A Minimal Share Market Model with Stochastic Volatility

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Abstract. The paper describes a continuous time share market model with a minimal number of factors. These factors are powers of Bessel processes. The asset prices are formed by ratios of the factors and have consequently leptokurtic return distributions. In this framework stochastic volatility with properties that are similar to those actually observed arises naturally. The model generates for the market index the well-known leverage effect due to negative correlation between the index and its volatility. It also incorporates possible default of an asset and thus models credit risk.

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1 Introduction

The modelling of financial markets is an important problem in modern finance. A successful share market model is of course one which must work in practice. This means, on the one hand its mathematical structure must be simple and reflect, with a minimal number of factors, the essential stylised empirical properties of the market. Also, it must be sufficiently tractable to allow for the implementation of efficient quantitative methods for pricing, hedging, calibration and risk management.

The well-known Black-Scholes-Merton (BSM) financial market model, see Black & Scholes (1973) and Merton (1973), assumes that geometric Brownian motions generate the asset price dynamics. The BSM model has become the benchmark model both in theory and practice for financial market analysis. Since each risky asset corresponds to one driving Wiener process the BSM is typically constructed as a complete market model with a minimal number of stochastic factors.

Therefore it can be considered an example of what we will call a *minimal market model*. More precisely, we consider a minimal market model to be one which involves only as many factors as there are traded risky assets. These factors should have a practical interpretation and must represent observable quantities.

However, although the importance of the BSM in the field cannot be underestimated, it is far from being satisfactory. For instance, in the context of option pricing practitioners have to correct for implied volatility smiles and skews. This is typically done in a rather subjective manner without major theoretical foundation. It would be preferable to have a simple continuous time model that avoids these *ad hoc* adjustments and provides a consistent approach to the management of both market and credit risk.

Over the last thirty years theoreticians and practitioners have searched for improved share market models. One important aspect of this search has been the requirement to model stochastic volatility. The BSM only covers the case of constant or deterministic time dependent volatility. For recent surveys on the modelling of stochastic volatility and empirical stylised facts in the context of asset price dynamics, see Ghysels, Harvey & Renault (1996) and Frey (1997).

Let us mention some major *stylised empirical facts* that should be incorporated in an advanced share market model:

- There is evidence of significant leptokurtosis in log-returns of most asset prices.
- Log-returns of indices and index benchmarked share prices are likely to be Student t distributed as is shown at the end of this paper.
- Persistence of high or low volatility over longer periods of time is typically observed.

- High threshold exceedances in log-returns and volatility are usually clustered.
- For indices there is evidence of substantial negative correlation between the index and its volatility, known as the leverage effect.
- Implied volatilities from options on indices are negatively skewed.
- Correlation between the index and share price is stochastic.

There is a rich literature on asset price modelling. A well studied group of asset price models is related to subordinated models, see e.g., Clark (1973), Praetz (1972), Blattberg & Gonedes (1974), Madan & Seneta (1990), Barndorff-Nielsen (1995), Eberlein & Keller (1995), Küchler et al. (1995), Hurst, Platen & Rachev (1997), Hurst (1997), Geman, Madan & Yor (1998) and Heyde (1999). These models use non-decreasing stochastic processes, usually with independent increments, to generate an independent operational time as the directing process. This directing stochastic process is then the source of stochastic volatility in these models. A major drawback of subordinated asset price models results from the fact that they assume in general independence between the asset price and its volatility. This contradicts the well documented leverage effect, see for instance, Black (1976), Jackwerth & Rubinstein (1995), Duffie (1998) and others. As a consequence these models generate implied volatility smiles for options on indices but in general no skews if one does not introduce extra modelling features. Furthermore, each asset is modelled typically with two stochastic factors. Thus, subordinated asset price models are often incomplete and therefore not minimal market models in the sense explained previously.

An extensive literature on ARCH and GARCH type asset price models uses discrete time stochastic processes. However, these models depend on a subjective choice of the time step size as an important parameter, see Engle & Bollerslev (1986) and more recently Andersen (1996). As Nelson (1990) has already pointed out, for vanishing time step size ARCH and GARCH type models are asymptotically equivalent to specific stochastic volatility diffusion models. In certain extensions ARCH type models are able to reflect several of the above mentioned stylised empirical facts including the leverage effect for specific derivatives, see, e.g., Duan (1995) and Härdle & Hafner (1998). However, this needs the introduction of extra modelling features and parameters. In their continuous time limits, see Nelson (1990), ARCH and GARCH type models involve more than one noise source per asset. This also means that they are in general incomplete and cannot be considered as minimal market models.

As emphasised in Barndorff-Nielsen & Shephard (1998), a financial market model should be a continuous time model and must not depend on subjective discretisations. In this paper we stress the fact that a successful model should be a minimal market model. Preference should be given to diffusion models with analytically tractable transition densities and the potential to model the default of assets. Together with the fact that a minimal market model should be a complete market model, this would permit the efficient evaluation of derivatives and calibration of the model. The difficulty is to identify such a diffusion model that matches closely the stylised empirical facts.

In this paper a minimal market model, the MMM, will be suggested, which exhibits the above mentioned properties. The model contains as many factors as there are traded risky assets which include also the market index. The factors in the model are represented by the index and index benchmarked share prices. They are modelled by powers of Bessel processes, which match well the empirical data as will be shown in the second part of the paper. Squared Bessel processes are also known as CIR or square root processes. These have analytically tractable transition densities and their functionals are well studied, see Revuz & Yor (1994). The asset prices are formed by ratios of these factors. At certain predictable default times the asset dynamics is stopped which relates to the modelling of some part of credit risk.

The MMM is a complete market model that naturally exhibits stochastic volatility. The model does not use any additional or separate stochastic volatility processes as is the case for most stochastic volatility models considered in the literature, see for instance, Hull & White (1987), Scott (1987), Wiggins (1987), Stein & Stein (1991), Heston (1993), Das & Sundaram (1998) or Fouque, Papanicolau & Sircar (1999) to mention just a few. The stochastic volatility results from the structure of the asset price dynamics.

The suggested share price dynamics is rather different from those of other asset price models because the share prices are defined as ratios of given processes. These ratios arise as a consequence of considerations about the relativity of asset prices that will be formulated in the next section.

2 Relativity of Asset Prices

Let us describe a fundamental property that is characteristic for any asset price and that is, that *all prices are relative*. There does not exist any absolute price. For example, a share price is usually nominated in the currency of the corresponding economy, however the corresponding shares might also be traded in a foreign currency at a foreign stock market and thus for another price. In any case, a share price expresses the value of a share in units of the corresponding reference currency. In principle, one could also consider the inverse of this quantity which then must also be a price. This price would represent the number of shares that would buy one unit of the respective currency. For an exchange rate and its inverse this fact seems obvious.

For the modelling of asset prices the above observation on the relativity of asset prices has important consequences. In the MMM an asset price is modelled as the ratio of two processes. The nominator is related to the units of the currency in which the share price is quoted and the denominator is expressed in relation to units of the asset that are bought for this price.

Our interpretation of the above principle is to consider that a share price is formed by the ratio of the share market index over the inverse of the index benchmarked share price. That is we have the relation

$$share\,price = \frac{index}{\left(\frac{share\,price}{index}\right)^{-1}}. \tag{2.1}$$

We note that this interpretation of the relativity principle naturally introduces the market index and index benchmarked share prices in our modelling. The index and the index benchmarked share prices are observable and have a clear economic interpretation.

3 Bessel Processes as Stochastic Factors

Following our previous remarks, the stochastic factors should be modelled by continuous stochastic processes and must remain strictly positive. The BSM uses the Wiener process to generate the logarithm of an asset price. This exploits in some sense the first order Wiener chaos. Positivity of asset prices is then enforced by identifying the asset price with the exponential of a linearly transformed Wiener process.

Positivity of factors and asset prices can also be naturally achieved by the use of building blocks from the second order Wiener chaos, which are closely related to squares of Wiener processes and thus to Bessel processes.

Let us introduce the *jth squared Bessel process* $Y^j = \{Y^j_t,\, t \in [0,T]\}$ which is the solution of the stochastic differential equation

$$dY^j_t = \kappa^j_t\,(1 - Y^j_t)\,dt + \gamma^j_t\,\sqrt{Y^j_t}\,dW^j_t \tag{3.1}$$

for $t \in [0,T]$ with reference level one and initial value $Y^j_0 > 0$, where $j \in \{0, 1, \ldots, N\}$. Here W^0, W^1, ..., W^N are possibly correlated standard Wiener processes on a given filtered probability space $(\Omega, \mathcal{F}, \underline{\mathcal{F}} = (\mathcal{F}_t)_{t\in[0,T]}, P)$. The *volatility parameter* $\gamma^j \,|\, [0,T] \to (0,\bar{\gamma}), \bar{\gamma} < \infty$, is taken to be a given deterministic, bounded function of time. We choose the *speed of adjustment parameter* κ^j_t at time t in the form

$$\kappa^j_t = \nu^j_t\,\frac{(\gamma^j_t)^2}{4}, \tag{3.2}$$

for $t \in [0,T]$. Here ν^j_t denotes the deterministic *dimension* of the jth squared Bessel process at time t, $j \in \{0, 1, \ldots, N\}$. We assume that the dimension is never less than two, that is

$$\nu^j_t \geq 2. \tag{3.3}$$

Then Y_t^j remains strictly positive P-a.s for all $t \in [0,T]$, where $j \in \{0,1,\dots,N\}$, see Revuz & Yor (1994). We remark that the transition densities of squared Bessel processes are those of gamma distributions.

In the financial literature, squared Bessel processes appear, for instance, as the short rate process in the Cox, Ingersoll & Ross (1985) interest rate model. They have also been used as factors in affine term structure models, see e.g., Duffie (1988). Furthermore, they have been applied in other financial modelling contexts, for instance, by Geman & Yor (1993), Heston (1993), Deelstra & Delbaen (1995), Delbaen & Shirakawa (1997), Göing-Jaeschke & Yor (1998) and Szatzschneider (1998).

For $j \in \{0,1,\dots,N\}$ we now introduce the jth *factor process* $F^j = \{F_t^j,\, t \in [0,T]\}$, with

$$F_t^j = \zeta_t^j \, \left(Y_t^j\right)^{p_t^j} \tag{3.4}$$

as the product of the p_t^j-th power of the jth squared Bessel process Y_t^j with the *jth average factor* ζ_t^j. Here $p_t^j \in (\underline{p},1)$, $\underline{p} > 0$, is the given, typically small but strictly positive power in the jth factor. We interpret F^0 as the asymptotic market index as will be explained in Section 6 in more detail. We will then see that the market index converges for $N \to \infty$ towards the asymptotic index F^0.

We assume that the jth average factor ζ^j fulfils the ordinary linear differential equation

$$d\zeta_t^j = \eta_t^j \, \zeta_t^j \, dt \tag{3.5}$$

with given jth initial value $\zeta_0^j > 0$ and deterministic, bounded jth *growth rate* $\eta^j = \{\eta_t^j,\, t \in [0,T]\}$ with $\eta_t^j \in [-\bar{\eta},\bar{\eta}]$, $\bar{\eta} < \infty$.

Since the factor Y^j, $j \in \{0,1,\dots,N\}$, can be considered as a standardised random factor with a mean reversion level of one the jth average factor ζ^j can increase or decrease the jth factor F^j to any desired level.

4 A Minimal Share Market Model

In the following we propose a minimal market model, the MMM, which incorporates the above mentioned factors. The main feature of the model is that its share prices are formed essentially by ratios of powers of squared Bessel processes.

Each asset is supposed to have some time value. The time value of the jth asset is determined by the deterministic, positive *jth riskless rate of return* f_t^j, $t \in [0,T]$, $j \in \{0,1,\dots,N\}$. The corresponding *jth riskless asset* B_t^j is then defined in the form

$$B_t^j = B_0^j \, \exp\left(\int_0^t f_s^j \, ds\right) \tag{4.1}$$

for $t \in [0,T]$ and $j \in \{0,1,2,\dots,N\}$ with given initial value $B^j_0 > 0$. We call f^0_t the *short term interest rate* and B^0_t the *money account* in the underlying currency. For $j \in \{1,2,\dots,N\}$ we interpret f^j_t as the jth *intrinsic dividend rate* and B^j_t as savings account in jth shares.

Let us denote by S^j_t the jth *share price* at time $t \in [0,T]$ for $j \in \{1,2,\dots,N\}$. We set for notational convenience $S^0_t = B^0_t$. In consideration of relation (2.1) we introduce the jth *index benchmarked asset price* $\bar{S}^j_t$ at time t by the ratio

$$\bar{S}^j_t = \frac{S^j_t}{F^0_t} \tag{4.2}$$

for $t \in [0,T]$ and $j \in \{0,1,\dots,N\}$.

It remains to further explain the structure of the index benchmarked asset price $\bar{S}^j_t$. We assume that the index benchmarked asset price is formed by the ratio of the jth riskless asset B^j_t over the jth factor F^j_t appropriately stopped. That is

$$\bar{S}^j_t = \frac{B^j_{\tau^j \wedge t}}{F^j_{\tau^j \wedge t}} \tag{4.3}$$

for all $t \in [0,T]$ and $j \in \{0,1,\dots,N\}$. Here the variable

$$\tau^j \wedge t = \min(t, \tau^j) \tag{4.4}$$

denotes the minimum of time t and the *predictable default time* τ^j of the jth asset, where

$$\tau^j = \inf\{s > 0 : \frac{B^j_s}{F^j_s} \geq C^j_s\} \tag{4.5}$$

for $t \in [0,T]$ and $j \in \{0,1,\dots,N\}$. The continuous function $C^j \mid [0,T] \to (0,\bar{C}]$, $\bar{C} < \infty$, represents a given, usually extremely high, deterministic *critical level* for the jth index benchmarked asset price. When the process $\frac{B^j}{F^j} = \left\{\frac{B^j_t}{F^j_t},\, t \in [0,T]\right\}$ reaches for the first time this critical level, then $\bar{S}^j_t$ is simply kept constant from this time onwards.

The introduction of the jth predictable default time not only captures the occurrence of a certain type of credit events, it is also important for ensuring the essential martingale property of index benchmarked asset prices under the pricing measure. In a forthcoming paper on pricing and hedging of derivatives under the MMM these issues will be discussed in detail. According to (4.2), (4.3) and (3.4) the jth share price has now the form

$$\begin{aligned} S_t^j &= F_t^0\,\bar{S}_t^j \\ &= F_t^0\,\frac{B^j_{\tau^j\wedge t}}{F^j_{\tau^j\wedge t}} \\ &= \frac{\zeta_t^0\,(Y_t^0)^{p_t^0}\,B^j_{\tau^j\wedge t}}{\zeta^j_{\tau^j\wedge t}\,\left(Y^j_{\tau^j\wedge t}\right)^{p_t^j}} \end{aligned} \tag{4.6}$$

for $t \in [0, T]$ and $j \in \{1, 2, \ldots, N\}$.

If we choose the index F^0 as numeraire, then it can be shown that the index benchmarked asset prices are martingales under a unique equivalent martingale measure. In this sense the market can be considered to be complete. This is important from the practical point of view since different choices of the pricing measure would result in different derivative prices. If we did not introduce the predictable default time in (4.3), then the index benchmarked asset prices would be only local martingales, see Karatzas & Shreve (1988), under the pricing measure. Thus the modelling of the predictable default events in the MMM is not only an important practical requirement but also mathematically essential to exclude potential arbitrage opportunities.

The use of unpredictable default times to model a more general class of credit events will be considered in a forthcoming paper. Furthermore, in future work the model will be extended to cover the cases when riskless rates of return, growth rates, powers, critical levels and default intensities are stochastic. In the following sections we will show that the stylised empirical facts that have been mentioned previously are well accounted for by the proposed model.

5 Properties of Volatility

As we have seen, the MMM does not involve separate volatility processes. The arising stochastic volatility is a natural consequence of the model.

According to (4.6) the jth share price is proportional to some ratio of powers of squared Bessel processes. This specific form of the share price is responsible for the probabilistic properties of the resulting stochastic volatility component, which we will study in the following under simplifying assumption. We assume from now on that the dimensions ν^j, powers p^j and time scale parameters γ^j are constant and the Wiener processes W^j are independent.

At first we consider the dynamics of the jth index benchmarked asset price $\bar{S}^j$. By application of the Itô formula we obtain from equation (4.3) together with

(4.1) (3.1), (3.4) and (3.5) the stochastic differential equation

$$d\bar{S}_t^j = \bar{S}_t^j \, a_t^j \, dt - \bar{S}_t^j \, \sigma_t^j \, dW_t^j \tag{5.1}$$

with *volatility*

$$\sigma_t^j = \frac{\gamma^j \, p^j}{\sqrt{Y_t^j}} \tag{5.2}$$

and *appreciation rate*

$$a_t^j = f_t^j - \eta_t^j + \frac{(\gamma^j)^2 \, p^j}{4} \left(\frac{2\,(p^j+1) - \nu^j}{Y_t^j} + \nu^j \right) \tag{5.3}$$

for $t \in [0, \tau^j \wedge T]$ and $j \in \{0, 1, \dots, N\}$.

According to (4.6), (3.4), (3.1), (3.5) and (4.1) it then follows by the Ito formula that the jth share price satisfies the stochastic differential equation

$$dS_t^j = S_t^j \left(a_t^j - a_t^0 + f_t^0 + \left(\sigma_t^0\right)^2 \right) dt + S_t^j \, \sigma_t^0 \, dW_t^0 - S_t^j \, \sigma_t^j \, dW_t^j \tag{5.4}$$

for $t \in [0, \tau^j \wedge T]$ and $j \in \{1, 2, \dots, N\}$.

We note from (5.2) that volatilities are stochastic in the MMM and proportional to the inverse of a Bessel process. This specific form of the stochastic volatility potentially explains most stylised empirical facts related to volatility.

For illustration we show in Figure 1 a simulated path of the asymptotic index F_t^0 and its volatility σ_t^0 over four years with typical parameters $\eta_t^0 = 0.07$, $p^0 = 0.05$, $\gamma^0 = 2.0$, $\zeta_0^0 = 0.1$, $Y_0^0 = 1$ and $\nu^0 = 4$. Looking at the sample path of the volatility component we can see that low and high volatilities persist for longer periods of time. The resulting volatility process exhibits natural persistence. High threshold exceedances are obviously clustered for volatilities but also for log-returns under the MMM as it is typically observed in practice.

We also note in Figure 1 negative correlation between the paths of the asymptotic index and its volatility. Let us make this observation more precise. By application of the Itô formula we obtain for the volatility σ_t^j from (5.2) and (3.1) the stochastic differential equation

$$d\sigma_t^j = \left(\frac{\sigma_t^j}{2} \left(\kappa^j + \left(1 - \frac{\nu^j}{4} \right) \frac{\left(\sigma_t^j\right)^2}{(p^j)^2} \right) - \frac{\left(\sigma_t^j\right)^3}{8\,(p^j)^2} \right) dt - \frac{\left(\sigma_t^j\right)^2}{2\,p^j} \, dW_t^j \tag{5.5}$$

for $t \in [0, T]$ and $j \in \{0, 1, \dots, N\}$.

The non-linear dynamics of the volatility in (5.5) is different from that of other stochastic volatility models in the literature. We derive for $j = 0$ from (3.4), (3.5) and (3.1) for the asymptotic index F_t^0 the stochastic differential equation

$$dF_t^0 = F_t^0 \left(f_t^0 - a_t^0 + \left(\sigma_t^0\right)^2 \right) dt + F_t^0 \, \sigma_t^0 \, dW_t^0 \tag{5.6}$$

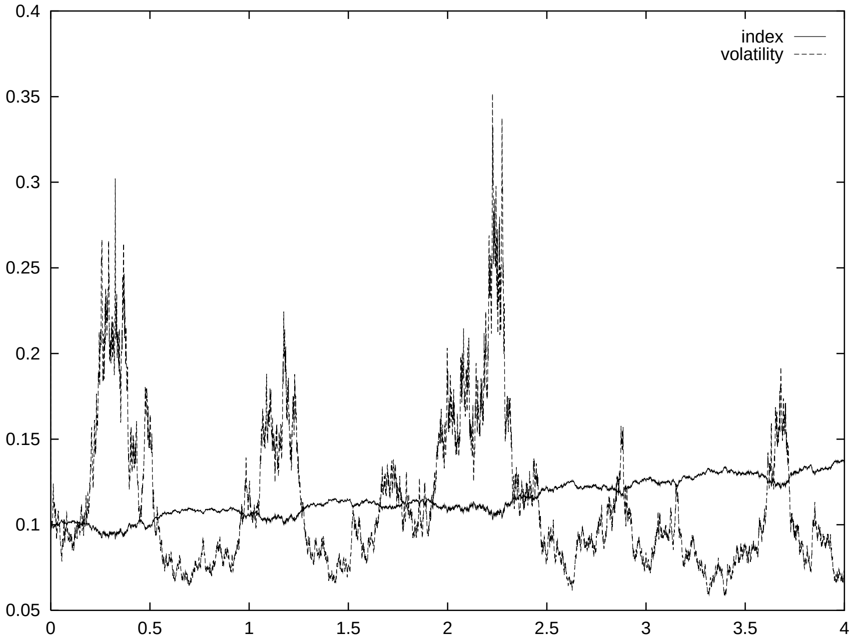

Figure 1: Simulated market index and its volatility.

for $t \in [0,T]$. From the stochastic differential equations (5.5) and (5.6) we see that the volatility σ^0 and the index F^0 are negatively correlated processes. This generates in the MMM the leverage effect.

In a forthcoming paper it will be shown that European call option prices on the share market index have negatively skewed implied volatilities. With the choice $p^0_t \approx 0.05$ these are close to those that can be actually observed for the S&P500.

For the MMM, say with dimension $\nu^0 = 4$ (see later Table 2 for empirical estimates of ν^0) and power p^0 we have under the invariant measure the ratio of the squared first moment over the second moment of F^0 in the form

$$R(p^0) = \frac{\left(E\left(F^0_\cdot\right)\right)^2}{E\left(\left(F^0_\cdot\right)^2\right)} = \frac{\left(\Gamma\left(p^0+2\right)\right)^2}{\Gamma\left(2\,p^0+2\right)}, \tag{5.7}$$

where $\Gamma(\cdot)$ denotes the gamma function. From S&P500 daily observations during 1977 - 1997 the estimated ratio of moments in (5.7) shows a value of $R(p^0) \approx 0.99844$ which corresponds according to (5.7) to a power $p^0 \approx 0.05$. This fits the value of the power p^0 that is needed to generate a realistic slope in the implied volatility skew, as has been mentioned above.

6 Market Forces

For $N \in \{1,2,3,\dots\}$ let us denote by q^j_N, $j \in \{0,1,\dots,N\}$ the number of units of the jth asset, that are in the market. The market has then the *index* *benchmarked initial capitalisation*

$$Q_{N,0} = \sum_{j=0}^{N} q_N^j \, \bar{S}_0^j. \tag{6.1}$$

We assume the following *diversification condition*, which excludes extreme concentrations of capital in single assets. We suppose that there exists a constant $K \in (0, \infty)$ such that

$$\frac{q_N^j \, \bar{S}_0^j}{Q_{N,0}} \in (0, K/(N+1)) \tag{6.2}$$

for all $j \in \{0, 1, \ldots\}$ and $N \in \{1, 2, 3, \ldots\}$.

We can now introduce the *market index* $I_{N,t}$ at time t in the form

$$I_{N,t} = \sum_{j=0}^{N} \frac{q_N^j}{Q_{N,0}} \, S_t^j \, \mathbf{1}_{\{t < \tau^j\}} \tag{6.3}$$

for $t \in [0, T]$. Here $\mathbf{1}_A$ is the indicator function for the event A.

Let us define the jth *market force* G_t^j at time t by the expression

$$G_t^j = q_N^j \, \bar{S}_t^j \, a_t^j \, \mathbf{1}_{\{t < \tau^j\}} \tag{6.4}$$

for $t \in [0, T]$ and $j \in \{0, 1, \ldots, N\}$, where a_t^j was given in (5.3).

We assume the *market force condition*

$$\sum_{j=0}^{N} G_t^j = 0 \tag{6.5}$$

for $t \in [0, T]$. This condition postulates that the *market forces offset each other.*

We prove in the Appendix that the expression

$$Z_{N,t} = E\left(\left(\frac{I_{N,t}}{F_t^0} - 1\right)^2\right) \tag{6.6}$$

converges for $N \to 0$ to zero. Thus, the market index I_N converges in a mean square sense to the asymptotic index F^0.

The concept of market forces turns out to be very useful. Further results that are related to dynamical asset allocation and optimal growth portfolios will be presented in a forthcoming paper.

7 Properties of Log-Returns

Let us now consider the log-return of the asymptotic index F^0 over a time interval $[t, t+\Delta]$ of small length Δ. We note from (5.6) and (5.2) that this log-return has approximately the form

$$\log\left(F^0_{t+\Delta}\right) - \log\left(F^0_t\right) \approx \sigma^0_t \left(W^0_{t+\Delta} - W^0_t\right) = \frac{\gamma^0\, p^0}{\sqrt{Y^0_t}} \left(W^0_{t+\Delta} - W^0_t\right) \tag{7.1}$$

when we neglect higher order terms in Δ. The right hand side of (7.1) represents a conditionally Gaussian distributed random variable with random variance $\frac{(\gamma^0\, p^0)^2}{Y^0_t}\,\Delta$. It can be shown that for dimension $\nu^0 > 2$, constant volatility parameter $\gamma^0 > 0$, and appropriate gamma distributed initial value the squared Bessel process Y^0 is ergodic. It then follows for all $t \in [0,T]$ that Y^0_t is gamma distributed with ν^0 degrees of freedom. Hence in the proposed market model the log-returns of F^0 are approximately Student t distributed with ν^0 degrees of freedom.

This is an important property of the log-returns of the asymptotic index and thus according to (6.6) also for the market index under the MMM. One could try to check or test this Student t feature in the log-returns of observed index data. Indeed this property has been empirically identified by studies on market indices in Hurst & Platen (1997), Hurst (1997) and Hurst, Platen & Rachev (1997). We will discuss some of these results in Section 9.

If we consider the log-return of the index benchmarked share price $\bar{S}^j_t$ over a small time interval $[t, t+\Delta]$, then we obtain according to (5.1) and (5.2) the expression

$$\begin{aligned}\log\left(\bar{S}^j_{t+\Delta}\right) - \log\left(\bar{S}^j_t\right) &\approx -\sigma^j_t \left(W^j_{t+\Delta} - W^j_t\right) \\ &= -\frac{\gamma^j\, p^j}{\sqrt{Y^j_t}} \left(W^j_{t+\Delta} - W^j_t\right) \end{aligned} \tag{7.2}$$

for $t \in [0,T]$ and $j \in \{1, 2, \ldots, N\}$. Obviously, the log-return of index benchmarked share prices has analogous distributional properties as the log-return of the asymptotic index in (7.1). Thus the term on the right hand side of (7.2) is for small Δ approximately Student t distributed with ν^j degrees of freedom. This feature of log-returns of index benchmarked share prices under the MMM is supported by results in Platen (1999) obtained from one hundred leading stocks of major stock markets. We will discuss these empirical findings in Section 11.

To illustrate some of these properties we plot in Figure 2 a simulated trajectory of a typical share price S^1 under the MMM together with the asymptotic index F^0 for the parameter set $\eta^0_t = \eta^1_t = f^1_t = 0.07$, $p^0 = p^j = 0.05$, $\zeta^0_0 = 0.1$, $\zeta^1_0 = 0.9$, $\nu^0 = \nu^1 = 4$, $y_0 = y_1 = 1$ and $\gamma^0 = \gamma^1 = 2.0$. This figure shows share price movements that are correlated with the market index. As in practice, this correlation turns out to be stochastic.

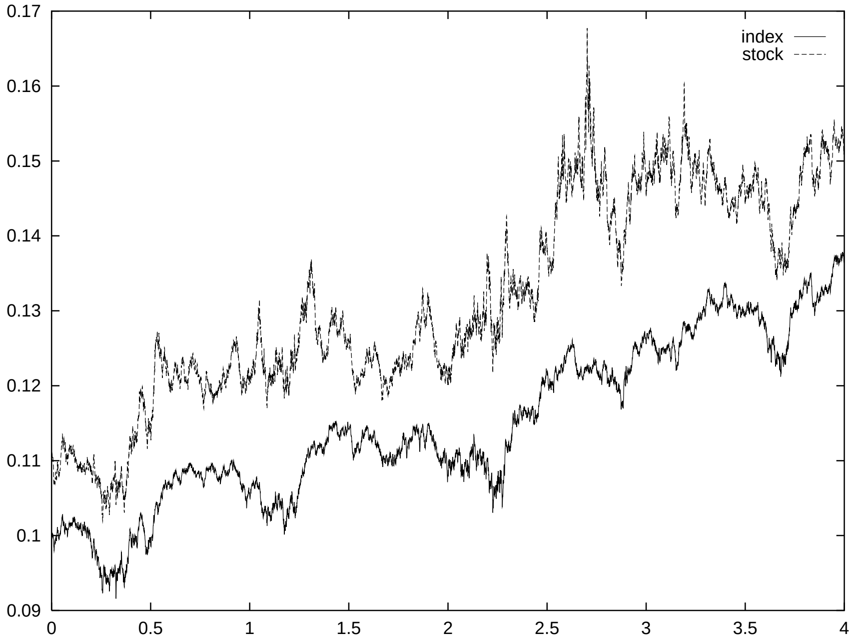

Figure 2: Simulated share price and index.

8 A Class of Leptokurtic Distributions

In the remaining part of this paper we discuss some results that empirically support the Student t feature for log-returns as theoretically suggested by the MMM. To provide an appropriate framework we choose a rather general class of distributions that includes the Student t distribution but also several important alternative distributions that have recently appeared in the literature in the context of asset price modelling.

We consider the class of *symmetric generalised hyperbolic distributions*, which are Gaussian mixture distributions. These distributions were extensively examined by Barndorff-Nielsen (1977, 1978) and Barndorff-Nielsen & Blaesild (1981). They arise when the mixing distribution is a generalised inverse Gaussian distribution.

The symmetric generalised hyperbolic distribution constitutes a four parameter family of distributions with parameters $\alpha, \delta \geq 0$ and $\eta, \lambda \in \Re$, where $\alpha \neq 0$ if $\lambda \geq 0$ and $\delta \neq 0$ if $\lambda \leq 0$, see Barndorff-Nielsen (1977, 1978). Its density is of the form

$$\begin{aligned} f(x) &= \frac{1}{\delta\,\sqrt{\Delta}\,K_\lambda(\alpha\,\delta)}\sqrt{\frac{\alpha\,\delta}{2\pi}}\left(1+\frac{(x-\eta\,\Delta)^2}{\delta^2\,\Delta}\right)^{\frac{1}{2}(\lambda-\frac{1}{2})} \\ &\quad \cdot K_{\lambda-\frac{1}{2}}\left(\alpha\,\delta\sqrt{1+\frac{(x-\eta\,\Delta)^2}{\delta^2\,\Delta}}\right). \qquad (8.1) \end{aligned}$$

Here λ and $\bar{\alpha} = \alpha\delta$ are the shape parameters determining the type of tail of the distribution. These shape parameters are chosen so that they are invariant under scale transformation. The parameter c is the scale parameter, where c^2 represents the mean of the mixing distribution and

$$c^2 = \begin{cases} \frac{2\lambda}{\alpha^2} & \text{if} \quad \delta = 0 \\ \frac{\delta^2 K_{\lambda+1}(\bar{\alpha})}{\bar{\alpha}\, K_\lambda(\bar{\alpha})} & \text{otherwise} . \end{cases} \tag{8.2}$$

The function $K_q(\cdot)$ is the modified Bessel function of the third kind with index q.

There exist a number of important asset price models that have been recently proposed in the literature and which generate log-returns with densities that are specific symmetric generalised hyperbolic densities.

The Student t distribution is one such symmetric generalised hyperbolic distribution. Its degrees of freedom are given by the expression

$$\nu = -2\lambda. \tag{8.3}$$

Here we have a negative shape parameter $\lambda < 0$. The other shape parameter $\bar{\alpha}$ is set to be zero for the Student t distribution. For larger degrees of freedom the Student t density has thinner tails and converges to the Gaussian density as $\nu \to \infty$.

Student t distributed log-returns appear in the independent increment model proposed by Praetz (1972) and Blattberg & Gonedes (1974). For the log-returns of an index this model comes closest to the model outlined in this paper. However, due to its structure no significant persistence and clustering of high threshold exceedances can be obtained for the volatility from such models.

When the shape parameter λ is fixed with $\lambda = -\frac{1}{2}$ and $\bar{\alpha} \geq 0$, then the normal-inverse Gaussian model of Barndorff-Nielsen (1995) arises. Another mixture distribution for log-returns appears in the variance gamma model of Madan & Seneta (1990) which is also supported in Geman, Madan & Yor (1998). This distribution arises when the shape parameters are such that $\bar{\alpha} = 0$ with some positive $\lambda > 0$. Another recent class of asset price models has been proposed by Eberlein & Keller (1995) and also by Küchler et al. (1995). These models generate log-returns that have a hyperbolic density. Here the shape parameter λ is fixed to $\lambda = 1$ and $\bar{\alpha} \geq 0$.

Most of these asset price models, including the Madan and Seneta model assume, in principle, independence between the asset and volatility noise. Consequently, these models cannot reflect without extra effort negative correlation between the index and its volatility as is needed to generate the leverage effect. Without such negative correlation implied volatilities show a smile but not a skew, which is typical for an index option.

In the following sections we will test log-returns of indices and index benchmarked share prices with respect to the above introduced class of symmetric generalised hyperbolic distributions.

9 Fitting a Best Distribution

The class of symmetric generalised hyperbolic distributions represents a rich class of leptokurtic distributions. Using the maximum likelihood ratio test one can identify a distribution of this class that fits best the log-returns of market indices. We emphasise that the maximum likelihood approach does not rely on higher moments, e.g. kurtosis. However, in its classical form, see Rao (1979), it assumes independence of the observations which we simply assume to perform the statistical analysis.

We define the likelihood ratio in the form

$$\Lambda = \frac{\mathcal{L}^*_{\text{model}}}{\mathcal{L}^*_{\text{general model}}}. \tag{9.1}$$

Here $\mathcal{L}^*_{\text{model}}$ represents the maximised likelihood function of a hypothesised distribution, for instance, the Student t distribution. With respect to this distribution the maximum likelihood estimate has been computed and is then used to obtain the corresponding likelihood function $\mathcal{L}^*_{\text{model}}$. On the other hand, $\mathcal{L}^*_{\text{general model}}$ denotes the maximised likelihood function for the general distribution, here the symmetric generalised hyperbolic distribution.

Under appropriate conditions it is well-known that the distribution of the test statistic

$$L_n = -2 \ln \Lambda \tag{9.2}$$

is for increasing number n of observations asymptotically chisquare distributed. Here the degrees of freedom equal the difference between the number of parameters in the general distribution and the hypothesised distribution. The symmetric generalised hyperbolic distribution is a four-parameter distribution and the Student t, normal-inverse Gaussian, variance-gamma and hyperbolic distributions are three-parameter distributions. Therefore, in the above cases L_n is for $n \to \infty$ asymptotically chisquare distributed with one degree of freedom.

It can then be shown that as $n \to \infty$

$$P\left(L_n < \chi^2_{1-\alpha,1}\right) \approx F_{\chi^2(1)}\left(\chi^2_{1-\alpha,1}\right) = 1 - \alpha, \tag{9.3}$$

where $F_{\chi^2(1)}$ denotes the chisquare distribution with one degree of freedom and $\chi^2_{1-\alpha,1}$ is its $100(1-\alpha)\%$ quantile. One can then check, say, for an $\alpha = 99\%$ significance level whether or not the test statistic L_n is in the 1% quantile of the chisquare distribution with one degree of freedom. If the relation

$$L_n < \chi^2_{0.01,1} \approx 0.0002 \tag{9.4}$$

is not fulfilled, then we reject the hypothesis that the suggested distribution is the true underlying distribution on a 99% significance level.

The above maximum-likelihood methodology offers the following definition of a *best distribution*. We choose the distribution with the smallest test statistic L_n as the best distribution in the given class of symmetric generalised hyperbolic distributions. This distribution then maximises the likelihood ratio Λ in (9.1).

10 Indices

Using the maximum likelihood approach it has been shown in Hurst (1997) and Hurst & Platen (1997) that for many stock market indices the log-returns are likely to be Student t distributed.

Let us describe some results which were based on Morgan Stanley International share market indices as provided by Datastream International. The study includes fifteen years daily data for Australia, Belgium, Canada, France, Germany, Hong-Kong, Italy, Japan, Netherlands, Singapore, Spain, Sweden, Switzerland, the UK and the USA. The observation period finishes at the beginning of 1997. In Table 1 we show the resulting values for the test statistic L_n, as defined in (9.2), for the four symmetric generalised hyperbolic distributions. These are the Student t, normal-inverse Gaussian, variance-gamma and hyperbolic distribution. We observe that for all fifteen indices the Student t distribution turns out to be the most likely one. We also note from Table 1, the hypothesis that the Student t distribution is the true underlying distribution cannot be rejected on a 99% significance level for ten of the fifteen indices. However, the other three distributions can be rejected in all cases on any reasonable significance level.

It is interesting to note that those indices for which the test statistic L_n is greater than the $\chi^2_{0.01,1}$ quantile, see (9.4), belong to markets that, it could be argued, still have potential to become more sophisticated or developed compared to, say the US market. According to Table 1 these are Belgium, Italy, Japan, Spain and Sweden.

In our discussion about different symmetric generalised hyperbolic distributions we mentioned in Section 8 that the shape of the tails of the different densities is characterised by the two parameters $\bar{\alpha}$ and λ. Let us visualise for different indices the corresponding estimated values of these two shape parameters in a two-dimensional graphic. Figure 3 plots the $(\bar{\alpha}, \lambda)$-estimates for the fifteen stock market indices. We note that the $\bar{\alpha}$ values, shown in horizontal direction, are rather small in most cases. The λ values, displayed in vertical direction, are all negative. This coincides with the fact that the Student t distribution is characterised by negative λ values and $\bar{\alpha} = 0$. Figure 3 visually supports the Student t hypothesis. If the best distribution were the normal inverse Gaussian one, most $(\bar{\alpha}, \lambda)$-points would have to be concentrated along the horizontal line at $\lambda = -0.5$. For the hyperbolic distribution they would have to be scattered along the line $\lambda = 1.0$. Finally, if the variance gamma model were superior, then the estimates in Figure 3 would need to be concentrated along the positive λ-axis.

Country	Student t	Normal Inverse Gaussian	Hyperbolic	Variance Gamma
Australia	0.0000	52.1444	85.4310	101.6556
Belgium	0.0182	28.9600	106.7174	116.2398
Canada	0.0000	30.2430	78.9614	91.8436
France	0.0000	24.5732	44.2510	52.8978
Germany	0.0000	35.4832	80.9622	96.5058
Hong Kong	0.0000	31.2074	108.1956	113.9280
Italy	0.6520	3.4306	14.5956	23.7080
Japan	0.8848	14.6750	77.8204	82.4098
Netherlands	0.0000	42.3646	88.9038	107.4172
Singapore	0.0000	54.0772	143.6444	143.7386
Spain	0.0290	15.0464	44.9322	54.9598
Sweden	0.0088	20.5876	53.2770	67.5088
Switzerland	0.0000	46.5656	115.0898	131.3266
UK	0.0000	47.5990	67.4152	87.2694
USA	0.0000	22.1012	58.3416	66.2470

Table 1: The L_n values for stock market indices.

We show in Table 2 the estimated degrees of freedom for the different indices. These range between 3.0 and 6.5 with an average that is close to 4.0, which refers to a λ-value of $\lambda = -2$, see (8.3).

Country	ν
Australia	4.8878
Belgium	3.0610
Canada	3.6318
France	4.7480
Germany	3.8506
Hong Kong	3.1044
Italy	4.3696
Japan	3.0687
Netherlands	4.0211
Singapore	3.2036
Spain	3.7402
Sweden	3.9750
Switzerland	3.5061
UK	6.4650
USA	3.5598
Total Mean	3.9462

Table 2: Degrees of freedom ν for the Student t distribution for indices.

11 Index Benchmarked Share Prices

As discussed in Section 7, the MMM suggests that log-returns of index benchmarked share prices, see (7.2), are approximately Student t distributed. This

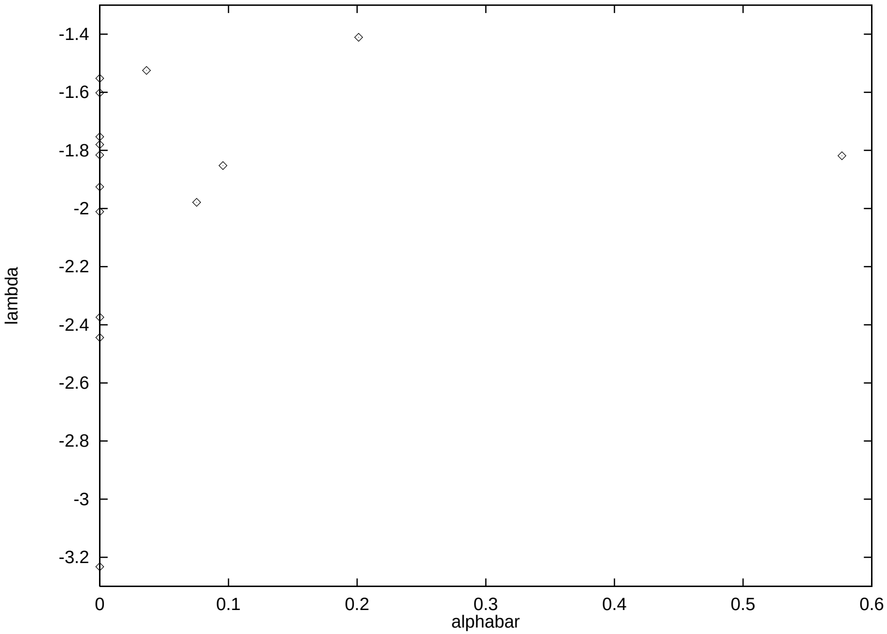

Figure 3: $(\bar{\alpha}, \lambda)$-plot for index log-returns.

has been supported by empirical results in Platen (1999) on log-returns of index benchmarked share prices from the Australian, German, Japanese, UK and US stock markets. From each of the five markets the twenty leading stocks were examined. For eleven years of observed daily log-returns, provided by Datastream International for the period from 1987 to 1997, the best distribution in the class of symmetric generalised hyperbolic distributions in the sense as described above was again identified as being the Student t distribution.

A summary of these findings is presented in Figure 4. Here we plot the estimated $(\bar{\alpha}, \lambda)$-parameter values for the one hundred examined stocks. It is obvious that most of the $(\bar{\alpha}, \lambda)$-estimates concentrate in the area close to the negative λ-axis, and again around $\lambda = -2$. As previously mentioned, this is the area that is typical for the Student t distribution with approximately four degrees of freedom.

Conclusion

The paper suggests a share market model that has a minimal number of factors, is complete and exhibits a natural way of handling stochastic volatility. Under this model the share market index dynamics is represented using a power of a squared Bessel process. The share prices are essentially ratios of powers of squared Bessel processes. The proposed model reflects major empirical stylised facts on stochastic volatility, including the leverage effect for indices, the persistence of volatility and the clustering of high threshold exceedances for volatilities and log-returns. The actually observed log-return distributions of indices and index benchmarked share prices closely match the corresponding distributions obtained from the pro-

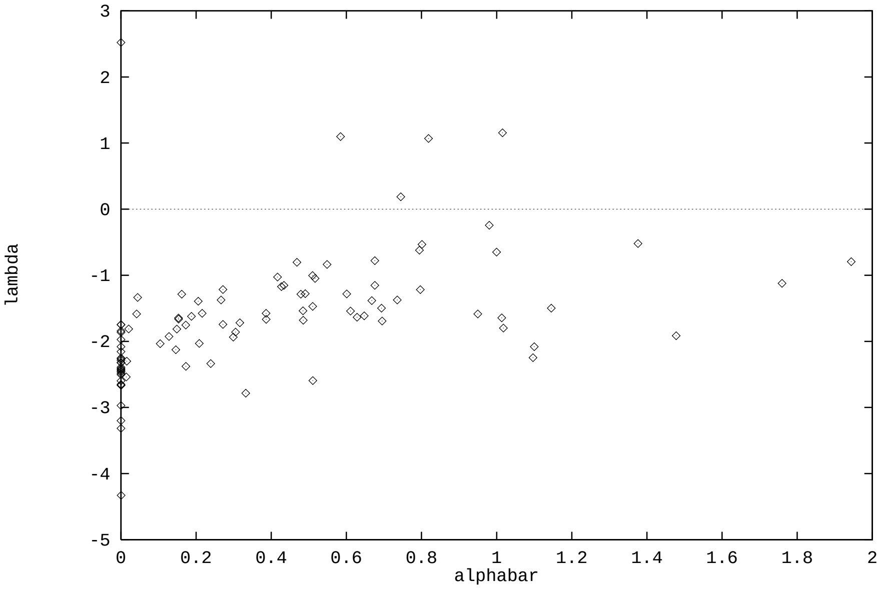

Figure 4: $(\bar{\alpha}, \lambda)$-plot for log-returns of index benchmarked share prices.

posed model.

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Appendix

We note from (6.3) and (4.6) that

$$\frac{I_{N,t}}{F_t^0} = \sum_{j=0}^{N} \frac{q_N^j}{Q_{N,0}} \cdot \bar{S}_t^j \, \mathbf{1}_{\{t<\tau^j\}}.$$

Then it follows from (5.1), (6.1) and (6.5) that

$$\begin{aligned}
\frac{I_{N,t}}{F_t^0} &= \sum_{j=0}^{N} \frac{q_N^j}{Q_{N,0}} \left(\bar{S}_0^j + \int_0^t \bar{S}_s^j \, a_s^j \, \mathbf{1}_{\{s<\tau^j\}} \, ds - \int_0^t \bar{S}_s^j \, \sigma_s^j \, \mathbf{1}_{\{s<\tau^j\}} \, dW_s^j \right) \\
&= 1 + \int_0^t \sum_{j=0}^{N} G_s^j \, ds + \sum_{j=0}^{N} \int_0^t \frac{q_N^j}{Q_{N,0}} \, \bar{S}_s^j \, \sigma_s^j \, \mathbf{1}_{\{s<\tau^j\}} \, dW_s^j \\
&= 1 + \sum_{j=0}^{N} \int_0^t \frac{q_N^j}{Q_{N,0}} \, \bar{S}_s^j \, \sigma_s^j \, \mathbf{1}_{\{s<\tau^j\}} \, dW_s^j \qquad (11.1)
\end{aligned}$$

From (6.6) we obtain with (11.1)

$$Z_{N,t} = E \left(\sum_{j=0}^{N} \int_0^t \left(\frac{q_N^j}{Q_{N,0}} \, \bar{S}_s^j \, \sigma_s^j \, \mathbf{1}_{\{s<\tau^j\}} \right)^2 ds \right).$$

Together with (5.2) and (3.4) we get

$$Z_{N,t} \leq \sum_{j=0}^{N} \int_0^T \left(\frac{q_N^j}{Q_{N,0}} \, \gamma^j \right)^2 E \left(\left(\frac{B_s^j}{\zeta_s^j} \right)^{-\frac{1}{p^j}} \left(\bar{S}_s^j \right)^{2+\frac{1}{p^j}} \, \mathbf{1}_{\{s<\tau^j\}} \right) ds.$$

Now it follows by the diversification condition (6.2) and by choosing upper bounds for the variables involved that

$$\begin{aligned}
Z_{N,t} &\leq \sum_{j=0}^{N} \int_0^T \left(\frac{K}{N+1} \right)^2 \bar{\gamma}^2 \, \exp\left(\frac{1}{\underline{p}} \, \bar{\eta} \, T \right) \bar{C}^{2+\frac{1}{\underline{p}}} \, ds \\
&\leq \frac{1}{N+1} \, T \, K^2 \, \bar{\gamma}^2 \, \exp\left(\frac{1}{\underline{p}} \, \bar{\eta} \, T \right) \bar{C}^{2+\frac{1}{\underline{p}}}
\end{aligned}$$

This proves that

$$\lim_{N\to\infty} Z_{N,t} = 0 \qquad (11.2)$$

for all $t \in [0,T]$.

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