RISK MANAGEMENT AND CAPITAL ALLOCATION WITH COHERENT MEASURES OF RISK

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Abstract. We present an application of coherent risk measures to the firm-wide risk management of portfolio selection and accounting of risk capital. Both regulators' and shareholders' concerns about use of capital are represented by measuring, under their respective sets of scenarios, expected future net worth and expected future excess of net worth over some required amount. For a firm with many divisions, the decomposition principle in linear optimisation of large scale systems, can be interestingly interpreted via an internal market. Firm's divisions compete internally for capital as well as for risk limits, providing hereby some accounting for risk capital, which is incentive-compatible with decentralized optimisation of the firm's objective.

Key words and phrases. accounting for risks, capital allocation, coherent capital assignment, coherent risk measure, concentration of risks, cost allocation, decentralization, decomposition principle, internal trading, measure of risk, net worth, performance measurement, regulators' constraints, risk-based capital, risk management, scenario, shareholders' constraints.

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1. Introduction

Capital is often understood both as a resource for the acquisition of financial assets as well as a protection against extreme adverse variations of these. Therefore the expression of "capital allocation" has two meanings which this paper intends to study and to explicit in some simple linear cases.

For that purpose we consider an investment firm dealing on financial markets via several desks or divisions. We use the notion of coherent risk measures to model the concerns of regulators via a finite set of scenarios under which it is required from the firm to have a positive expected future net worth. We also incorporate the concerns of capital providers with another set of scenarios under each of which the expected future net worth of the firm must be at least equal to some preassigned goal.

The paper provides in Sections 4 and 5 a way for decentralizing the regulator- and shareholder-constrained portfolio choice, by allowing the different desks to trade among themselves the capital and the limits assigned to them. They then make their own decisions of portfolio choice with their modified capital and risk limits. This carries some similarities with decomposition methods of convex programming (see [Ro], Section 28). The assumption of a finite number of scenarios allows us the explicit use of linear duality and it appears that, at the optimum, the shadow prices faced by the central manager for capital and limits of risks equal the internal prices faced by the desks. This is in touch with the classical large scale linear systems optimisation (see [BHM], Chapter 12, [Da], [DW] and [HH], Kapitel VII.)

Section 6 examines the use of the scenario representation of coherent risk measures in two examples of risk capital assignment.

2. The concerns of regulators and shareholders

The framework is as in [ADEH] with a finite set Ω of states of nature, an acceptance set $\mathcal{A}$ representing the future net worths of the firm which the regulator accepts today (date 0). This set is defined by a set $\mathcal{R}$ of scenarios, i.e. probabilities on Ω such that a future net worth X at the future date 1 is acccepted at the initial date 0 by the regulator if and only if for each $\mathbb{R} \in \mathcal{R}$, $\mathbf{E}_{\mathbb{R}}[X] \geq 0$. (For simplicity we assume here zero interest rate between dates 0 and 1.)

A representative potential shareholder is supposed to have on his own, first, a non tradable future wealth W and, second, some tradable asset with future market value Z. He may sell it at the initial date 0 and trust the proceeds of the sale to the investment firm which promises to repay him the random amount X at date 1. This firm may have access to more markets than the individual investor does. The set $\mathcal{B}$ of the investment firm's positions X which are "acceptable for the shareholder" is the set of X for which $\mathbf{E}_{\mathbb{P}_{sh}}[u(W+X)] \geq \mathbf{E}_{\mathbb{P}_{sh}}[u(W+Z)]$, where $\mathbb{P}_{sh}$ and u are the shareholder's subjective probability and utility function for wealth at date T.

Natural restrictions on the set $\mathcal{B}$, are similar to restrictions on regulators' acceptance set (see [ADEH], Section 2.2) but we do not require $\mathcal{B}$ to be conical. There exists a set $\mathcal{S}$ of scenarios and a family $b_{\mathbb{S}}$, $\mathbb{S} \in \mathcal{S}$ of constants such that a future net worth X at date 1 is accceptable at the initial date to the shareholders if and only if for each $\mathbb{S} \in \mathcal{S}$, $\mathbf{E}_{\mathbb{S}}[X] \geq b_{\mathbb{S}}$. We denote $\rho_{\mathcal{A},\mathcal{B}}$ the subadditive mapping from $\mathbf{R}^{\Omega}$ into the reals defined by

$$\rho_{\mathcal{A},\mathcal{B}}(X) = \inf\{m \mid m + X \in \mathcal{A} \cap \mathcal{B}\}.$$

Notice that the notion of benchmark as in [De] is an example of approaching shareholders' concerns.

3. A Linear Problem for the Management of the Firm

3.1 The Firm's Manager's Primal Problem.

The regulator's risk measure ρ is supposed to correspond to an acceptance set given by a *finite* set $\mathcal{R}$ of scenarios. The initial capital of the firm is denoted by K, and the acceptance set $\mathcal{B}$ of shareholders providing this capital is supposed to be given by a *finite* set $\mathcal{S}$ of scenarios. There are D desks, each of them with a linear space $\mathcal{X}_d$ of dimension n_d of available securities, and the firm has access to the sum $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2 + \cdots + \mathcal{X}_D$. Current securities prices are given by a functional π_0 and, all agents are supposed to have the same probability distribution $\mathbb{P}$ for future prices of securities.

Given an initial price function π_0 on $\mathcal{X}$, the firm's manager's problem is to solve

$$\text{(3.1 F)} \qquad \inf_{X\in\mathcal{X}} -\mathbf{E}_{\mathbb{P}}[X], \text{ such that for each } \mathbb{R}\in\mathcal{R},\ \mathbf{E}_{\mathbb{R}}[X]\geq 0,$$

$$\text{for each } \mathbb{S}\in\mathcal{S},\ \mathbf{E}_{\mathbb{S}}[X]\geq b_{\mathbb{S}}, \text{ and } \pi_0(X)=K.$$

If $X_{d,1}\dots X_{d,n_d}$ generate $\mathcal{X}_d$, the problem above is written coordinate wise as

$$\inf_{x^{d,i}} -\sum_{1\leq d\leq D}\sum_{1\leq i\leq n_d} x^{d,i}\mathbf{E}_{\mathbb{P}}[X_{d,i}]$$

$$\text{such that for each } \mathbb{R}\in\mathcal{R},\ \sum_{1\leq d\leq D}\sum_{1\leq i\leq n_d} x^{d,i}\mathbf{E}_{\mathbb{R}}[X_{d,i}]\geq 0,$$

$$\text{for each } \mathbb{S}\in\mathcal{S},\ \sum_{1\leq d\leq D}\sum_{1\leq i\leq n_d} x^{d,i}\mathbf{E}_{\mathbb{S}}[X_{d,i}]\geq b_{\mathbb{S}}, \text{ and } \sum_{1\leq d\leq D}\sum_{1\leq i\leq n_d} x^{d,i}\pi_0(X_{d,i})=K.$$

This linear problem has therefore $n = n_1\cdots + n_D$ variables without sign constraints, $\#\mathcal{R}+\#\mathcal{S}$ inequality constraints and one equality constraint.

3.2 The Firm's Manager's Dual Problem.

The dual of the problem (3.1 F) has variables $\lambda_{\mathbb{R}}$, $\mu_{\mathbb{S}}$, ϕ, all but the last constrained to be positive, and n equality constraints. It is stated as

$$\text{(3.2 F')} \qquad \sup_{\lambda_{\mathbb{R}},\,\mu_{\mathbb{S}},\,\phi} \sum_{\mathbb{S}\in\mathcal{S}} \mu_{\mathbb{S}} b_{\mathbb{S}} + \phi\cdot K$$

$$\text{such that for each } (d,i),\ 1\leq d\leq D,\ 1\leq i\leq n_d,$$

$$\sum_{\mathbb{R}\in\mathcal{R}} \lambda_{\mathbb{R}}\mathbf{E}_{\mathbb{R}}[X_{d,i}] + \sum_{\mathbb{S}\in\mathcal{S}} \mu_{\mathbb{S}}\mathbf{E}_{\mathbb{S}}[X_{d,i}] + \phi\cdot\pi_0(X_{d,i}) = -\mathbf{E}_{\mathbb{P}}[X_{d,i}].$$

4. From Firm to Desks

4.1 Allocation of Capital and of Risk Limits.

The firm's manager may attempt a decentralisation of problem (3.1 F) by assigning to desk d, $d = 1, \dots, D$, some amount of capital K_d, some limit $\rho_{\mathbb{R},d}$ for each regulators' scenario $\mathbb{R} \in \mathcal{R}$, some limit $b_{\mathbb{S},d}$ for each shareholders' scenario $\mathbb{S} \in \mathcal{S}$, in such a way that

$$K = K_1 + \cdots + K_D, \quad \text{for each } \mathbb{R},\ \rho_{\mathbb{R},1} + \cdots + \rho_{\mathbb{R},D} = 0,$$
$$\text{and for each } \mathbb{S},\ b_{\mathbb{S},1} + \cdots + b_{\mathbb{S},D} = b_{\mathbb{S}}.$$

Desk d' problem, *if* it stands alone, is then

(4.1 DA)
$$\inf_{x_{d,i}, 1 \le i \le n_d} - \sum_{1 \le i \le n_d} x_{d,i} \mathbf{E}_{\mathbb{P}} [X_{d,i}]$$
$$\text{such that for each } \mathbb{R} \in \mathcal{R},\ \sum_{1 \le i \le n_d} x_{d,i} \mathbf{E}_{\mathbb{R}} [X_{d,i}] \ge \rho_{\mathbb{R},d},$$
$$\text{for each } \mathbb{S} \in \mathcal{S},\ \sum_{1 \le i \le n_d} x_{d,i} \mathbf{E}_{\mathbb{S}} [X_{d,i}] \ge b_{\mathbb{S},d}, \text{ and } \sum_{1 \le i \le n_d} x_{d,i} \pi_0(X_{d,i}) = K_d.$$

4.2 The Primal Problems of the Desks, with Tentative Internal Markets.

Let us suppose that desks are allowed to trade among themselves their initial capitals as well as amounts by which they may overshoot or undershoot their respective limits under the various scenarios under consideration: this allows them access to a wider range of individually feasible portfolios, under some global market clearing conditions. Terms of trades are taken tentatively at some values noted k, $l_{\mathbb{R}}$, $m_{\mathbb{S}}$.

The problem of desk d is

(4.2 DT)
$$\inf_{x_{d,i}, 1 \le i \le n_d, \delta_{\mathbb{R},d}, \mathbb{R} \in \mathcal{R}, \varepsilon_{\mathbb{S},d}, \mathbb{S} \in \mathcal{S}, k_d} - \sum_{1 \le i \le n_d} x_{d,i} \mathbf{E}_{\mathbb{P}} [X_{d,i}] + \sum_{\mathbb{R} \in \mathcal{R}} \delta_{\mathbb{R},d} l_{\mathbb{R}} + \sum_{\mathbb{S} \in \mathcal{S}} \varepsilon_{\mathbb{S},d} m_{\mathbb{S}} + k_d k$$
$$\text{such that for each } \mathbb{R} \in \mathcal{R},\ \sum_{1 \le i \le n_d} x_{d,i} \mathbf{E}_{\mathbb{R}} [X_{d,i}] + \delta_{\mathbb{R},d} \ge \rho_{\mathbb{R},d},$$
$$\text{for each } \mathbb{S} \in \mathcal{S},\ \sum_{1 \le i \le n_d} x_{d,i} \mathbf{E}_{\mathbb{S}} [X_{d,i}] + \varepsilon_{\mathbb{S},d} \ge b_{\mathbb{S},d},$$
$$\text{under the new budget constraint } \sum_{1 \le i \le n_d} x_{d,i} \pi_0(X_{d,i}) = K_d + k_d.$$

Moreover, the study of individual desks' problems, is only relevant if internal markets clear, i.e. under the conditions that

(4.2 MCR)
$$\text{for each } \mathbb{R} \in \mathcal{R},\ \sum_{1 \le d \le D} \delta_{\mathbb{R},d} = 0,$$

(4.2 MCS)
$$\text{for each } \mathbb{S} \in \mathcal{S},\ \sum_{1 \le d \le D} \varepsilon_{\mathbb{S},d} = 0,$$

(4.2 MCK)
$$\text{and } \sum_{1 \le d \le D} k_d = 0.$$

4.3 The Duals of the Tentative Desk Problems with Internal Trades.

Problem (4.2 DT) with number d, $1 \leq d \leq D$, parametrized as in Section 4.2 by k, $l_{\mathbb{R}}$, $m_{\mathbb{S}}$, has the dual

$$\text{(4.3 DT')} \qquad \sup_{\lambda_{\mathbb{R},d},\, \mathbb{R}\in\mathcal{R},\, \mu_{\mathbb{S},d},\, \mathbb{S}\in\mathcal{S},\, \phi_d} \sum_{\mathbb{R}\in\mathcal{R}} \lambda_{\mathbb{R},d}\rho_{\mathbb{R},d} + \sum_{\mathbb{S}\in\mathcal{S}} \mu_{\mathbb{S},d} b_{\mathbb{S},d} + \phi_d K_d$$

with all variables, except ϕ_d, positive and with the equality constraints

$$\text{for each } i,\ 1 \leq i \leq n_d,$$

$$\sum_{\mathbb{R}\in\mathcal{R}} \lambda_{\mathbb{R},d}\mathbf{E}_{\mathbb{R}}\left[X_{d,i}\right] + \sum_{\mathbb{S}\in\mathcal{S}} \mu_{\mathbb{S},d}\mathbf{E}_{\mathbb{S}}\left[X_{d,i}\right] + \phi_d \pi_0(X_{d,i}) = -\mathbf{E}_{\mathbb{P}}\left[X_{d,i}\right]$$

and for each $\mathbb{R} \in \mathcal{R}$ and each $\mathbb{S} \in \mathcal{S}$, $\lambda_{\mathbb{R},d} = l_{\mathbb{R}}$, $\mu_{\mathbb{S},d} = m_{\mathbb{S}}$, and $\phi_d = -k$.

5. Global and Decentralized Solutions to the Firm's Portfolio Selection Problem

The market clearing conditions imposed upon the problems (4.2 DT) ensure that adding the securities chosen by the desks in feasible solutions, gives a feasible solution of the firm's problem (3.1 F). We have conversely the following result.

Proposition 1. *If the values of the parameters k, $l_{\mathbb{R}}$, $m_{\mathbb{S}}$ common to each desk's primal and dual problem with internal trades, are chosen so as to be a feasible solution of the firm's dual problem (3.2 F'), then they are also a common feasible solution for each desk's dual problem (4.3 DT'). Under the same assumption, each feasible solution of the firm's primal problem is the sum of portfolios which are part of feasible solutions of the desks' primal problems (4.2 DT). These solution, taken together, satisfy the market clearing conditions (4.2 MCR, MCS, MCK).*

Proof of Proposition 1. (1) Choosing for each d, each $\mathbb{R}$, and each $\mathbb{S}$, the values $\lambda_{\mathbb{R},d} = l_{\mathbb{R}} = \lambda_{\mathbb{R}}$, $\mu_{\mathbb{S},d} = m_{\mathbb{S}} = \mu_{\mathbb{S}}$, $\phi_d = -k = \phi$, where the $\lambda_{\mathbb{R}}$, $\mu_{\mathbb{S}}$, ϕ are a feasible solution of (3.2 F'), allows one to guarantee the constraints of each (4.3 DT') out of the validity of the constraints in (3.2 F').

(2) Given a feasible solution $x^{d,i}$, $1 \leq d \leq D$, $1 \leq i \leq n_d$ of the firm's primal problem (3.1 F), we shall construct a feasible solution for each primal problem (4.2 DT) by choosing first $x_{d,i} = x^{d,i}$. This will ensure that the desks' different portfolios add up to the portfolio feasible for the firm.

(3) For each binding scenario (of type $\mathbb{R}$ or of type $\mathbb{S}$) constraint of (3.1 F), we chose for each d a value of $\lambda_{\mathbb{R},d}$ or $\mu_{\mathbb{S},d}$ making the corresponding scenario constraint of (4.2 DT) an equality.

(4) Finally, for each d, the value of k_d is chosen so as to validate the budget constraint in (4.2 DT).

(5) The market clearing conditions (4.2 MCR) and (4.2 MCS) are satisfied as a result of the decomposition of capital and limits imposed in Section 4.1. "Walras' Law" ensures that the aggregate internal demand for capital is zero, that is (4.2 MCK).

Proposition 2. *If, in Proposition 1, the feasible solution for the firm's dual problem (3.2 F') is optimal, it is also optimal for each dual (4.3 DT'). If, moreover, the feasible solution for the firm's primal (3.1 F) is optimal, then the solutions built in the proof of Proposition 1 are optimal for each primal (4.2 DT) respectively.*

Proof of Proposition 2. (1) We prove the optimality of the duals' common feasible solution by applying the complementary slackness result to it and to the various primals' feasible solutions constructed in Proposition 1. All dual problems (4.3.DT') having only equality conditions, the complementary slackness conditions just have to be verified for the non-binding constraints in the *primal* problems (4.2 DT).

(2) Point (3) in the proof of Proposition 1 makes sure that, when a scenario-linked constraint is non-binding in *one* of the primals, it is also non-binding in the firm's primal. By optimality of $x^{d,i}$, $1 \leq d \leq D$, $1 \leq i \leq n_d$ in (3.2 F'), it is known that the corresponding dual variable is zero for the firm, hence for *all* the desks.

Parametrizing the problems (4.2 DT) with an optimal solution of the dual problem (3.2 F') of the firm, we conclude, by Proposition 2, that the optimal value of (3.1 F) is not larger than the sum of the optimal values of the problems (4.2 DT). Therefore, adding the securities chosen by the desks in optimal solutions of the problems (4.2 DT), gives an optimal solution of the firm's problem (3.1 F). Internal trading, at the correct prices, has allowed the desks to reach an optimal portfolio for the firm. The decomposition principle of [DW] should allow to reach these correct prices in a finite number of iterations.

6. Coherent Risk Capital Assignments

While *measuring* individual risks via scenarios is done on a risk by risk basis and has the subadditivity property built-in, *accounting* for the specific risks within the firm, must be made in a way additive with respect to these risks, depending on the firm's global portfolio. The capital allocated for a risk should not depend on which desk or set of desks carry it, but only on the risk and its complement in the total portfolio (see [Be], Chapters 16-19, [Do], Chapter 11, for a non-mathematical presentation of the problem).

We reserve for another paper the study of the actual assignments of capital resulting from the internal trading of the preceding Sections. We study the concept of coherent risk capital assignment independently of questions of optimality, in two examples.

Definition. *Given the regulators' as well as the sharehoders' acceptance sets $\mathcal{A}$ and $\mathcal{B}$, a coherent risk capital accounting/assignment system for the firm is a mapping Ψ from $I\!R^\Omega \times I\!R^\Omega$ into the reals. The number $\Psi_X(Y) = \Psi(X,Y)$ is the amount of risk capital charged/assigned to any desk carrying position Y, when the firm's global position is X. The following properties are required*

(1) *for each $X \in I\!R^\Omega$, $\Psi_X(X) = \rho_{\mathcal{A},\mathcal{B}}(X)$.*
(2) *for each $X \in I\!R^\Omega$, the function Ψ_X is additive and smaller than the function $\rho_{\mathcal{A},\mathcal{B}}$.*

Remark 1. The second condition has the flavor of game theoretic concepts used in [BHR]. It has been early advocated by the second author of the present paper, to ensure that no group of desks is accounted for more capital than the risk measure of the position it is taking. It is also one formalization of the benefits of diversification.

Remark 2. Some risk measurement systems allow for a negative measure of certain "risks". The second condition above requires then, that a desk with a negative measure of risk be accounted for a negative risk capital. The "covariance principle" for allocating capital, defined by $\Psi(X,Y) = \rho(X) \cdot \dfrac{cov(X,Y)}{var(X)}$, does not satisfy this for a general risk measure ρ.

We define a coherent risk capital accounting/assignment system Ψ by the following: for each portfolio X of the firm, let $\mathbb{T}^*_X$ be one scenario in $\mathcal{A} \cup \mathcal{B}$ which maximizes $\mathbf{E}_{\mathbb{T}}[-X]$ for $\mathbb{T}$ in $\mathcal{A} \cup \mathcal{B}$. Then $\Psi(X,Y) = \Psi_X(Y)$ is defined as $\mathbf{E}_{\mathbb{T}^*_X}[-Y]$. It is clear indeed that $\Psi(X,Y) \leq \sup_{\mathbb{T} \in \mathcal{A} \cup \mathcal{B}} \{\mathbf{E}_{\mathbb{T}}[-Y]\}$.

The "expected shortfall allocation principle" introduced in [S], together with the use of the (coherent) individual risk measure given by the "worst conditional expectation (see [ADEH], Sections 3.3 and 5.1,) provides another example of coherent capital assignment, at elast under the assumption that "TailVaR" and "Worst conditional expectation" are equal (see [ADEH], Proposition 5.3), as shown by the scenario representation of the latter measure.

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