P(\underline{X}_t \geq x) = G(-x,t;-\alpha), \qquad P(X_1(t)\in dy, \underline{X}_1(t) \geq x) = g(-y,-x,t;-\alpha_1)dy.$$

PROOF: C.f. Theorem 1 of He, Keirstead, and Rebholz (1998) and the proof given here.

Corollary 2 *Let*

$$S_t = S_0 e^{\left(\mu-\frac{1}{2}\sigma^2\right)t+\sigma W_t}$$

where μ, σ $(\sigma > 0)$ are constants. W is a Brownian motion defined on some probability space. For the first hitting time $\tau_m := \inf\{t \geq 0 | S_t \leq m\}$, $(m < S_0)$ it holds that

$$P(\tau_m \leq t) = N\left(\frac{\ln\frac{m}{S_0} - \left(\mu-\frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right) + e^{2\frac{\mu-\frac{1}{2}\sigma^2}{\sigma^2}\ln\frac{m}{S_0}} N\left(\frac{\ln\frac{m}{S_0} + \left(\mu-\frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right),$$

$$P(\tau_m \in dt) = \frac{-\ln\frac{m}{S_0}}{\sqrt{2\pi\sigma^2 t^3}} e^{-\frac{1}{2}\frac{\left(\ln\frac{m}{S_0}-(\mu-\frac{1}{2}\sigma^2)t\right)^2}{\sigma^2 t}} dt.$$

PROOF: Note that

$$\tau_m : \ = \ \inf\left\{t \geq 0 \,\middle|\, S_t \leq m\right\} \ = \ \inf\left\{t \geq 0 \,\middle|\, \ln \frac{S_t}{S_0} \leq \ln \frac{m}{S_0}\right\}.$$

Let X_t denote the logarithm of the normalized asset price, i.e.

$$X_t := \ln \frac{S_t}{S_0} = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t$$

and set $\alpha = \mu - \frac{1}{2}\sigma^2$. The stopping time τ_m is related to the first hitting time of a one-dimensional Brownian motion with drift α. With $x := \ln \frac{m}{S_0} < 0$ it follows

$$P(\tau_m \leq t) = P(\underline{X}_t \leq x) = 1 - P(\underline{X}_t \geq x).$$

According to Proposition 7, we have

$$\begin{aligned} 1 - P(\underline{X}_t \geq x) &= 1 - G(-x, t; -\alpha) \\ &= 1 - N\left(\frac{-x + \alpha t}{\sigma\sqrt{t}}\right) + e^{\frac{2\alpha x}{\sigma^2}} N\left(\frac{x + \alpha t}{\sigma\sqrt{t}}\right) \\ &= N\left(\frac{x - \alpha t}{\sigma\sqrt{t}}\right) + e^{\frac{2\alpha x}{\sigma^2}} N\left(\frac{x + \alpha t}{\sigma\sqrt{t}}\right). \end{aligned}$$

Inserting α and x gives the distribution function.To derive the density function, define

$$f(t) := N\left(\frac{x - \alpha t}{\sigma\sqrt{t}}\right) + e^{\frac{2\alpha x}{\sigma^2}} N\left(\frac{x + \alpha t}{\sigma\sqrt{t}}\right).$$

Taking partial derivatives gives

$$\begin{aligned} f'(t) &= N'\left(\frac{x - \alpha t}{\sigma\sqrt{t}}\right) \times \left(\frac{-\alpha\sigma\sqrt{t} - \frac{\sigma}{2\sqrt{t}}(x - \alpha t)}{\sigma^2 t}\right) \\ &\quad + e^{\frac{2\alpha x}{\sigma^2}} \times N'\left(\frac{x + \alpha t}{\sigma\sqrt{t}}\right) \times \left(\frac{\alpha\sigma\sqrt{t} - \frac{\sigma}{2\sqrt{t}}(x + \alpha t)}{\sigma^2 t}\right) \end{aligned}$$

Using $N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, we get

$$\begin{aligned} e^{\frac{2\alpha x}{\sigma^2}} N'\left(\frac{x + \alpha t}{\sigma\sqrt{t}}\right) &= \frac{1}{\sqrt{2\pi}} e^{\frac{2\alpha x}{\sigma^2} - \frac{1}{2}\left(\frac{x+\alpha t}{\sigma\sqrt{t}}\right)^2} \\ &= \frac{1}{\sqrt{2\pi}} e^{\frac{1}{\sigma^2}[2\alpha x - \frac{1}{2t}(x+\alpha t)^2]} \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\alpha t)^2}{\sigma^2 t}} \\ &= N'\left(\frac{x - \alpha t}{\sigma\sqrt{t}}\right). \end{aligned}$$

Inserting this in the above equation for $f'(t)$ gives

$$\begin{aligned} f'(t) &= N'\left(\frac{x-\alpha t}{\sigma\sqrt{t}}\right) \times \frac{-\sigma x}{\sqrt{t}\sigma^2 t}(x-\alpha t+x+\alpha t) \\ &= \frac{-x}{\sqrt{2\pi\sigma^2 t^3}} e^{-\frac{1}{2}\frac{(x-\alpha t)^2}{\sigma^2 t}}. \end{aligned}$$

Using $\alpha = \mu - \frac{1}{2}\sigma^2$ and $x = \ln \frac{m}{S_0}$, which implies $\frac{\partial x}{\partial m} = \frac{S_0}{m}$, gives the result. □

B First Hitting Time – Two Dimensional Case

The distribution of the first hitting time of a two-dimensional arithmetic Brownian motion is given in He, Keirstead, and Rebholz (1998) and Zhou (2001):

Proposition 7 *Let* $X_t^{(j)} = \alpha_j t + \sigma_j W_t^{(j)}$ $(j = 1, 2)$, *where* α_j *and* σ_j *are constants.* $W^{(1)}$, $W^{(2)}$ *are two correlated Brownian motions with* $\langle W^{(1)}, W^{(2)} \rangle_t = \rho t$. *Then, the probability that* $X^{(1)}$ *and* $X^{(2)}$ *will not hit the upper boundaries* $x^{(1)} > 0$ *and* $x^{(2)} > 0$ *up to time* t *is given by*

$$Q\left(\overline{X}_t^{(1)} \le x^{(1)}, \overline{X}_t^{(2)} \le x^{(2)}\right) = \frac{2}{\alpha t} e^{a_1 x_1 + a_2 x_2 + bt} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \cdot e^{-\frac{r_0^2}{2t}} \int_0^{\alpha} \sin\left(\frac{n\pi\theta}{\alpha}\right) g_n(\theta) d\theta$$

where $\overline{X}_t := \max_{0 \le s \le t} X_s$. *The parameters are defined by*

$$a_1 = \frac{-\alpha_1\sigma_2 + \rho\alpha_2\sigma_1}{(1-\rho^2)\sigma_1^2\sigma_2} \qquad a_2 = \frac{-\alpha_2\sigma_1 + \rho\alpha_1\sigma_2}{(1-\rho^2)\sigma_2^2\sigma_1}$$

$$d_1 = a_1\sigma_1 + a_2\sigma_2\rho \qquad d_2 = a_2\sigma_2\sqrt{1-\rho^2}$$

and by

$$\begin{aligned} b &= \alpha_1 a_1 + \alpha_2 a_2 + \frac{1}{2}\sigma_1^2 a_1^2 + \frac{1}{2}\sigma_2^2 a_2^2 + \rho\sigma_1\sigma_2 a_1 a_2 \\ \alpha &= \begin{cases} \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \text{if } \rho < 0 \\ \pi + \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \text{otherwise} \end{cases} \\ \theta_0 &= \begin{cases} \tan^{-1}\left(\frac{\frac{x2}{\sigma_2}\sqrt{1-\rho^2}}{\frac{x1}{\sigma_1} - \rho\frac{x2}{\sigma_2}}\right) & \text{if } (.) > 0 \\ \pi + \tan^{-1}\left(\frac{\frac{x2}{\sigma_2}\sqrt{1-\rho^2}}{\frac{x1}{\sigma_1} - \rho\frac{x2}{\sigma_2}}\right) & \text{otherwise} \end{cases} \\ r_0 &= \frac{x_2}{\sigma_2} / \sin(\theta_0). \end{aligned}$$

The function g_n is defined as

$$g_n(\theta) = \int_0^{\infty} re^{-\frac{r^2}{2t}} e^{d_1 r \sin(\theta-\alpha) - d_2 r \cos(\theta-\alpha)} I_{\frac{n\pi}{\alpha}}\left(\frac{rr_0}{t}\right) dr.$$

PROOF: Cf. Proposition 1 of Zhou (2001) and the proof given there. □

Corollary 3 *For two stocks $S^{(k)}$ and $S^{(l)}$ with volatilities σ_k and σ_l and correlation $\rho_{k,l}$, the distribution function of the first hitting time $\min\{\tau_{m,k}, \tau_{m,l}\}$ under the risk-neutral measure is given by*

$$\begin{aligned} & Q\left(\min\{\tau_{m,k}, \tau_{m,l}\} \le t\right) \\ = \; & 1 - \frac{2}{\alpha t} e^{a_k \ln\left(\frac{S_0^{(k)}}{m}\right) + a_l \ln\left(\frac{S_0^{(l)}}{m}\right) + bt} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \cdot e^{-\frac{r_0^2}{2t}} \int_0^{\alpha} \sin\left(\frac{n\pi\theta}{\alpha}\right) g_n(\theta) d\theta \end{aligned}$$

where

$$a_k = \frac{(r-0.5\sigma_k^2)\sigma_l - \rho_{k,l}(r-0.5\sigma_l^2)\sigma_k}{(1-\rho_{k,l}^2)\sigma_k^2\sigma_l} \qquad a_l = \frac{(r-0.5\sigma_l^2)\sigma_k - \rho_{k,l}(r-0.5\sigma_k^2)\sigma_l}{(1-\rho_{k,l}^2)\sigma_l^2\sigma_k}$$

$$d_k = a_k\sigma_k + a_l\sigma_l\rho_{k,l} \qquad d_l = a_l\sigma_l\sqrt{1-\rho_{k,l}^2}$$

and

$$\begin{aligned} b &= -(r-0.5\sigma_k^2)a_k - (r-0.5\sigma_l^2)a_l + \frac{1}{2}\sigma_k^2 a_k^2 + \frac{1}{2}\sigma_l^2 a_l^2 + \rho_{k,l}\sigma_k\sigma_l a_k a_l \\ g_n(\theta) &= \int_0^{\infty} re^{-\frac{r^2}{2t}} e^{d_k r \sin(\theta-\alpha) - d_l r \cos(\theta-\alpha)} I_{\frac{n\pi}{\alpha}}\left(\frac{rr_0}{t}\right) dr. \end{aligned}$$

$I_v(z)$ is the modified Bessel function of order v. α, θ_0, and r_0 are given in Proposition 7 in Appendix B for the case where $k = 1$ and $l = 2$.

PROOF: The stock prices are given by

$$S_t^{(j)} = S_0 e^{(r-\frac{1}{2}\sigma_j^2)t + \sigma_j W_t^{(j)}} \qquad j = k, l.$$

The first hitting time of the lower boundary $m_j < S_0^{(j)}$ by the geometric Brownian motion $S_t^{(j)}$ is

$$\tau_m^{(j)} \; := \; \inf\left\{t \ge 0 \,\middle|\, S_t^{(j)} \le m_j\right\} \; = \; \inf\left\{t \ge 0 \,\middle|\, -\ln\frac{S_t^{(j)}}{S_0^{(j)}} \ge \ln\frac{S_0^{(j)}}{m_j}\right\}.$$

With the definition of the arithmetic Brownian motion

$$X_t^{(j)} := -\ln \frac{S_t^{(j)}}{S_0^{(j)}} = -\left(r - \frac{1}{2}\sigma_j^2\right) t - \sigma_j W_t^{(j)},$$

the first hitting time can be rewritten as

$$\tau_m^{(j)} \quad = \quad \inf \left\{ t \geq 0 \,\middle|\, X_t^{(j)} \geq \ln \frac{S_0^{(j)}}{m_j} \right\}.$$

Using the relation

$$\{\tau_{m,j} > t\} = \left\{ \overline{X}_t^{(j)} < \ln \frac{S_0^{(j)}}{m_j} \right\}$$

we can conclude

$$Q\left(\min\{\tau_{m,k}, \tau_{m,l}\} > t\right) \quad = \quad Q\left(\overline{X}_t^{(k)} < \ln \frac{S_0^{(k)}}{m_k}, \overline{X}_t^{(l)} < \ln \frac{S_0^{(l)}}{m_l} \right).$$

Since both, $\ln \frac{S_0^{(k)}}{m_k} > 0$ and $\ln \frac{S_0^{(l)}}{m_l} > 0$, the result follows from Proposition 7. □

C Price of the Minimum Option on Two Underlyings

Stulz (1982) gives the pricing formula for the minimum of two underlying assets by using Magrabe's (1978) formula to exchange one asset for an other.

Proposition 8 *For two risky assets $S^{(1)}$ and $S^{(2)}$ with volatilities σ_1 and σ_2 and correlation ρ, the price of the corresponding minimum option is given by*

$$\begin{aligned} C_t^{Min,(n=2)} &= E_Q\left[e^{-r(t_N - t)} \min\left\{S_{t_N}^{(1)}, S_{t_N}^{(2)}\right\} \mid \mathcal{F}_t\right] \\ &= S_t^{(1)} - S_t^{(1)} N(d_1) + S_t^{(2)} N(d_2) \end{aligned}$$

where

$$d_1 = \frac{\ln \frac{S_t^{(1)}}{S_t^{(2)}} + \frac{1}{2}\tilde{\sigma}^2 (t_N - t)}{\tilde{\sigma}\sqrt{t_N - t}}, \qquad d_2 = d_1 - \tilde{\sigma}\sqrt{t_N - t}, \qquad \tilde{\sigma} = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}.$$

PROOF: The risk-neutral pricing equation is

$$E_Q\left[e^{-r(t_N - t)} \left(S_{t_N}^{(1)} - \max\left\{S_{t_N}^{(1)} - S_{t_N}^{(2)}, 0\right\}\right) \mid \mathcal{F}_t\right]$$

The rest of the proof immediately follows from Magrabe (1978), who gives an explicit formula for the option to exchange one asset for another. □

D Price of the Minimum Option on Several Underlyings

To price a minimum option on several underlyings, we rely on the results of Johnson (1987). Johnson (1987) argues that the price of a minimum option on several assets can be derived using as well Magrabe's (1987) solution for pricing exchange options.

Proposition 9 *Let n denote the number of underlying risky assets and N_n is the n-dimensional normal distribution. Then the $t_0 = 0$ price of a minimum call-option with strike K and maturity T is*

$$\begin{aligned} C_{t_0}^{min} = \quad & S_1 N_n(d_1(S_1, K, \sigma_1^2), -d_1'(S_1, S_2, \sigma_{12}^2), ..., \\ & \qquad -d_1'(S_1, S_n, \sigma_{1n}^2), -\rho_{112}, -\rho_{113}, ..., \rho_{123}, ...) \\ & + S_2 N_n(d_1(S_2, K, \sigma_1^2), -d_1'(S_2, S_1, \sigma_{12}^2), ..., \\ & \qquad -d_1'(S_2, S_n, \sigma_{2n}^2), -\rho_{212}, -\rho_{223}, ..., \rho_{213}, ...) \\ & + ... \\ & + S_n N_n(d_1(S_n, K, \sigma_n^2), -d_1'(S_n, S_1, \sigma_{n2}^2), ..., \\ & \qquad -d_1'(S_n, S_{n-1}, \sigma_{n-1n}^2), -\rho_{n1n}, -\rho_{n2n}, ..., \rho_{n12}, ...) \\ & - K e^{-rT} N_n(d_2(S_1, K, \sigma_1^2), d_2(S_2, K, \sigma_2^2), ..., \\ & \qquad d_2(S_n, K, \sigma_n^2), \rho_{12}, \rho_{13}, ...). \end{aligned}$$

The functions d_1, d_1' and d_2 are defined by

$$d_1'(S_i, S_j, \sigma_{ij}^2) = \frac{\log \frac{S_i}{S_j} + \frac{1}{2}\sigma_{ij}^2 T}{\sigma_{ij}\sqrt{T}}, \quad d_2(S_i, K, \sigma_i^2) = \frac{\log \frac{S_i}{K} + \left(r - \frac{1}{2}\sigma_i^2\right)T}{\sigma_i\sqrt{T}}, \quad d_1 = d_2 + \sigma_i\sqrt{T}.$$

The correlations are given by

$$\rho_{iij} = \frac{\sigma_i - \rho_{ij}\sigma_j}{\sigma_{ij}}, \quad \rho_{ijk} = \frac{\sigma_i^2 - \rho_{ij}\sigma_i\sigma_j - \rho_{ik}\sigma_i\sigma_k + \rho_{jk}\sigma_k\sigma_j}{\sigma_{ij}\sigma_{ik}}.$$

PROOF: C.f. Johnson (1987). □

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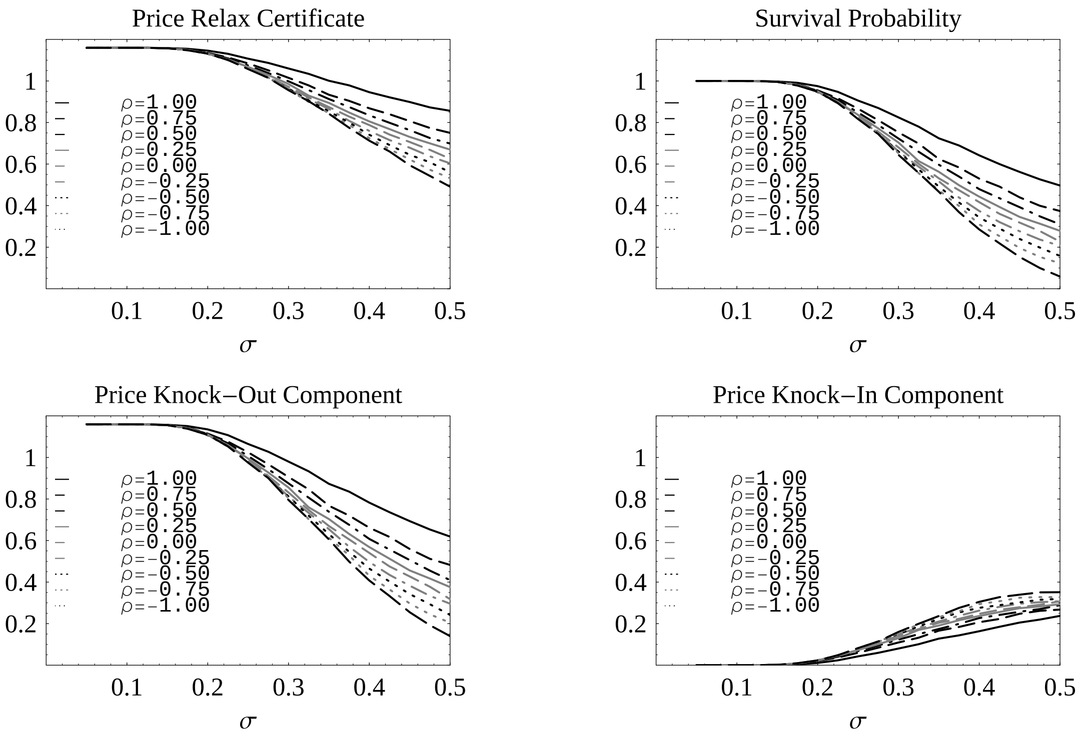

Figure 1: Relax certificate on two underlyings

The figure gives the price, the survival probability, the price of the knock-out part and the price of the knock-in part of a relax certificate as a function of the volatility of the two stocks for varying correlations. The parameters are $m = 0.5$, $\delta = 0.11$, $\underline{\underline{T}} = \{1, 2, 3\}$, $S_0^{(1)} = S_0^{(2)} = 1$ and $r = 0.05$.

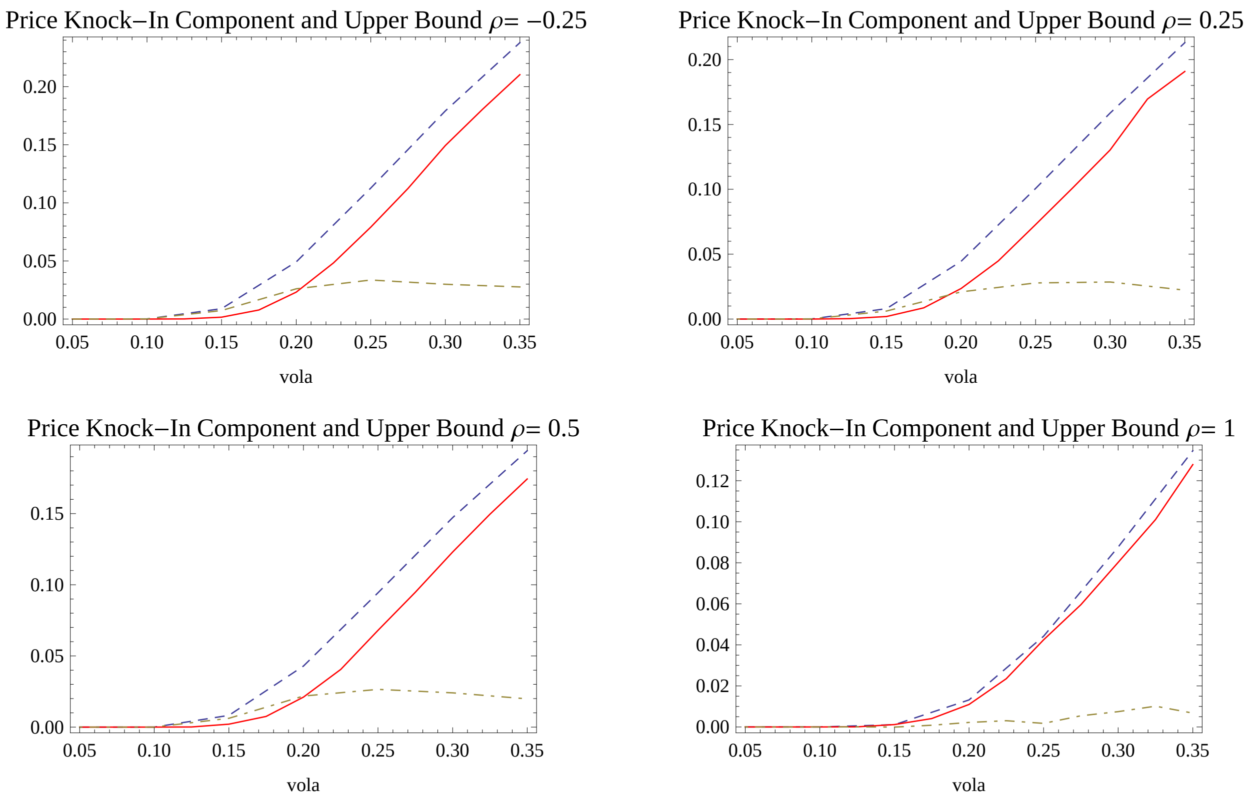

Figure 2: Exact price vs. upper bound of knock-in part

The figure compares the exact price of the knock-in price (solid line) with the upper price bound derived in Proposition 5 (dashed line). The dash-dotted line shows the difference between the upper price bound and the exact price.

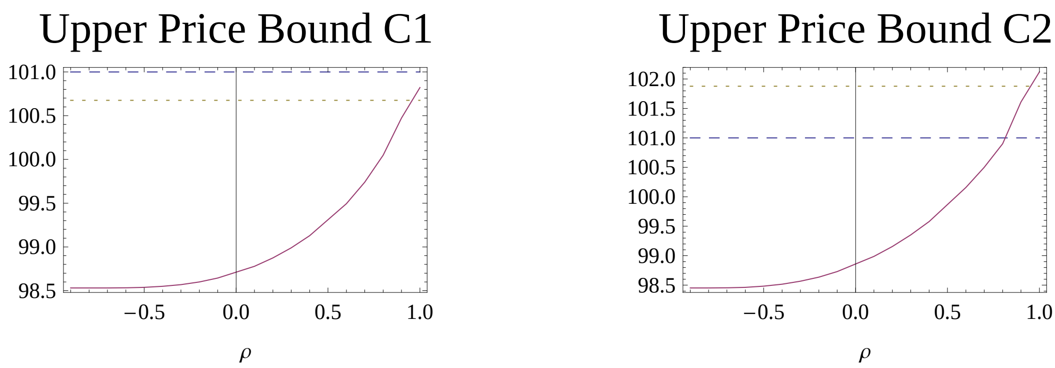

Figure 3: Upper price bound of C1 and C2

The figure shows the issue price (dotted line), the price of a relax certificate on one underlying (dashed line) and the upper price bounds for C1 and C2 (solid line) which result from Theorem 1 as a function of the correlation between the underlyings. The implied volatilities are given in Table 2.