Subordinated Stock Price Models: Heavy Tails and Long–Range Dependence in the High–frequency Deutsche Bank Price Record

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Abstract

Following a previous study on subordinated exchange rate models, we investigate the main properties of the high–frequency Deutsche Bank price record in the setting of stochastic subordination and stable modeling, focusing on heavy–tailedness and long memory, together with their dependence on the sampling period. We find that the market time process has increments described well by Gamma distributions, and that the log price process in intrinsic time can be approximated, at different time scales, by α–stable Lévy processes. Most importantly, long–range dependence with strong intensity is present in the market time process, with an estimated Hurst index $H \approx 0.9$. Finally, the stable domain of attraction offers a good fit of the returns in physical time, which display weak long memory. As a consequence, we propose as a realistic model for the stock prices a process $Z(t)$ subordinated to an α-stable Lévy motion $S(t)$ by a long–memory intrinsic time process $T(t)$ with Gamma–distributed increments.

1 Introduction

As a follow–up paper to our study Marinelli, Rachev and Roll (1999), we investigate the probabilistic structure of the Deutsche Bank price record at very short time scales (high–frequency data) in the setting of stochastic subordination[1], i.e. we model the log price process in physical time as a compound process $Z(t) = S(T(t))$, where $S(t)$ is a stochastic process indexed on a stochastically "deformed" time scale represented by a process $T(t)$. In this setting, $T(t)$ models the market activity, which changes over time, as it is well known, and $S(t)$ is the stock (log) price process in market time.

We focus on the properties of heavy tailedness and long–range dependence[2], studying their behavior with respect to the time lag, for all the processes involved in the setting of subordination.

From the statistical study of the observed increments of the market time process, we find empirical evidence of strong long memory in the underlying process $T(t)$, while the returns in intrinsic time, i.e. the increments of $S(t)$, appear not to posses long memory. These features of $T(t)$ and $S(t)$ characterized the corresponding processes for USD–CHF exchange rates, where we found that long memory in $S(t)$, if present, had very low intensity. Moreover, it results from our data set that intra–daily returns are heavy tailed, and appear to be in the domain of attraction of a stable law[3] with characteristic exponent $\alpha < 1.5$. We also find evidence, although not conclusive, of the presence of long–range dependence structures of weak intensity in the return process in physical time.

The outcome of our analysis for stock prices is consistent with our previous results on exchange rates. In fact, we shall see that the three processes $T(t)$, $S(t)$ and $Z(t)$ for USD–CHF exchange rate and Deutsche Bank stock price share most of their features.

Based on our empirical findings, we introduce a model of the temporal evolution of stock prices based on the hypothesis that the log price process is driven by a stable process[4] in the market time, subordinated by a long–range dependent stochastic process (indexed on the physical time), with Gamma–distributed increments.

The presented empirical findings show that asset returns are in a completely different class than that predicted by the classical model driven by geometric

[1]See Bertoin (1998) for an introduction to the theory of subordinated processes, and Hurst, Platen and Rachev (1997, 1999) for a discussion of their adoption in financial modeling.

[2]See Beran (1994) for an introduction to the theory of long–range dependence and for a survey of the available statistical methods.

[3]For the definition and basic properties of stable laws, we refer to Samorodnitsky and Taqqu (1994). A survey of their applications in financial modeling can be found in Mittnik, Rachev and Paolella (1998).

[4]See Samorodnitsky and Taqqu (1994) for a treatise on stable processes.

Brownian motion: most notably, returns are heavy tailed and long–range dependent. As a consequence, derivative pricing and other theories based on distributional assumptions, like value–at–risk and portfolio theory, have to be reviewed to take into account the higher risk involved with returns strongly deviating from gaussianity.

2 The data set

We consider the high–frequency Deutsche Bank price record registered at the Frankfurt Stock Exchange from May 1, 1996 to April 30, 1998. The data set consists of 66255 observations for a total of 498 business days, so we have an average of around 133 ticks per day. In the period of our records, the business hours of the Frankfurt Stock Market were from 10:30am to 1:30pm, while they were extended to 8:30am to 4:30pm only on July 1, 1998. We have therefore approximately 44 quotes an hour, or equivalently one quote every 1.35 minutes. Figure 1 displays the empirical paths of the processes $S(t)$, $T(t)$ and $Z(t)$ for our data set.

We have obtained this data set from the Karlsruher Kapitalmarktdatenbank, which collects a great variety of financial data for the German market.[5]

3 Statistical Analysis of the Market Time Process

We define the market time (also called *trading time* or *intrinsic time*) as

$$T(t) = \{\text{Number of transactions up to time } t\},$$

and we model it as an increasing $\mathbb{R}_0^+$–valued stochastic process defined on $\mathbb{R}_0^+$, and such that $T(0) = 0$. The trading time can be considered as a measure of market activity, and it is introduced to explain the empirical observation that price changes are higher when market activity is more intense. For further details on the application of stochastic subordination to the modeling of financial prices, see Clark (1973), Ghysels, Gouriéroux and Jasiak (1995), and Hurst, Platen and Rachev (1997, 1999). Mandelbrot, Fisher and Calvet (1997) have proposed recently, in the context of their Multifractal Model of Asset Returns, to model the market time with the cumulative distribution function of a multifractal measure.

The properties of the market time in the study of high–frequency financial data are closely related to the notions of *quote time* (see Evertsz (1995)) and of *tick frequency* (see Guillaume et al. (1997)). Quote time consists in the

[5]See Appendix A for more details on the format of the data, and Lüdecke (1997) for informations on the data bank.

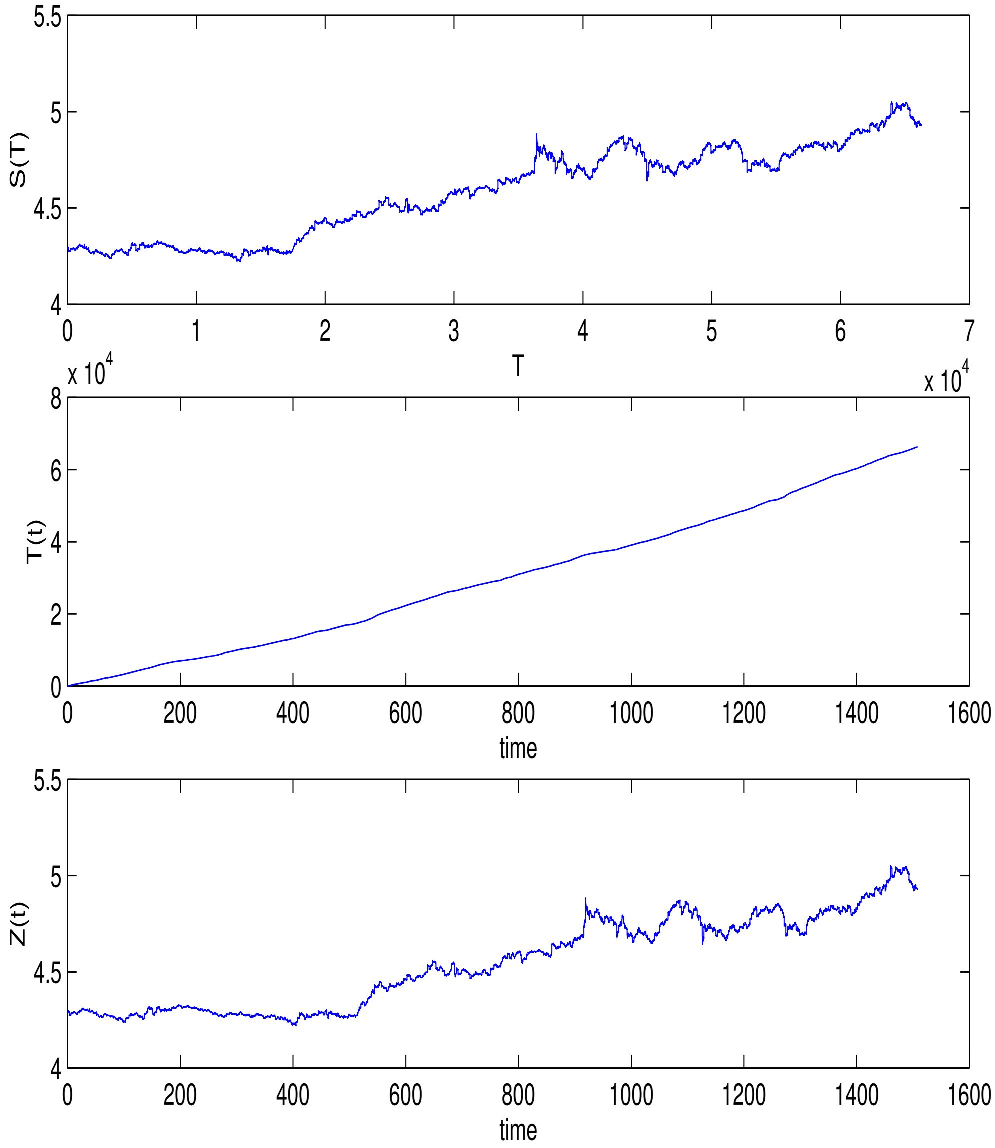

Figure 1: Tick–by–tick logarithmic price levels, trace of the trading time $T(t)$, and log price levels in physical time for the Deutsche Bank stock, from top to bottom. Times are expressed in hours.

mapping of the times $\{t_i\}_{i=1,\dots,N}$ at which the stock price is registered to the set $\{1,\dots,N\}$, i.e. prices in quote time coincide, in our settings, with prices in intrinsic time. Tick frequency at time t is defined as

$$f(t;\Delta t) = \frac{1}{\Delta t}\mathrm{card}\Big\{x(t_j)\,|\,t_j \in (t-\Delta t, t]\Big\},$$

where t_j are the times at which the quotes are registered, and $x(t_j)$ are the corresponding prices. The tick frequency is often considered a proxy to the transaction volume. In the setting of subordination, it holds

$$f(t;\Delta t) = \frac{T(t)-T(t-\Delta t)}{\Delta t}.$$

In this section we investigate the probabilistic structure of the increments of the trading time process $T(t)$ for different lags, and we look for the presence of long–range dependence structures.

3.1 Probabilistic structure

A natural choice of underlying distributions for the increments of the intrinsic time process $T(t)$ is the family of infinitely divisible distributions.[6] Here we consider only exponential, Gamma, lognormal laws and stable subordinators (i.e. maximally skewed α-stable laws with $\alpha < 1$), together with Weibull law (which is not infinitely divisible, but as we will see, provides a good fit).

For increments of $T(t)$ with a sampling period $\Delta t = 2,\ 5,\ 15,\ 30$ minutes, 1 hour and 1 day, i.e. for

$$\Delta T(t) = T(t) - T(t-\Delta t),$$

we obtain the following descriptive statistical data

	2 min	5 min	15 min	30 min	1 h	1 day
$\hat{\mu}$	1.4644	3.6612	10.9836	21.9671	43.9343	133.0422
$\hat{\sigma}$	1.0060	2.0881	5.1872	9.4774	17.4650	46.9927
$\hat{\beta}$	1.1231	0.8723	0.7652	0.7684	0.7332	0.6021
$\hat{\kappa}$	4.4089	3.9521	3.8329	4.0429	4.2443	4.0642

where μ is the mean, σ the standard deviation, β the skewness, and κ the kurtosis. Recall that β is a measure of the asymmetry of the distribution, and κ of the tail thickness. Note that the value of the sample kurtosis $\hat{\kappa}$ suggests

[6]For the definition and basic properties of infinitely divisible laws, see Feller (1971), Section 6.3 or Janicki and Weron (1994), Section 4.4.

that the increments of $T(t)$ have light tails. In fact, this empirical observation is supported, as we shall see, by the likelihood ratio test, which indicates as best fit of the increments at different time lags a distribution with finite moments of all orders.

Table 1 shows the values of the parameter estimates and of the logarithmic likelihood for all chosen fits of the 2–minute increments. Moreover, we report in Table 2 the likelihood values ($-\log \mathcal{L}$) of all parametrizations for time lags of 2, 5, 15, 30 minutes and 1 hour, while full tables with parameter estimates can be found in the Appendix. Note that the Gamma distribution offers the best fit in all cases, except for $\Delta t = 5$ and 10 minutes, for which the Weibull distribution performs slightly better. Moreover, the Weibull distribution proves to be inadequate as a model for daily increments of $T(t)$. In fact, its likelihood value "explodes", a phenomenon that we already encountered in the fit of weekly increments of $T(t)$ for the USD–CHF exchange rate (see Marinelli, Rachev and Roll (1999)).

Figure 2 displays the empirical PDFs of the increments of $T(t)$ and their best two fits (i.e., Gamma and Weibull) for sampling periods from 2 to 30 minutes.

The probabilistic structure of the increments of $T(t)$, as we have already pointed out, is in agreement with the low value of their sample kurtosis, since both Gamma and Weibull laws admit finite moments of every order. However, in contrast with the Gamma distribution, Weibull law unfortunately does not belong to the class of infinitely divisible distributions.

Moreover, due to the fact that the empirical kurtosis of the increments of $T(t)$ is very low, and that the family of stable subordinators performs poorly in comparison to the other fits, we are lead to exclude an underlyng distribution with heavy tails.

It is evident, from both the likelihood scores and the empirical fits, how the Gamma law consistently provides an accurate fit. In our previous study on exchange rates, we found that the Weibul distribution provides the best fit, with Gamma law approaching the Weibull likelihood score with increasing time lags.

The whole picture is then complicated by the fact that, as we show in the following, the increments of $T(t)$ cannot be assumed to be independent. In fact, both classical R/S analysis and Lo's modified R/S statistics favor the hypothesis that long–range dependence structures are present in the market time process $T(t)$.

Distribution	Parameter estimates	$-\log \mathcal{L}$
Exponential	$\hat{\alpha} = 1.464434$	$6.2500 \cdot 10^4$
Gamma	$\hat{a} = 2.073898$ $\hat{b} = 0.706131$	$5.6772 \cdot 10^4$
Lognormal	$\hat{\mu} = 0.121410$ $\hat{\sigma} = 0.774806$	$5.8144 \cdot 10^4$
Stable	$\hat{\alpha} = 0.9993$ $\hat{\beta} = 1$ $\hat{\sigma} = 0.4684$ $\hat{\mu} = -399.326$	$6.1146 \cdot 10^4$
Weibull	$\hat{a} = 0.475704$ $\hat{b} = 1.519623$	$5.7065 \cdot 10^4$

Table 1: Fits of the 2–minute increments of the market time process $T(t)$

Distribution	2 m	3 m	5 m	10 m	15 m	30 m	1 h
Exponential	62500	53895	41580	27061	20487	12334	7212
Gamma	56772	48989	37554	24163	18205	10937	6399
Lognormal	58144	50421	38668	24709	18529	11081	6453
Stable	61146	–	–	–	–	–	–
Weibull	57065	49006	37463	24138	18212	10964	6421

Table 2: Negative log likelihood values for the fits of the increments of $T(t)$ at time lags up to 1 hour.

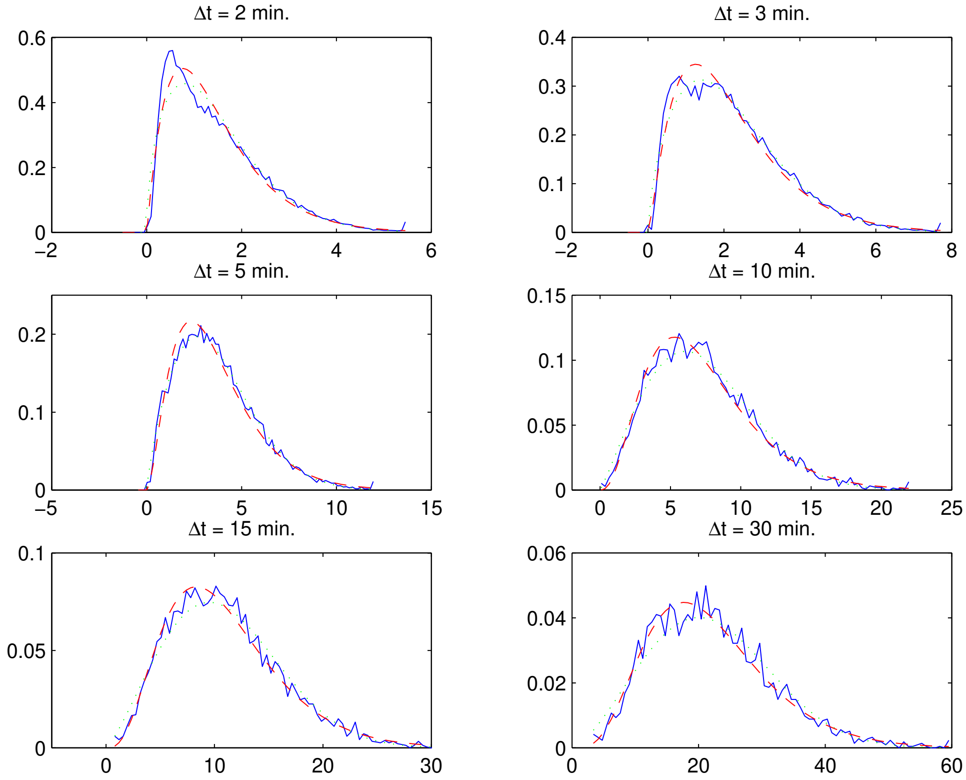

Figure 2: Empirical PDF (solid) of the increments of the market times process $T(t)$ for different time lags, together with Gamma (dashed) and Weibull fit (dotted).

3.2 Long–range dependence in $T(t)$

Classical R/S analysis applied to the increments of $T(t)$ strongly evidences the presence of long memory in the data. In fact, the estimated Hurst index H belongs to the range $0.8 - 0.9$ for all considered time lags. In the following table, we summarize our findings:

	2 m	3 m	4 m	5 m	10 m	15 m	30 m	1 h	1 d
H	0.89	0.89	0.88	0.89	0.88	0.87	0.86	0.84	0.82

Note that the intra–hourly estimates of H are almost equal, while they tend to slightly decrease at lower frequencies (i.e., for longer sampling periods), as it is reasonable to expect. Since R/S analysis is sensitive to the presence of short–

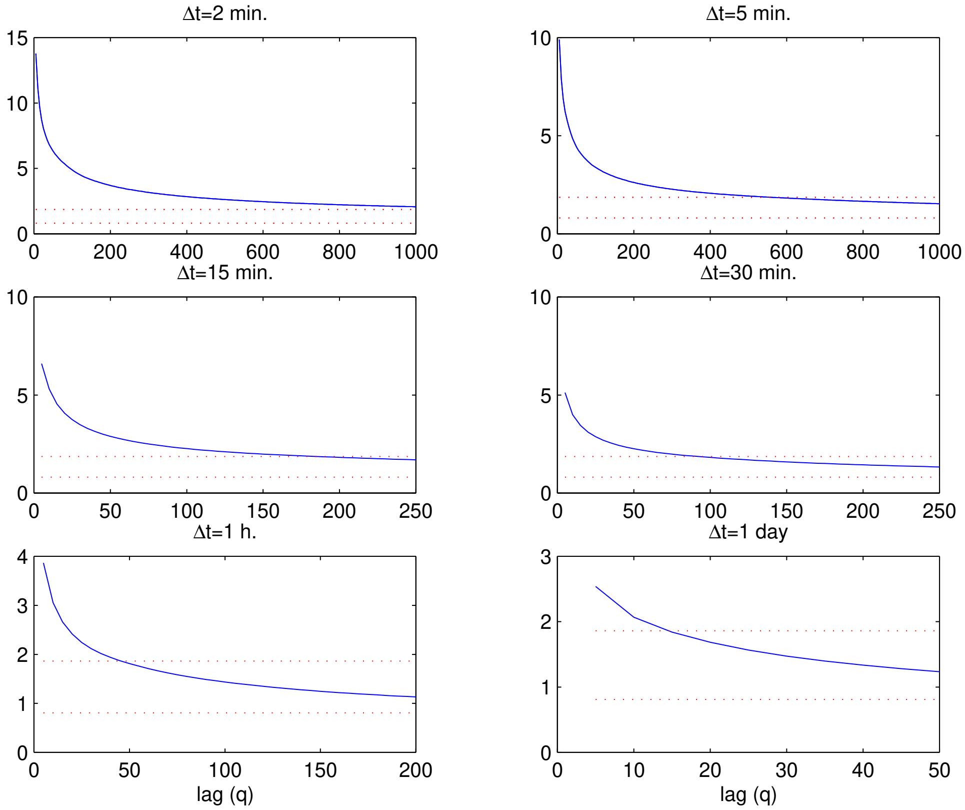

Figure 3: Graphs of the Lo's statistic $V_N(q)$ against q (the lag) for the increments of $T(t)$ at different sampling periods. Between dotted lines is the 95% confidence region of Lo's test.

range dependence, the small variation in H might also be seen as a "weakening" of the effects of short memory as the time lags increases.

We have also applied Lo's test to the increments of $T(t)$, and a plot of the modified R/S statistic $V_N(q)$ against q for different choices of the sampling period is given in Figure 3. The values of the lag q beyond which $V_N(q)$ enters the confidence region of Lo's test are quite large compared to the sizes of the corresponding time series, so this behavior is not to be considered in contradiction with the estimates of H obtained through classical R/S analysis. For more details, see Lo (1991) and Teverovsky, Taqqu and Willinger (1998, 1999).

The plots also show a behavior of $V_N(q)$ consistent with that of fractional ARIMA (see Teverovsky, Taqqu and Willinger (1998)), which suggests, at a

Lag	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\mu}$
1	1.55	$-9.91 \cdot 10^{-2}$	$3.64 \cdot 10^{-4}$	$-2.66 \cdot 10^{-5}$
2	1.54	$6.78 \cdot 10^{-2}$	$5.09 \cdot 10^{-4}$	$1.58 \cdot 10^{-5}$
4	1.46	$6.48 \cdot 10^{-2}$	$7.01 \cdot 10^{-4}$	$2.05 \cdot 10^{-5}$
8	1.31	$7.24 \cdot 10^{-2}$	$9.98 \cdot 10^{-4}$	$7.81 \cdot 10^{-5}$
16	1.22	$8.67 \cdot 10^{-2}$	$1.47 \cdot 10^{-3}$	$2.96 \cdot 10^{-4}$

Table 3: Stable fits of the increments of $S(t)$ for different sampling periods

level of qualitative analysis, that the increments of $T(t)$ could possibly be generated by a FARIMA process with Gamma distributed innovations.

4 Statistical Analysis of the tick–by–tick Price Process

In this section we study the probabilistic structure at different time scales of the price process in intrinsic time $S(t)$. Recall that, having defined $T(t)$ as the number of transactions up to time t, then the tick–by–tick price record can be viewed as a sample path of the process $S(t)$.

4.1 Probabilistic Structure

The 1–quote returns (i.e., in our setting, the increments of $S(t)$ with unit sampling period) are clearly not drawn from a stable random variable, as it can be inferred by the shape of their empirical PDF displayed in Figure 4. One should also note that kernel probability density estimation has smoothing effects that could hide multimodality in the underlying distribution, with two local maxima besides the central peak of the PDF.

The stable law offers a reasonably good fit of the central portion of the density, but unfortunately it cannot fully explain the shape of the empirical PDF, and of a possible multimodality: in fact, the stable distribution is unimodal (for a proof, see Yamazato (1978) and Zolotarev (1986)).

The estimated index of stability α is fairly "stationary" for sampling periods up to 4 quotes, while it decreases to approximately 1.3 and 1.2 for sampling periods of 8 and 16 quotes. However, there is no reason to believe that α would be constant for tick–by–tick data at the shortest time scales, nor that it would decrease for longer sampling periods. Moreover, we observed a similar phenomenon in the study of the tick–by–tick returns of the USD–CHF exchange rate. The variability of the index of stability α might be due to the fact that

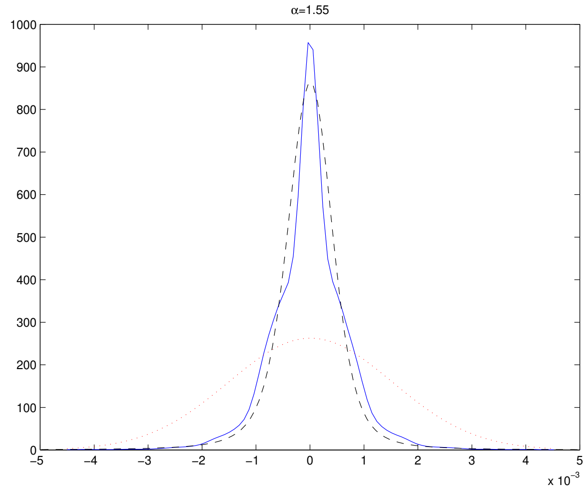

Figure 4: Empirical PDF (solid) of the 1–quote returns, together with stable (dashed) and normal fit (dotted).

the increments of the process $S(t)$ are in fact not independent, or that they are not identically distributed, or that the underlying distribution is not purely stable, but instead could be a mixture with, speaking loosely, a dominant stable component. We shall see, however, that long memory in $S(t)$, if present, is very weak. It is also interesting to note that the estimated skewness parameter β is very close to zero independently of the sampling period, meaning that, in the setting of stable modeling, the tick–by–tick returns are almost perfectly symmetrically distributed even at different time scales.

4.2 Long–range dependence

The estimated values of the Hurst index for the increments of $S(t)$ at all considered sampling periods are very close to the theoretical value $H_0 = 0.5$ distinguishing processes with no long memory: namely, they are in the range 0.52 – 0.53. Recall that the classical R/S method is robust with respect to heavy–tailed time series: in fact, $R(n)$ asymptotically behaves like $n^H = n^{d+1/\alpha}$, and $S(n)$ like $n^{1/\alpha-1/2}$, hence $R/S(n)$ asymptotically behaves like $n^{d+1/2}$, independently of the value of α. What we actually estimate with the classical R/S method is therefore $d+1/2$, and not the self–similarity index H.[7] Since $d=0$ distinguishes processes with no long memory, the obtained values suggest that long–range dependence, if at all present, has very weak intensity.

[7]See Mandelbrot and Taqqu (1979) and Taqqu and Teverovsky (1998).

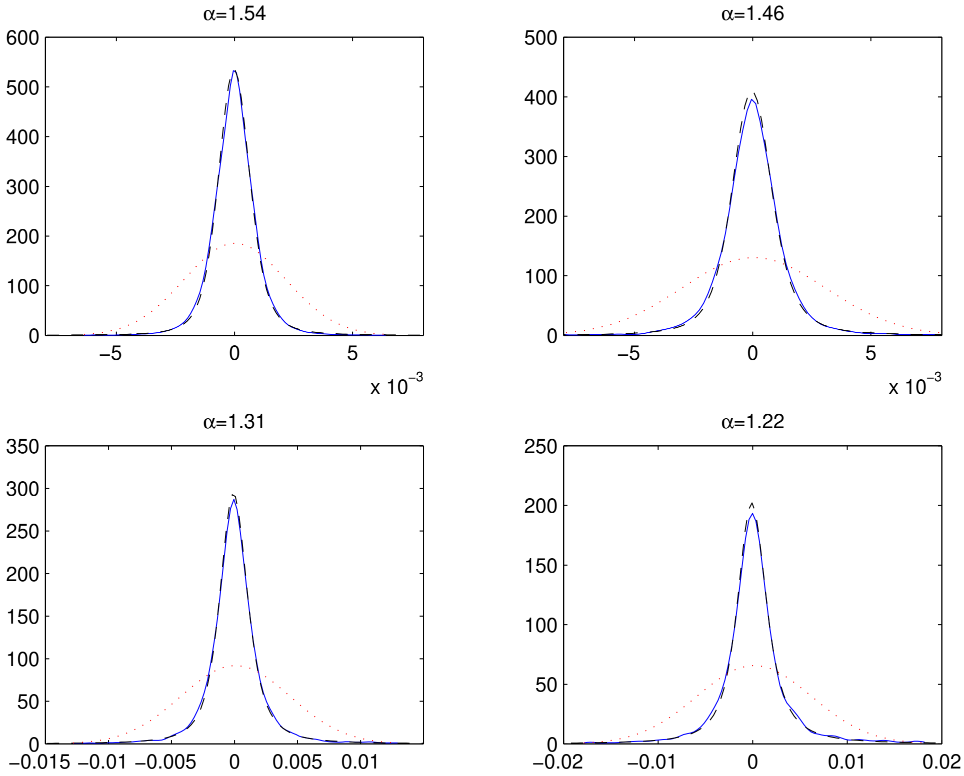

Figure 5: Empirical PDF (solid) of the 2, 4, 8 and 16–quote returns (in left–right, top–down order), together with stable (dashed) and normal fit (dotted).

Moreover, Lo's modified R/S statistic $V_N(q)$ falls inside the acceptance region for rejecting long memory for all values of the lag q, even for very small ones, as evidenced in Figures 6 and 7. However, since Lo's $V_N(q)$ statistics implicitly requires the existence of the second moment of the distribution generating the data, Lo's test might not be robust with respect to heavy–tailed time series, even if a small simulation study suggests that it provides useful results when applied to stable noise (see Marinelli, Rachev and Roll (1999)).

On the basis of the fact that stable laws offers a very good fit of the increments of $S(t)$, and that there is strong evidence for rejecting the hypothesis of long–range dependence in the underlying generating process, we are led to propose as a suitable model for the log price in market time, on a first level of approximation, an α–stable Lévy process. The clearest limitation of this model is that it does not allow the stability index α of the increments to change with

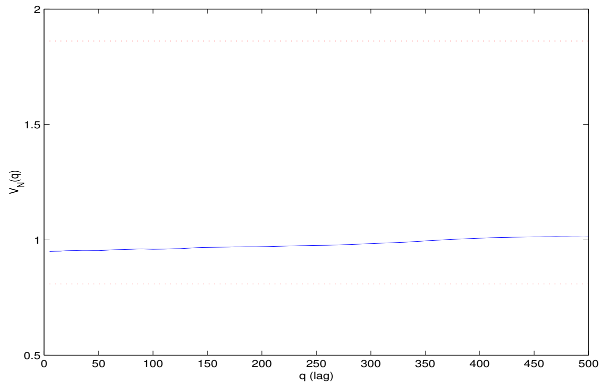

Figure 6: Lo's statistic $V_N(q)$ versus q for the 1–quote increments of the tick–by–tick price record.

respect to the time scale. A possible explanation of this behavior is that the there could be other components in the dynamics of tick–by–tick prices that "add up" to the stable component. Another suggestive hypothesis could be that the underlying process is in fact a GARCH process with stable innovations.[8]

5 Statistical Analysis of the Price Process in Physical Time

In this section we investigate what probabilistic structure have the increments of the compound process $Z(t) = S \circ T\,(t) = S(T(t))$, which describes the log price in physical time. Since we have seen that $S(t)$ and $T(t)$ are properly approximated by, respectively, an α–stable Lévy process and a process with Gamma distributed increments, we would expect the subordinated process $Z(t)$ to have increments in the domain of attraction of stable laws. In fact, a stable process subordinated to an intrinsic time process with Gamma distributed increments has ν–stable distributed increments (see Klebanov and Rachev (1996)). Moreover, if $S(t)_{t\geq 0}$ is an α–stable process and $T(t)$ has Weibull distributed increments, then one can prove easily that the subordinated process $Z(t) = S(T(t))$ has marginal distributions that are in the domain of attraction of an α–stable law (see also Marinelli (1999)).

[8]For an account of the theory of stable–GARCH processes and of their applications in financial modeling see Panorska, Rachev and Mittnik (1995) and Mittnik, Paolella and Rachev (1997).

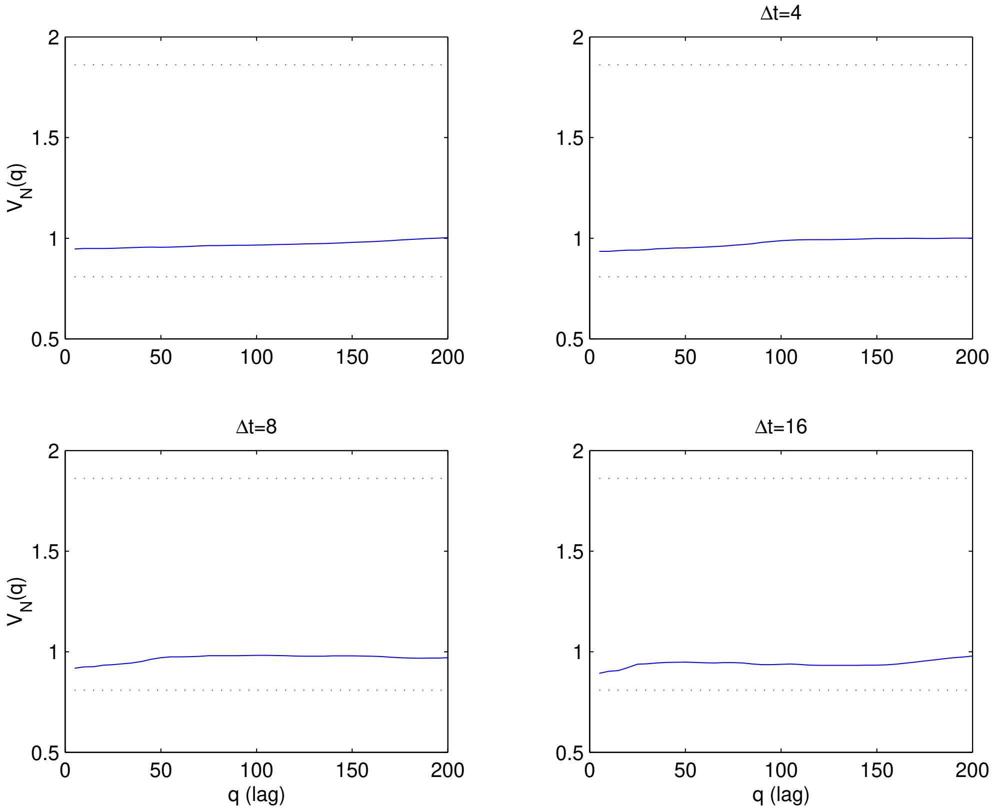

Figure 7: Lo's statistic $V_N(q)$ versus q for the increments of the tick–by–tick price record: sampling periods of 2, 4, 8 and 16 quotes, in left to right, top–down order.

5.1 Probabilistic structure

As expected, the stable domain of attraction offers a good description of the increments of the compound process $Z(t) = S(T(t))$, as evidenced by Figures 8 and 9.

The estimated index of stability is close to 1.3 for time lags of 2, 3, 4 and 5 minutes, while it decreases to about 1.2 for lags of 10 and 15 minutes, and to about 1.15 for lags of 30 minutes and 1 hour. Similarly to our analysis of USD–CHF exchange rates, the estimated indices of stability α of $S(t)$ and $Z(t)$ are different. We conjecture that the increase in tail thickness from $S(t)$ to $Z(t)$ is induced by the long–range dependence structures of the intrinsic time process $T(t)$. Moreover, we cannot exclude the presence of explicit dependece

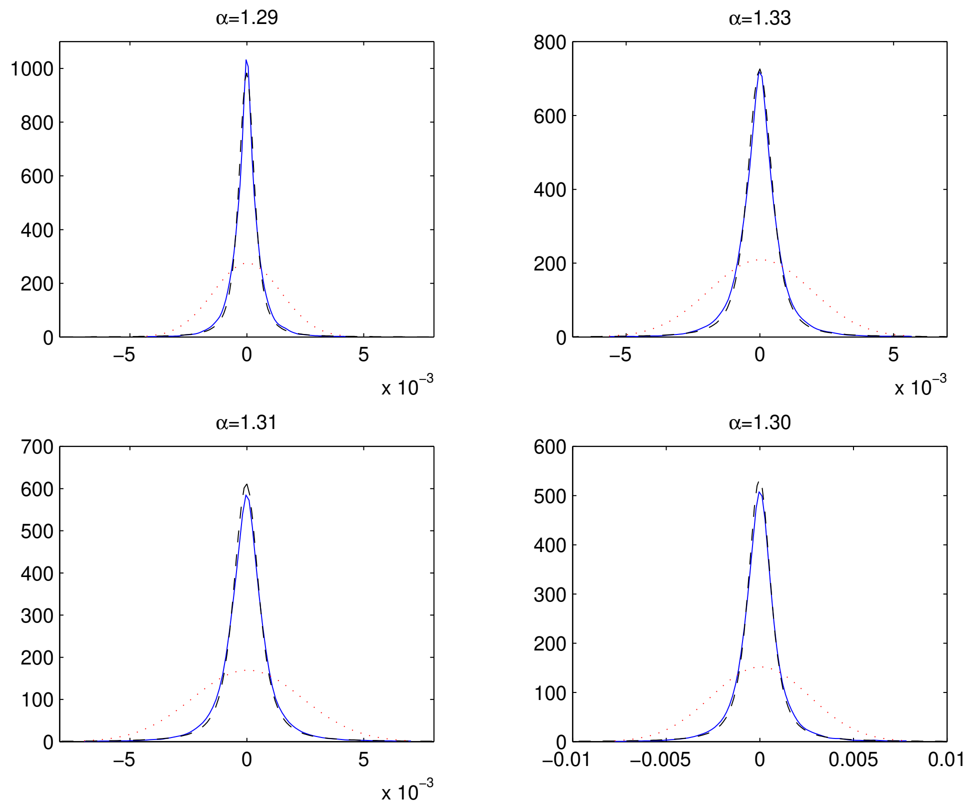

Figure 8: Empirical PDF (solid) of the 2, 3, 4 and 5–minute returns (in left–right, top–down order), together with stable (dashed) and normal fit (dotted).

between the processes $T(t)$ and $S(t)$, that in our setting are assumed to be independent.

5.2 Long–range dependence

The estimated Hurst index $H = d + 1/2$ is almost constantly equal to 0.58, and to 0.61 for daily returns. This indicates the possible presence of long–range dependence, although of weak intensity. On the other hand, Lo's test consistently favors the hypothesis of absence of long memory, as evidences by Figure 10. In essence, there is no conclusive evidence of presence or absence of dependece structures in the underlying generating process, but we can safely assert that long memory is weak, but stronger than that of $S(t)$. This is most likely due to the fact that $Z(t)$ "inherits" long range dependence structures from both $S(t)$ and $T(t)$, the latter of which was found to have strongly dependent

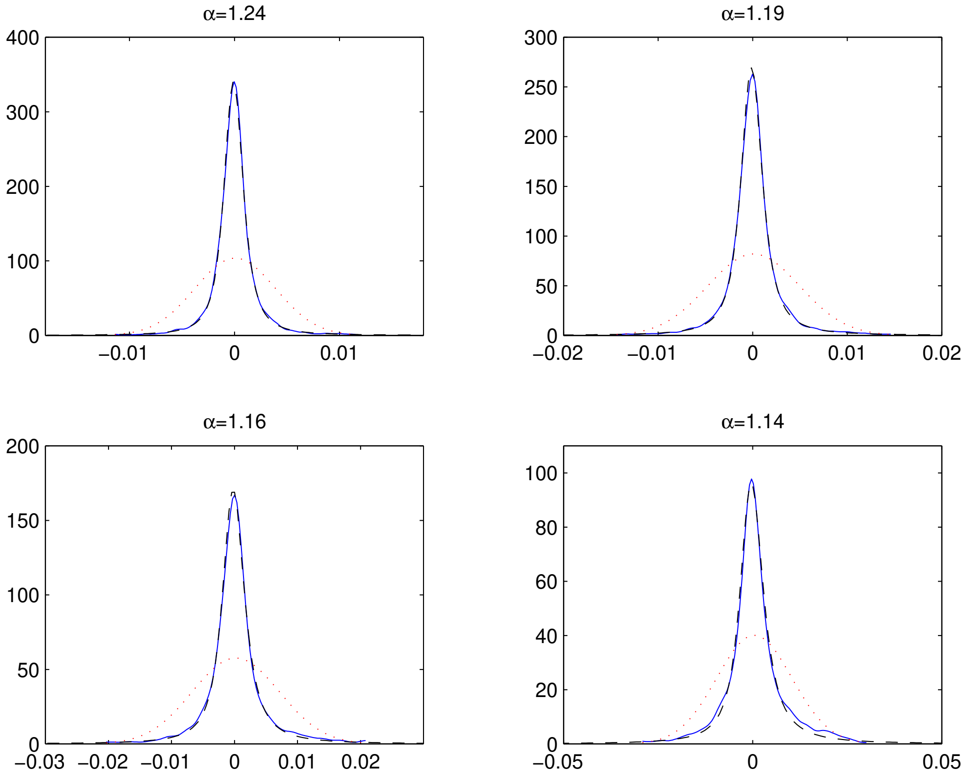

Figure 9: Empirical PDF (solid) of the 10, 15, 30 and 60–minute returns (in left–right, top–down order), together with stable (dashed) and normal fit (dotted).

increments.

6 Daily data

We collect in a separate section the results for daily data, since we have only about 500 samples, significantly less than the sizes of the other time series we have studied in this paper.

Note that the estimated index of stability $\alpha = 1.64$ for daily returns is greater than the estimates at shorter time scales, an expected behavior that was observed in our previous study on exchange rates as well.

The classical R/S estimate $H = 0.61$ (corresponding to $d = 0.11$), however, is in line with the results obtained at higher frequencies, showing that long–

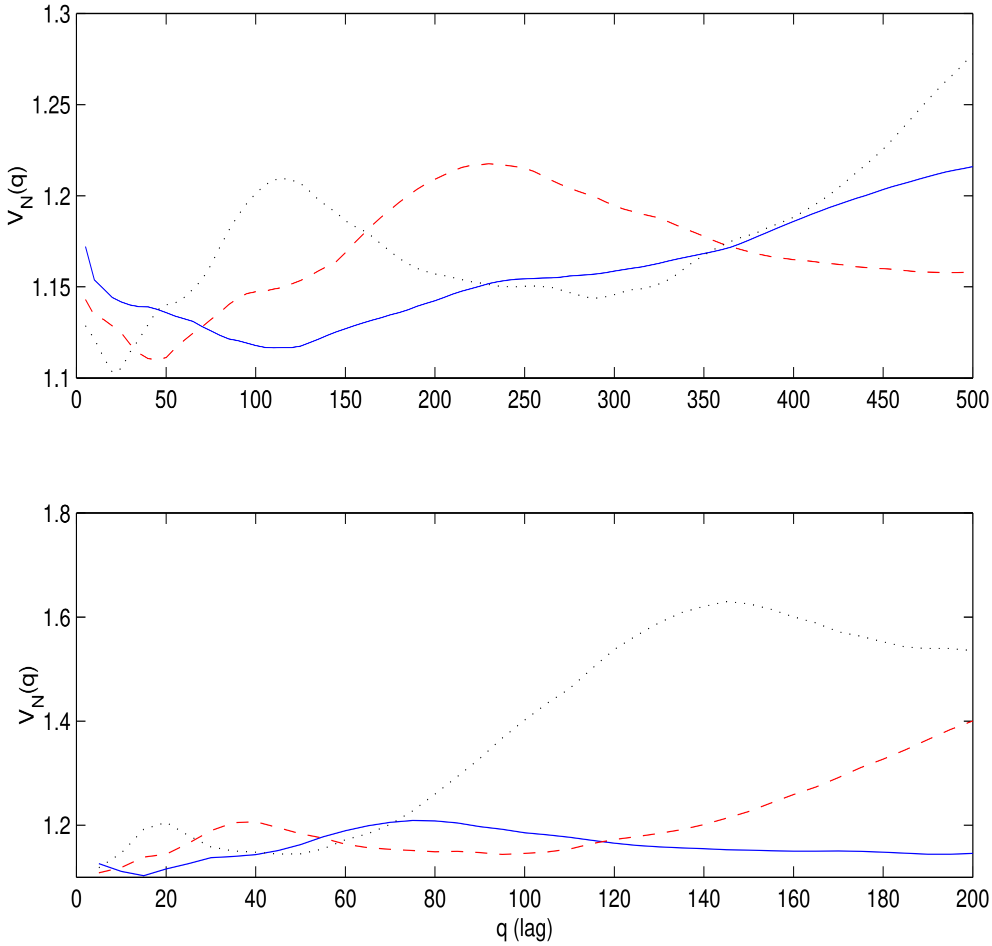

Figure 10: Lo's statistic $V_N(q)$ versus q for the increments of the price record in physical time $Z(t)$. In the top panel, for sampling periods of 2, 5, and 10 minutes (solid, dashed, and dotted respectively). In the bottom panel, for sampling periods of 15, 30, and 60 minutes (same graphical conventions).

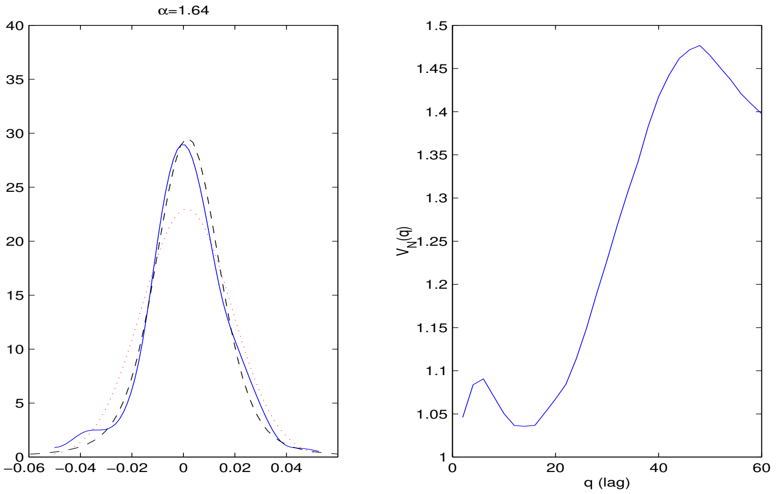

Figure 11: Left panel: empirical PDF of the daily increments of $Z(t)$, with stable (dashed) and normal fit (dotted). Right panel: Lo's statistic $V_N(q)$ against q for the the daily increments of $Z(t)$.

range dependence, even if of weak intensity, is persistent over different business days.

7 Conclusions

We have investigated the probability structure and the presence of long–range dependence in a data set of high–frequency financial data, proposing interpretations of the observed phenomena, in the setting of stochastic deformation of the physical time (stochastic subordination). We have shown that the trading time $T(t)$ has increments described very well by a Gamma distribution, and that there is strong empirical evidence for the presence of long–range dependence structures in the market time process. Moreover, stable laws offer very good fits of returns both in intrinsic time and physical time, for different sampling periods. While Lo's test categorically excludes the presence of long memory in the increments of $S(t)$, classical R/S analysis provides estimates of d very close to zero, hence practically rejecting hypothesis that long–range dependence is present in the tick–by–tick returns.

These empirical findings suggest that a reasonable model for the description of stock prices is an α-stable Lévy motion subordinated by a long–range dependent intrinsic time process with Hurst index $H \approx 0.9$ and Gamma distributed increments.

In our previous study of high–frequency exchange rates, we obtained very similar results: in particular, we found $S(t)$ to be well approximated by an α–stable process, possibly fractionally stable, and the market time $T(t)$ to be a process with strongly dependent increments, well fit by the Gamma distribution (although it was outperformed by the Weibull law). These characteristics seem therefore to be of general nature, and not only peculiar features of the time series we have considered. We have not given formal explanations of why, for instance, the intrinsic time $T(t)$ has strong long range dependent increments, but we conjecture that, under the point of view of stochastic subordinated models, heavy tailedness comes from the price process in intrinsic time, and long–range dependence has its origin in the market time process.

Moreover, models of the type we have discussed, which account for the presence of long–range dependence and heavy–tailed distributions in financial data, should be used as more realistic base for risk management (value–at–risk), derivative pricing, and portfolio theory.

Appendix

A The Karlsruher Kapitalmarktdatenbank

The tick-by-tick data set we study is part of the database called "Fortlaufende Kurse + Umsatzdaten", available from the Karlsruher Kapitalmarktdatenbank (KKMDB). More informations are available at the Internet URL

`http://finance.wiwi.uni-kalrsruhe.de/kkmdb`.

Each record in the database consists of 7 fields, i.e.

1. Security code (e.g., 804010 in the case of Deutsche Bank).
2. The date.
3. The stock market where the record was registered (in our case, it is always the Frankfurt Stock Exchange).
4. The transaction time, in the format hh:mm:ss.ss.
5. A flag, taking a value of either E, V or S, where E denotes an opening price, V a variable intra–day price, and S a closing price.
6. The transaction price in DEM, with an accuracy of 4 decimal digits.
7. The transaction volume.

B Tables

Distribution	Parameter estimates	$-\log \mathcal{L}$
Exponential	$\hat{\alpha} = 2.1967$	53896
Gamma	$\hat{a} = 2.3280$ $\hat{b} = 0.9436$	48989
Lognormal	$\hat{\mu} = 0.5570$ $\hat{\sigma} = 0.7377$	50421
Weibull	$\hat{a} = 0.2274$ $\hat{b} = 1.6444$	49006

Table 4: Fits of the 3–minute increments of the market time process $T(t)$

Distribution	Parameter estimates	$-\log \mathcal{L}$
Exponential	$\hat{\alpha} = 3.6611$	41581
Gamma	$\hat{a} = 2.7733$ $\hat{b} = 1.3201$	37555
Lognormal	$\hat{\mu} = 1.1068$ $\hat{\sigma} = 0.6778$	38668
Weibull	$\hat{a} = 0.07482$ $\hat{b} = 1.8296$	37463

Table 5: Fits of the 5–minute increments of the market time process $T(t)$

Distribution	Parameter estimates	$-\log \mathcal{L}$
Exponential	$\hat{\alpha} = 7.3224$	27062
Gamma	$\hat{a} = 3.5967$ $\hat{b} = 2.0358$	24163
Lognormal	$\hat{\mu} = 1.8455$ $\hat{\sigma} = 0.5866$	24710
Weibull	$\hat{a} = 0.01208$ $\hat{b} = 2.0898$	24138

Table 6: Fits of the 10–minute increments of the market time process $T(t)$

Distribution	Parameter estimates	$-\log \mathcal{L}$
Exponential	$\hat{\alpha} = 10.9836$	20487
Gamma	$\hat{a} = 4.1542$ $\hat{b} = 2.6439$	18205
Lognormal	$\hat{\mu} = 2.2712$ $\hat{\sigma} = 0.5389$	18529
Weibull	$\hat{a} = 3.554 \cdot 10^{-3}$ $\hat{b} = 2.2391$	18212

Table 7: Fits of the 15–minute increments of the market time process $T(t)$

Distribution	Parameter estimates	$-\log \mathcal{L}$
Exponential	$\hat{\alpha} = 21.9671$	12334
Gamma	$\hat{a} = 5.0868$ $\hat{b} = 4.3184$	10937
Lognormal	$\hat{\mu} = 2.9880$ $\hat{\sigma} = 0.4806$	11081
Weibull	$\hat{a} = 3.81 \cdot 10^{-4}$ $\hat{b} = 2.4525$	10964

Table 8: Fits of the 30–minute increments of the market time process $T(t)$

Distribution	Parameter estimates	$-\log \mathcal{L}$
Exponential	$\hat{\alpha} = 43.9343$	7212
Gamma	$\hat{a} = 6.0628$ $\hat{b} = 7.2465$	6399
Lognormal	$\hat{\mu} = 3.6980$ $\hat{\sigma} = 0.4330$	6453
Weibull	$\hat{a} = 3.122 \cdot 10^{-5}$ $\hat{b} = 2.6602$	6420

Table 9: Fits of the hourly increments of the market time process $T(t)$

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