

# Optimal Consumption, Portfolio Selection and Life Insurance for Financial Planning\*

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Revised May 1999

## Abstract

This paper examines the question of lifetime personal financial planning—how should individuals determine their optimal consumption, portfolio selection and life insurance needs? Although Richard (1975) provides the theoretical basis for such a model, no numerical results from this model have been produced. The paper uses the Markov chain approximation method of Kushner (1977) to determine numerical results for Richard's model. This approximation method is general, and handles constraints to the model; solutions are developed with a borrowing constraint. The results are interpreted in light of financial planners' traditional rules of thumb for both investment in risky assets over one's lifetime and life insurance purchases.

*Journal of Economic Literature* Classification Numbers: C63, D91, G11, G22, J26.

*Keywords*: optimal portfolio selection, financial planning, life insurance, stochastic control, Markov chain.

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\*This paper is based on Chapters 2–4 of my Ph.D. dissertation at the University of New South Wales. I would like to thank Geoffrey Kingston, John Piggott, Keith McLaren, Chris Adam, Michael Sherris and Hazel Bateman for comments and discussion. Financial support under ARC Grant A79331125 is gratefully acknowledged.

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# 1 Introduction

This paper examines lifetime personal financial planning—how should individuals determine their optimal consumption, portfolio selection and life insurance needs over their life cycles? Personal financial planning, which encompasses the saving and insurance decisions of individuals, has been confined to high net worth individuals in the past. However, there are a number of reasons to expect it to become much more widespread in the future.

Governments and companies throughout the world appear to be shifting in increasing numbers towards accumulation (or defined contribution) type retirement schemes in a move to make individuals responsible for their own retirement.<sup>1</sup> Clearly, as those in control of pension funds shift risk back onto individuals through the rise of defined contribution arrangements, individuals will have more personal responsibilities. For the unsophisticated, the required acumen may be lacking. What advice can modern financial economics give to these people as they embark on their financial planning?

The literature offers up a number of contributions on optimal consumption and portfolio selection in a lifetime (finite-horizon) setting, beginning principally with Samuelson (1969), Merton (1969, 1971). A large literature has developed, with many of its facets mentioned in Merton (1990) and Duffie (1992).

Merton (1971) introduced mortality to these models, incorporating a parametric survival model of mortality into his formulation. While much of the literature that has followed focuses only on optimal consumption and portfolio selection, Richard (1975) extended this model to consider additionally the optimal amount of life insurance. In doing so, he substituted the parametric survival model of

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<sup>1</sup>See, for example, World Bank (1994) and Bodie & Papke (1992).

Merton (1971) with a more realistic tabular, or nonparametric, survival model along the lines of Yaari (1965).

Richard’s model contains key elements of personal financial planning. However, due to the complexity of the model, it is not at all straightforward to implement. The contribution of this paper is to provide a general approach for obtaining numerical solutions of the model. This approach readily allows for the incorporation of different functional forms, as well as permitting a variety of constraints to be imposed.

The numerical solution of Richard allows the financial planning implications of the model to be fully examined. The issue of age-phased reduction in risky assets will be considered. Such advice is often given by financial planners. Indeed, Jagannathan & Kocherlakota (1996, p. 11) quote a rule of thumb for age-phasing—that the percentage of one’s wealth in bonds should be no more than one’s age. Richard’s model supports the age-phasing proposition.<sup>2</sup>

In addition, the optimal amount of life insurance given by Richard can be compared to that recommended by personal financial planners. Here, authors have advocated the ‘human life value’ (henceforth HLV) of Huebner (1964).<sup>3</sup> Essentially, Huebner argues for individuals taking life insurance to the value of their future earnings, thus protecting their human capital, and protecting this asset in such a way that death leaves their family’s net worth unchanged.<sup>4</sup> Within the framework of the Richard model, however, such a rule of thumb can be shown

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<sup>2</sup>After initially suggesting an investor’s time horizon doesn’t matter in investment decisions, Paul A. Samuelson spent a number of years examining the rationale for age-phasing in a search to reconcile observed age-phasing with his belief in the optimal behaviour of a constant proportion of wealth invested in risky assets. See, for example, Samuelson (1963), Samuelson (1989*a*), Samuelson (1989*b*), Samuelson (1991) and Samuelson (1994).

<sup>3</sup>This is also the view that is taught to students of life insurance. See Black & Skipper (1987, pp. 201–202, 204–205).

<sup>4</sup>Mathematically, the HLV of someone aged  $t$  can be expressed as  $\int_t^\infty Y(\theta)e^{-r(\theta-t)}d\theta$ , where  $r$  is the discount rate used and  $Y(\theta)$  is the person’s income at time  $\theta$ .

to be questionable. Rather, an appropriate amount of life insurance should be based on current consumption, or a multiple thereof.<sup>5</sup>

The paper is organized as follows. Section 2 discusses the Richard model. The approach to solving the model is given in section 3. Section 4 treats the results of the model and analyses the findings. Some concluding remarks are made in section 5. Appendix A describes an alternative approach to the model’s solution, which verifies the paper’s results. Appendix B details some computational considerations.

## 2 The Model

Richard (1975) models a multi-period utility maximizing investor with objective<sup>6</sup>

$$\max E\left[\int_{\tau}^T U(C(t), t)dt + B(Z(T), T)\right], \quad (1)$$

where  $T$  is the investor’s uncertain time of death, and  $U$ ,  $C$ ,  $Z$  and  $B$  are the investor’s utility, consumption, legacy at death and utility from bequest. The investor is able to choose between two securities, one risky and one risk-free, with the price of the risky asset,  $Q$ , following geometric Brownian motion

$$\frac{dQ(t)}{Q(t)} = \alpha dt + \sigma dq(t), \quad (2)$$

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<sup>5</sup>In this way, we get yet another “consumption-based” rule from modern financial theory.

<sup>6</sup>One problem that has been discussed concerning this objective function (Borch 1990, pp. 257–260) has been that it doesn’t allow for the spouse or beneficiary predeceasing the insured. For simplicity, we assume that in the event the spouse dies before the insured, the insured immediately finds someone whom he or she wishes to insure at the same amount.

The resolution of this issue is not straightforward; Borch does not attempt it. We leave the issue for future research. However, it must be borne in mind that our solution results in over-insurance to the extent this wrinkle matters.

where  $dq(t)$  is a Wiener increment.

The investor's change in wealth is given by the stochastic differential equation

$$\begin{aligned} dW(t) = & -C(t)dt - P(t)dt + Y(t)dt + rW(t)dt + \\ & (\alpha - r)\pi(t)W(t)dt + \sigma\pi(t)Wdq(t), \end{aligned} \quad (3)$$

where  $P(t)$ ,  $Y(t)$ ,  $W(t)$  are, respectively, the investor's life insurance premium paid, income (assumed to be non-stochastic), and wealth at time  $t$ . From equation (2), the mean return on risky investment is  $\alpha$ , with standard deviation  $\sigma$ , while the risk-free investment returns  $r$ ; the investor places a proportion  $\pi$  of wealth in the risky asset.

Richard's model necessarily incorporates the probability of death of an investor. Let the investor's age-at-death,  $X$ , a continuous random variable, have a cumulative distribution function given by  $F(x)$  and probability density function of  $f(x)$ . Consequently,  $S(x) = 1 - F(x)$  gives the probability that the investor lives to age  $x$ . The function  $S(x)$  is known as the survival function. The conditional probability density function (the probability the investor dies at exact age  $x$ , having survived to that age) is given by  $f(x)/S(x)$ , and is known as the *force of mortality* by demographers and actuaries, or as the *hazard rate* or *intensity rate* by reliability theorists (Elandt-Johnson & Johnson 1980).

The investor buys instantaneous term life insurance to the amount of  $Z(t) - W(t)$ . For this, a premium of  $P(t)$  is paid. If we denote the force of mortality by

$\mu(t)$ , then the amount of premium paid for actuarially fair insurance will be<sup>7</sup>

$$P(t) = \mu(t)(Z(t) - W(t)). \quad (4)$$

The investor's problem is to solve equation (1), subject to budget constraint (3) and initial wealth condition  $W(0) = W_0$ , by optimal choice of controls  $C, \pi$  and  $Z$ .  $U$  is assumed to be strictly concave in  $C$  and  $B$  is assumed strictly concave in  $Z$ . Equation (1) can be re-expressed as

$$J(W, \tau) = \max_{C, Z, \pi} E_\tau \int_\tau^\omega \left( \frac{S(T)}{S(\tau)} \right) \mu(T) \left[ \int_\tau^T U(C(t), t) dt + B(Z(T), T) \right] dT, \quad (5)$$

where  $\omega$  represents the limiting age of the underlying mortality table, i.e.,  $X \in [0, \omega]$ . Applying Fubini's theorem, equation (5) becomes

$$J(W, \tau) = \max_{C, Z, \pi} E_\tau \int_\tau^\omega \frac{S(T)}{S(\tau)} [\mu(T)B(Z(T), T) + U(C(T), T)] dT \quad (6)$$

after swapping the order of integration over the triangle  $T \geq t, t \geq \tau$  in  $\Re^2$ . The Hamilton-Jacobi-Bellman (HJB) equation is therefore

$$\begin{aligned} 0 = & \max_{C, Z, \pi} \left\{ \mu(t)B(Z(t), t) + U(C(t), t) - \mu(t)J + J_t \right. \\ & \left. + [\pi\alpha W + (1 - \pi)rW + Y - C - P]J_W + \frac{1}{2}\sigma^2\pi^2 W^2 J_{WW} \right\}. \end{aligned} \quad (7)$$

As  $Y(t)$  in equation (7) is non-stochastic, Richard demonstrates that (7) is equivalent to an equation involving capitalized  $Y(t)$ . That is, *adjusted wealth* is

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<sup>7</sup>Richard actually considered the more general case of there being some sort of 'loading' to mortality. This means mortality rates are increased to above their true levels, ensuring profitability for the life insurer. For the purposes of this paper, the simpler case of actuarial fairness is sufficient, with the 'loaded' model being an straightforward extension.

defined as

$$\tilde{W}(t) \equiv W(t) + b(t), \quad (8)$$

where  $b(t)$  is defined as the capitalized value of future income:

$$b(t) = \int_t^\omega Y(\theta) \frac{S(\theta)}{S(t)} e^{-r(\theta-t)} d\theta. \quad (9)$$

The standard approach (Richard 1975, Bodie, Merton & Samuelson 1992) is to remove  $Y(t)$  from (7) and substitute  $\tilde{W}(t)$  for  $W(t)$ . Income is thus treated as a traded asset. As Bodie et al. (1992) rightly point out, the individual never actually ‘sells’ his or her human capital, but rather enters the (assumed) complete market in traded securities to accomplish the same thing. For example, the riskless asset could be sold short, with the proceeds invested in the risky asset; the short sale would be designed so future liabilities from the sale are offset by future wage payments.

Richard provides an algebraic solution to the above model for CRRA utility. He demonstrates that when

$$U(C(t), t) = h(t) \frac{C^\gamma(t)}{\gamma}, \quad \gamma < 1, h > 0, C > 0 \quad (10)$$

$$B(Z(t), t) = m(t) \frac{Z^\gamma(t)}{\gamma}, \quad \gamma < 1, h > 0, Z > 0 \quad (11)$$

the optimal controls are given by<sup>8</sup>

$$C^*(W, t) = \left( \frac{h(t)}{\hat{a}(t)} \right)^{1/(1-\gamma)} [W + b(t)], \quad (12)$$

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<sup>8</sup>Note that equation (13) simplifies equation (32) of Richard, due to the consideration of actuarially fair insurance.

$$Z^*(W, t) \equiv W + \frac{P^*(W, t)}{\mu(t)} = \left( \frac{m(t)}{\hat{a}(t)} \right)^{1/(1-\gamma)} [W + b(t)] \quad \text{and} \quad (13)$$

$$\pi^*(W, t)W = \frac{\alpha - r}{(1 - \gamma)\sigma^2} [W + b(t)] = \tilde{\pi}^* \tilde{W} \quad (14)$$

where

$$\hat{a}(t) = \left\{ \int_t^\omega k(\theta) \frac{S(\theta)}{S(t)} \exp \left[ \frac{\gamma}{1 - \gamma} \left( \frac{(\alpha - r)^2}{2(1 - \gamma)\sigma^2} + r \right) \right] d\theta \right\}^{1-\gamma} \quad (15)$$

and<sup>9</sup>

$$k(t) = \left\{ \left[ \frac{1}{\mu(t)} \right]^{\gamma/(1-\gamma)} [\mu(t)m(t)]^{1/(1-\gamma)} + h^{1/(1-\gamma)}(t) \right\}. \quad (16)$$

The solutions are linear in adjusted wealth, a familiar result for HARA (hyperbolic absolute risk aversion) utility functions (Merton 1971). Interestingly, for  $h(t) = m(t)$  optimal consumption and bequest amounts will be identical. The solution for  $\tilde{\pi}^*$  indicates investment in the risky asset should be a constant fraction of adjusted wealth. This is an example of the well-known result that optimal investment behaviour over the life cycle, for utility functions that display constant relative risk aversion, is “myopic”, with individuals always investing a constant proportion of wealth in the risky asset and ignoring the future distribution of asset returns.

Numeric solutions to this model are not available. Although it is possible to approximate the integral in (15) (see Appendix A for details), a more general approach is to use a probabilistic approximation to solve the control problem. Such an approach readily handles different functional forms for  $U$ ,  $B$ ,  $h$  and  $m$ . It also makes the inclusion of constraints to the model relatively straightforward.

Richard’s model allows individuals to borrow an unlimited sum at the risk-

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<sup>9</sup>Equation (39) of Richard, which gives the formula for  $k(t)$ , contains a typographical error. Equation (16) above is this author’s corrected version—for the case of actuarially fair life insurance.



free rate,  $r$ . A better model would include some sort of borrowing restraint. Drawing on the work of Fleming & Zariphopoulou (1991), one possibility would be to allow individuals to borrow up to the amount of their *financial* wealth,  $W(t)$  ( $\equiv \tilde{W}(t) - b(t)$ ), at the risk-free rate, but at a higher rate  $R$  for borrowings in excess of this amount.<sup>10</sup> Hence, (3) is modified to become:

$$\begin{aligned} d\tilde{W}(t) = & -C(t)dt - P(t)dt + r\tilde{W}(t)dt + (\alpha - r)\hat{\pi}(t)\tilde{W}(t)dt \\ & - (R - r)\max[\hat{\pi}(t)\tilde{W}(t) - W(t), 0]dt + \sigma\hat{\pi}(t)\tilde{W}(t)dq(t), \end{aligned} \quad (17)$$

where  $\hat{\pi}(t)$  is the *modified* proportion of adjusted wealth investment in risky assets and  $\max[\hat{\pi}(t)\tilde{W}(t) - W(t), 0]$  is the amount of money the individual has borrowed. The second last term of equation (17) captures the essence of costly borrowing by reducing the returns to adjusted wealth.

Finally, noting that discounted values of the functions of interest can produce very small values, we may facilitate the numerical solution of the model by converting the discounted HJB equation (7) to current values. If we set  $h(t) = e^{-\rho t}$ , so  $U(C(t), t) = e^{-\rho t}\tilde{U}(C(t))$ , and  $m(t) = e^{-\rho t}\phi(t)$ , so  $B(Z(t), t) = e^{-\rho t}\phi(t)\tilde{B}(Z(t))$ , where  $\rho$  represents an individual's rate of time preference and  $\phi(t)$  is a function relevant to bequest determination, then equation (7) becomes

$$\begin{aligned} 0 = & \max_{C, Z, \tilde{\pi}} \left\{ \mu(t)\phi(t)\tilde{B}(Z(t)) + \tilde{U}(C(t)) - \mu(t)\tilde{J} - \rho\tilde{J} + \tilde{J}_t \right. \\ & \left. + [\tilde{\pi}\alpha\tilde{W} + (1 - \tilde{\pi})r\tilde{W} - C - P]\tilde{J}_W + \frac{1}{2}\sigma^2\tilde{\pi}^2W^2\tilde{J}_{WW} \right\}. \end{aligned} \quad (18)$$

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<sup>10</sup>This could be seen as reflecting the presence of moral hazard problems associated with borrowing against one's future income.

### 3 Solving the Model

The numerical solution of finite horizon stochastic optimal control problems is well described in Kushner (1977, Chapter 7) and Kushner & Dupuis (1992, Chapter 12). The “explicit” solution approach involves a finite difference approximation to the HJB equation (18), a second-order linear parabolic partial differential equation, as described below.

Firstly, consider the coefficient of  $\tilde{J}_W$  in (18). Partition the terms that make up this coefficient into a positive group,  $d^+ = (\alpha - r)\tilde{\pi}\tilde{W} + r\tilde{W} + \mu\tilde{W}$ , and a negative group  $d^- = C + \mu Z$ , where  $d = d^+ + d^-$ ,  $d$  being the coefficient of  $\tilde{J}_W$ . Let us approximate the partial derivatives in equation (18) as follows:

$$\begin{aligned} f_t(x, t) &\rightarrow \frac{f(x, t + \delta) - f(x, t)}{\delta} \\ f_x(x, t) &\rightarrow \frac{f(x + h, t + \delta) - f(x, t + \delta)}{h} \quad \text{for } d^+ \\ f_x(x, t) &\rightarrow \frac{f(x, t + \delta) - f(x - h, t + \delta)}{h} \quad \text{for } d^- \\ f_{xx}(x, t) &\rightarrow \frac{f(x + h, t + \delta) + f(x - h, t + \delta) - 2f(x, t + \delta)}{h^2} \end{aligned}$$

and write (18) as follows, where  $V(\cdot, \cdot)$  represents the solution to the finite difference equation:

$$\begin{aligned} 0 = & \max_{C, \tilde{\pi}, Z} \left\{ \mu\phi\tilde{B}(Z(t)) + \tilde{U}(C(t)) - (\mu + \rho)V(\tilde{W}, t) + \right. \\ & \frac{V(\tilde{W}, t + \delta) - V(\tilde{W}, t)}{\delta} + \frac{V(\tilde{W} + h, t + \delta) - V(\tilde{W}, t + \delta)}{h} d^+ \\ & \left. - \frac{V(\tilde{W}, t + \delta) - V(\tilde{W} - h, t + \delta)}{h} d^- + \right. \\ & \left. \frac{1}{2}(\sigma\tilde{\pi}W)^2 \frac{V(\tilde{W} + h, t + \delta) + V(\tilde{W} - h, t + \delta) - 2V(\tilde{W}, t + \delta)}{h^2} \right\}. \quad (19) \end{aligned}$$

Equation (19) can be written as

$$\begin{aligned}
V(\tilde{W}, t) = & \max_{C, \tilde{\pi}, Z} \frac{1}{1 + \mu\delta + \rho\delta} \left\{ \tilde{U}(C(t)) + \mu\phi(t)\tilde{B}(Z(t)) \right. \\
& V(\tilde{W}, t + \delta) \left[ 1 - \frac{\delta}{h}d - \frac{\delta}{h^2}\sigma^2 \right] + \\
& \left. V(\tilde{W} + h, t + \delta) \left[ \frac{\delta}{h}d^+ + \frac{\delta}{h^2}\frac{\sigma^2}{2} \right] + V(\tilde{W} - h, t + \delta) \left[ \frac{\delta}{h}d^- + \frac{\delta}{h^2}\frac{\sigma^2}{2} \right] \right\}
\end{aligned}$$

or, more conveniently,

$$\begin{aligned}
V(\tilde{W}, t) = & \max_{C, \tilde{\pi}, Z} \frac{1}{1 + \mu\delta + \rho\delta} \left\{ \left[ \sum_{\theta=-1}^1 p(\tilde{W}, \tilde{W} + \theta h) V(\tilde{W} + \theta h, t + \delta) \right] \right. \\
& \left. + \delta \left[ \tilde{U}(C(t)) + \mu\phi(t)\tilde{B}(Z(t)) \right] \right\} \quad (20)
\end{aligned}$$

where the  $p(\cdot, \cdot)$  may be interpreted as transition probabilities of a Markov chain, locally consistent with equation (3), and given by:<sup>11</sup>

$$p(\tilde{W}, \tilde{W} + h) = \frac{\delta}{h^2} \left\{ \frac{1}{2}(\sigma\tilde{\pi}\tilde{W})^2 + h \left[ (\alpha - r)\tilde{\pi}\tilde{W} + r\tilde{W} + \mu\tilde{W} \right] \right\} \quad (21)$$

$$p(\tilde{W}, \tilde{W} - h) = \frac{\delta}{h^2} \left\{ \frac{1}{2}(\sigma\tilde{\pi}\tilde{W})^2 + h [C + \mu Z] \right\} \quad (22)$$

$$p(\tilde{W}, \tilde{W}) = 1 - p(\tilde{W}, \tilde{W} + h) - p(\tilde{W}, \tilde{W} - h) \quad (23)$$

The boundary condition is  $V(\tilde{W}, \omega) = \phi(\omega)\tilde{B}(Z(\omega))$ . Thus, the solution to the investor's stochastic control problem (equation (1)) is approximated by the solution to equation (20) as  $h \rightarrow 0$  and  $\delta \rightarrow 0$  together. The convergence of this approximation method has been established by viscosity solution techniques (Fitzpatrick & Fleming 1990, Fitzpatrick & Fleming 1991).

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<sup>11</sup>We must, through choice of  $\delta$  and  $h^2$ , ensure  $0 \leq p(\cdot, \cdot) \leq 1$ .

$\alpha = 0.10$	$h = 0.02$
$r = 0.05$	$N = 1\,000$
$\rho = 0.05$	$\gamma = -0.5$
$\sigma = 0.30$	$R = 0.06$
Mortality: A1967–70 <sup>a</sup>	$\omega = 110$

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<sup>a</sup>From Neill (1977, Appendix III). This is a table of assured lives mortality; no sex distinction is present. Many annuity and assurance values are tabulated, making it particularly useful in interpreting the solution results.

Table 1: Parameters used in the numerical solution of the model.

Equation (20) was solved on a grid by backward iteration, using a computer. Further details may be found in Appendix B.

Imposing the borrowing constraint, discussed in Section 2 above, is a straightforward matter of solving equation (20), subject to the required constraint.

## 4 Results and Implications

This section lays out the results of the numerical solutions of the Richard model, as well as some implications of its results. Firstly, we consider the parametrization of the model. Consideration is given to the resulting life insurance implications, and how these suggest the HLV rule of thumb is not an appropriate rule for life insurance purchases. Numeric results are then discussed, and we consider the life cycle dynamics of the model with a simple illustrative example showing its age-phasing consequences. Introducing realistic constraints concludes the section.

Using the parameters set out in Table 1, the Richard model was solved using the methodology of the preceding section. Details on the verification of the solution may be found in Appendix A. The functional forms used for  $U(C, t)$  and  $B(Z, t)$  were those of equations (10) and (11);  $h(t)$  was set to  $e^{-\rho t}$  and  $m(t)$  were set to  $e^{-\rho t}\phi(t)$ . From equation (13), we can see that if  $\phi(t) = \xi^{1-\gamma}(t)$ , then  $Z^*(t) = \xi(t)C^*(t)$ . Drawing on prevailing social norms, a reasonable value to

provide a spouse following the death of his or her partner would be an amount sufficient to provide two-thirds of the deceased's current income for life. This notion has in fact been enshrined in pension benefits regulations in Canada, for example, where the surviving spouse of a deceased pensioner is provided with between 50% and 66 2/3% pension continuation.<sup>12</sup> Hence, we set  $\xi(t)$  to  $2/3\bar{a}_t^r$ , giving  $Z^*$  as  $2/3C^*\bar{a}_t^r$ , where<sup>13</sup>

$$\bar{a}_t^r = \int_t^\omega \frac{S(\theta)}{S(t)} e^{-r(\theta-t)} d\theta. \quad (24)$$

In this way, the bequest value is made to be a stock, rather than a flow variable.

The above serves to underline the dependence of individual's life insurance needs on their bequest function. Under the assumptions made, the inconsistency of this model with the HLV concept of financial planning is apparent: individuals' insurance needs are dependent on their personal characteristics, and in particular, their level of consumption. An approach to life insurance provision consistent with this model would be based on some function of investor consumption. The HLV rule of thumb is, however, based on the flow of future income. Intuitively we can see the HLV concept providing too much insurance for the case of high income individuals with only moderate consumption tastes, or providing too little for those with high rates of time preference.

The solutions for a mildly risk averse investor ( $\gamma = -0.5$ ) with time preference  $\rho = 0.05$  are displayed in Table 2.<sup>14</sup> The constancy of  $\tilde{\pi}^*$  is apparent and its values

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<sup>12</sup>See, for example, s. 45(2) of Ontario's Pension Benefits Act, 1987.

<sup>13</sup>For simplicity, we assume partners are close in age and have similar preferences. In the numerical work  $\bar{a}_t^{0.05}$  is approximated by a linear function,  $20 - (19/80)(x - 30)$ , which, while not exactly matching  $\bar{a}_t^{0.05}$  values, gives both a reasonable approximation to the function and avoids the inconvenience of a zero boundary condition.

<sup>14</sup>The value of  $\gamma = -0.5$  corresponds to an Arrow-Pratt measure of relative risk aversion of 1.5. Recent numerical estimates (Hansen & Singleton 1983, Mankiw 1985) place this measure

Age	$E(\tilde{W})$ (\$)	$E(W)$ (\$)	$C^*$ (\$)	$\tilde{\pi}^*$ (%)	$Z^*$ (\$)	Life Insurance (\$)	Invested in Risky (\$)	HLV Life Insurance (\$)
30	1 000 000	10 000	59 100	35.9	792 000	782 000	359 000	1 017 000
40	1 062 000	221 000	67 500	36.0	797 000	576 000	382 000	876 000
50	1 070 000	459 000	76 600	36.4	792 000	333 000	389 000	645 000
60	962 000	704 000	82 700	36.7	714 000	10 000	353 000	269 000
70	711 000	711 000	79 600	36.8	560 000	-151 000	262 000	0
80	356 000	356 000	56 100	36.5	305 000	-51 000	130 000	0
90	81 000	81 000	19 600	36.8	76 000	-5 000	30 000	0
100	7 000	7 000	2 500	36.8	6 000	-1 000	3 000	0

Table 2: Numerical solution of Richard (1975),  $\gamma = -0.5$ ,  $\rho = 0.05$ , and life insurance levels from HLV concept.

are very close to the theoretical value given by equation (14). Table 2 gives results for an individual whose lifetime wealth path follows that expected.<sup>15</sup>

The table indicates that for a 30 year old individual with wealth of 1% of adjusted wealth (\$10 000) and a constant of income of about \$61 000 to age 65 (implies  $b(30)=\$990\,000$ ),<sup>16</sup> the individual will consume \$59 100, invest \$359 000 in risky assets, and take insurance to the value of around \$782 000. An 80 year old individual, on the other hand with wealth of \$356 000 and no future income, would consume \$56 100, invest \$130 000 in risky assets, and *sell* insurance (or take bets on his or her death) to the value of \$51 000. (This quirk will be expanded on below.) At all ages, the level of life insurance suggested by the Richard (1975) model is less than than given by the HLV rule of thumb.

The optimal proportion of adjusted wealth invested in the risky asset is con-

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between 1 and 3. Lucas (1990) advances empirical and theoretical arguments based on cross-country interest rate differentials, why the measure should be less than 2.

<sup>15</sup>Expected wealth holdings over an individual's lifetime were generated by computer simulation (Purcal forthcoming).

<sup>16</sup>This follows from  $60\,636.07 \int_{30}^{65} [S(\omega)/S(30)] e^{-r(\omega-30)} d\omega = 990\,000$ , using the interest and mortality rates given in Table 1.

Age	Proportion of financial wealth invested in risky, $\pi^*$ (%)
30	3 590
35	328
40	173
45	114
50	83
55	64
60	49
>65	37

Table 3: Age-phasing path for numerical solution of Richard (1975),  $\gamma = -0.5$ ,  $\rho = 0.05$ .

stant over time, a feature of models with individuals with utility functions that exhibit constant relative risk aversion and an underlying stochastic process with constant coefficients. Note, however, as a proportion of financial wealth (as opposed to *adjusted* wealth), the proportion invested in risky falls as investors age—our 30 year old invests 3 590% of financial wealth in risky while the 80 year old invests only 37%. Bodie et al. (1992) point out that a major practical manifestation of this is the leveraged purchase of the family home. This is an example of age-phasing, which advocates individuals should reduce their exposure to risky assets over time. Table 3 gives the age-phasing path generated by our solution to the model. Thus, Richard’s model lends support to this common tenet of financial planning, suggesting also that young investors may be highly leveraged.

It is interesting to observe in Table 2 that, for our parametrization of the model, one needs a considerable amount of life insurance when young—this amount decreases over one’s working life to become zero, and even negative around, and after, the time of retirement. A negative premium implies individu-

als are able to sell life insurance (for a life aged exactly their age) or, alternatively, they are able to take bets on their own deaths, to be paid out of terminal wealth.

As Fischer (1973) pointed out, this state of affairs is equivalent to the purchase of an annuity. Thus, the model has also provided a motivation for retirement income. Table 2 shows optimal annuity payment levels start at zero near retirement and rise to a peak, then gradually falling off.

It is in the area of retirement income that a weakness of the current set up of the model is apparent. The utility function used allows consumption to move to very low levels at higher ages. Clearly, to motivate reasonable retirement income needs, a more general HARA utility function with a positive level of subsistence consumption would be more appropriate.

As discussed above, a more appropriate model of consumption and investment behaviour would also include a borrowing constraint. The constraint, equation (17), was incorporated into the model. The results for  $R = 0.06$  are shown in Table 4. The effect on behaviour is as one would expect: when the constraint binds we see investment falling to a rate less than when borrowing is unconstrained. Here  $\hat{\pi}^*$  falls from its unconstrained level of around 37% to around 30% when the constraint is binding (i.e., when financial wealth is less than 37% of adjusted wealth).

Through developing the numerical solution for the borrowing constrained model, it became apparent that the investor's portfolio selection behaviour follows certain rules, set out in Table 5.<sup>17</sup> The results in the  $\hat{\pi}^*$  column of Table 4 do not exactly reflect the results of Table 5. This is because the solution for  $\hat{\pi}^*$  derives from equations of the form  $-[(\alpha - r)W/(\sigma W)^2][v'(W)/v''(W)]$ ,

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<sup>17</sup>These results are identical to those explored theoretically in Fleming & Zariphopoulou for an optimal investment/consumption model with the same borrowing constraint, but with infinitely-lived investors, no life insurance and no income.



Age	$\tilde{W}^a$ (\$)	$W^b$ (\$)	$C^*$ (\$)	$\hat{\pi}^*$ (%)	$Z^*$ (\$)	Life Insurance (\$)	Invested in Risky (\$)	$W/\tilde{W}$ (%)
30	1 000 000	10 000	54 500	31.0	731 000	721 000	310 000	1.0
31	1 008 000	25 000	55 100	30.4	730 000	705 000	306 000	2.5
32	1 016 000	42 000	55 800	29.8	730 000	688 000	302 000	4.1
33	1 023 000	58 000	56 500	29.1	730 000	672 000	298 000	5.7
34	1 031 000	77 000	57 100	28.6	729 000	652 000	294 000	7.4
35	1 039 000	97 000	57 900	28.0	730 000	633 000	291 000	9.3
36	1 042 000	113 000	58 500	27.6	728 000	615 000	287 000	10.9
37	1 048 000	134 000	59 300	27.1	729 000	595 000	284 000	12.8
38	1 053 000	154 000	60 000	26.8	728 000	574 000	282 000	14.7
39	1 058 000	176 000	60 900	26.5	729 000	553 000	280 000	16.7
40	1 062 000	199 000	61 700	26.3	729 000	530 000	279 000	18.7
41	1 066 000	223 000	62 600	26.1	729 000	506 000	279 000	20.9
42	1 073 000	250 000	63 600	26.0	731 000	481 000	279 000	23.3
43	1 075 000	274 000	64 500	26.0	730 000	456 000	279 000	25.5
44	1 075 000	298 000	65 300	27.8	730 000	432 000	298 000	27.7
45	1 077 000	324 000	66 200	30.2	730 000	406 000	325 000	30.1
46	1 079 000	351 000	67 000	32.6	727 000	376 000	351 000	32.6
47	1 079 000	379 000	67 800	34.4	725 000	346 000	371 000	35.1
48	1 078 000	407 000	68 500	35.0	722 000	315 000	377 000	37.7
49	1 076 000	434 000	69 200	35.5	718 000	284 000	382 000	40.3
50	1 070 000	459 000	69 500	35.8	710 000	251 000	383 000	42.9
60	962 000	727 000	71 100	36.6	614 000	-113 000	353 000	75.6
70	711 000	711 000	63 800	36.8	449 000	-262 000	261 000	100.0
80	356 000	356 000	42 300	36.8	230 000	-126 000	131 000	100.0
90	81 000	81 000	14 000	36.8	54 000	-27 000	30 000	100.0
100	7 000	7 000	2 000	36.8	4 000	-3 000	2 000	100.0

<sup>a</sup>These are the  $E(\tilde{W})$  values from Table 2.

<sup>b</sup>In an effort to reduce computation time,  $b(t)$  was approximated by a quadratic equation. Hence the difference between this column and the corresponding column in Table 2.

Table 4: Numerical solution of Richard (1975),  $\gamma = -0.5$ ,  $\rho = 0.05$ ,  $R = 0.06$ .

Market	$\hat{\pi}^*(t)$
$\frac{(\alpha-r)}{\sigma^2(1-\gamma)} > \frac{(\alpha-R)}{\sigma^2(1-\gamma)} > \frac{W}{\tilde{W}}$	$\frac{(\alpha-R)}{\sigma^2(1-\gamma)}$
$\frac{(\alpha-r)}{\sigma^2(1-\gamma)} > \frac{W}{\tilde{W}} > \frac{(\alpha-R)}{\sigma^2(1-\gamma)}$	$\frac{W}{\tilde{W}}$
$\frac{W}{\tilde{W}} > \frac{(\alpha-r)}{\sigma^2(1-\gamma)} > \frac{(\alpha-R)}{\sigma^2(1-\gamma)}$	$\frac{(\alpha-r)}{\sigma^2(1-\gamma)}$

Table 5: Effect of borrowing constraint.

$-[(\alpha - R)W/(\sigma W)^2][v'(W)/v''(W)]$  or  $W/\tilde{W}$ , where  $v(W)$  is the value function corresponding to the HJB equation (18) above. Due to the HARA form of the direct utility function,  $U(C)$ , it is readily established that  $v'(W)/v''(W) = -W/(1 - \gamma)$  (Fleming & Zariphopoulou 1991, p. 808), giving the values that appear in Table 5. However, for the purposes of the approximation method,  $v'(W)/v''(W)$  is determined by finite difference approximations. These do not always produce constant values of  $-W/(1 - \gamma)$ . Closer correspondence would be achieved by working over a finer grid.

The investor's life cycle proportion-in-risky investment path ( $\hat{\pi}^*$ ) from Table 4 is represented diagrammatically in Figure 1. In region A of the figure the investor has to borrow funds to make his or her optimal investment in the risky asset; the investor's optimal investment in risky is  $(\alpha - R)/[\sigma^2(1 - \gamma)] = 30\%$ . Region A of the figure corresponds to ages 30–44 in Table 4. In region B, optimal investment is at rate  $W/\tilde{W}$ —this level of investment is better than investing at the lower rate of  $(\alpha - R)/[\sigma^2(1 - \gamma)]$ , or borrowing to invest at the higher rate of  $(\alpha - r)/[\sigma^2(1 - \gamma)]$ . In region B the level of investment tracks the growth in financial wealth. Region B corresponds to ages 45–47 in Table 4.

In region C of Figure 1 the investor no longer has to borrow to invest at level

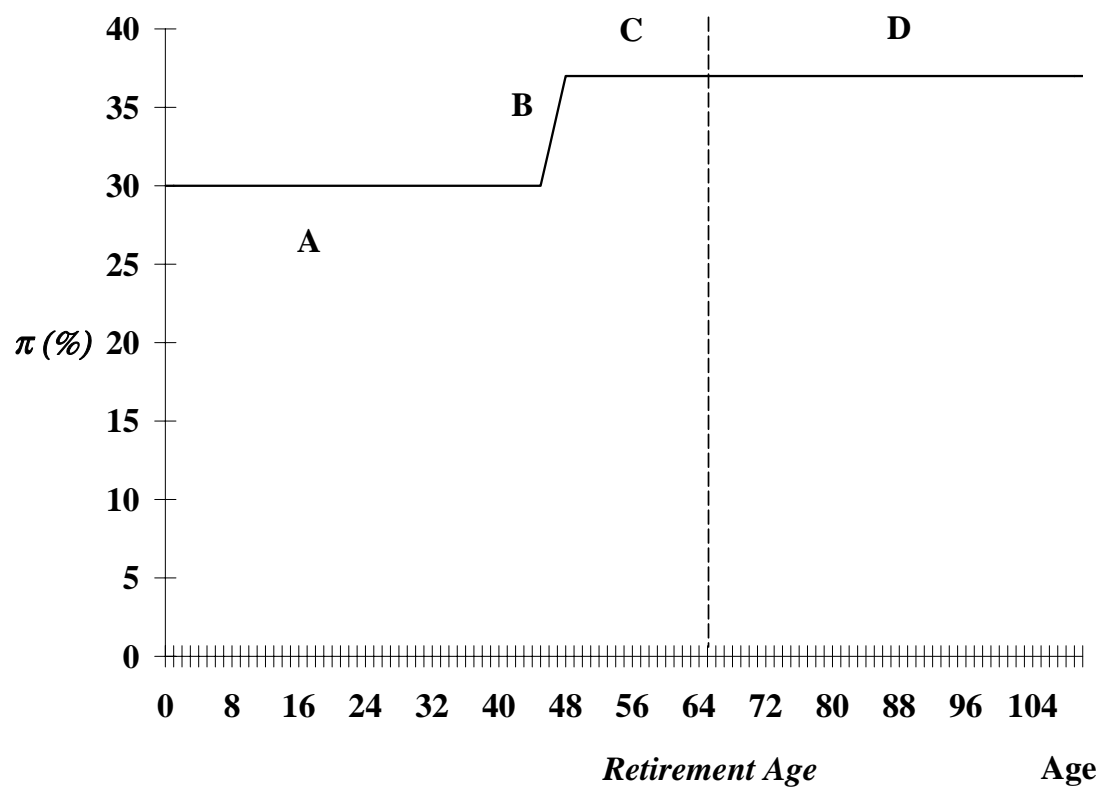


Figure 1: Investment in risky asset over time with costly borrowing,  $R = 0.06$ .

$(\alpha - r)/[\sigma^2(1 - \gamma)]$ . His or her level of financial wealth is a sufficient proportion of adjusted wealth to be able to draw on those funds for all desired risky investment. This occurs between ages 48 and 65 in Table 4. In region D the investor has retired, and no longer has a human capital component to his or her adjusted wealth. As a result, in region D the investor would invest in risky at level  $(\alpha - r)/[\sigma^2(1 - \gamma)]$  regardless of his or her wealth situation.

In Table 4, investors' levels of consumption and life insurance over their lifetimes are lower than the unconstrained levels of Table 2.

## 5 Conclusion

This paper has examined the Richard (1975) model as a preliminary step to developing a relevant model for financial planning. Using a probabilistic approach, numerical solutions to the Richard model were provided.

The Richard model, with its non-stochastic income, generated an age-phased reduction in investment in risky assets over an individual's life, lending support to this traditional piece of financial planners' advice. For the parametrization of the model considered above, we are led to the situation of individuals being highly leveraged in their youth.

The life insurance behaviour implied by the model was also examined, showing that, for the CRRA parametrization of the model considered, optimal life insurance purchase is related to consumption levels. This finding calls into question the usefulness of the HLV concept of financial planning, which focuses on future income streams. In addition, the model provides details about optimal annuity purchase around retirement, suggesting a hump shaped pattern of annuity receipts.

Consideration was given to making the Richard model more realistic. The imposition of a borrowing constraint, reflecting costly borrowing, reduced investment in the risky asset in a way that was able to be quantified.

The solution method employed is quite powerful, able to deal with complications like the constraints mentioned above, as well as different functional forms.

Future work aims to inject further realism into the model in a variety of ways. By recasting the model into a more general HARA utility framework, with a positive subsistence consumption level, we should be able to motivate more realistic levels of retirement income. Currently, the model permits low levels of consumption at older ages.

Bodie et al. (1992) considered uncertain wage income, thus exposing individuals to future income risk, with consequences for consumption and investment behaviour. In their model, wage income was perfectly correlated with the risky asset's diffusion. Using the techniques discussed in this paper, and drawing on the recent work of Duffie & Zariphopoulou (1993) and Duffie, Fleming, Soner & Zariphopoulou (1997), it is possible to consider the more interesting case of imperfect correlation between the risky asset and wages.

## **A Verification of Results**

The text above has discussed a numerical approach to approximate solutions to Richard's optimal stochastic control problem. It is important to verify the accuracy of any numerical approximation; this is the purpose of this Appendix.

The results of the numerical solution have been verified in the following man-

ner. Equation (15) may be expressed as

$$\begin{aligned}
\hat{a}(t) &= \left\{ (e^{-\rho t})^{1/(1-\gamma)} \int_t^\omega (\mu(\theta) + 1) \frac{S(\theta)}{S(t)} \exp \left[ \left( \frac{1}{1-\gamma} \rho - \frac{\gamma}{1-\gamma} \left\{ \frac{1}{2} \left( \frac{\alpha-r}{\sigma} \right)^2 \frac{1}{1-\gamma} + r \right\} \right) (t-\theta) \right] d\theta \right\}^{1-\gamma} \\
&= e^{-\rho t} \{ \bar{A}_t^\zeta + \bar{a}_t^\zeta \}^{1-\gamma},
\end{aligned} \tag{25}$$

where  $\bar{A}_t^\zeta$  is the expected present value of a term life insurance paying \$1 on death. The term  $\bar{a}_t^\zeta$ , the expected present value of an annuity, payable continuously for life, is given by equation (24). Furthermore, equation (25) assumes  $h(t) = m(t) = e^{-\rho t}$ . Both  $\bar{A}_t^\zeta$  and  $\bar{a}_t^\zeta$  are evaluated at the continuously compounded rate of interest

$$\zeta = \left[ \frac{1}{1-\gamma} \rho - \frac{\gamma}{1-\gamma} \left\{ \frac{1}{2} \left( \frac{\alpha-r}{\sigma} \right)^2 \frac{1}{1-\gamma} + r \right\} \right]. \tag{26}$$

It is a requirement for the solution of the model that  $\zeta$  be non-negative.

Approximations are readily available for  $\bar{a}_t^\zeta$  in the actuarial literature. One such approximation is<sup>18</sup>

$$\bar{a}_t \approx \left( \sum_{\theta=t+1}^\omega e^{-\zeta\theta} \frac{S(\theta)}{S(t)} \right) + \frac{1}{2} - \frac{1}{12} [\mu(t) + \zeta].$$

The theoretical relationship between  $\bar{A}_t^\zeta$  and  $\bar{a}_t^\zeta$ ,  $\bar{A}_t^\zeta = 1 - \zeta \bar{a}_t^\zeta$ , can then be used to evaluate  $\bar{A}_t^\zeta$ .<sup>19</sup> The resulting solution values compare favourably with those generated by the numerical methods of section 3, for  $h(t) = m(t) = e^{-\rho t}$ .

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<sup>18</sup>See, for example, Neill (1977).

<sup>19</sup>This relationship is proved in Neill (1977, p. 84).

## B Computational Details

Equation (20) was solved using backward iteration over a grid of the state variable. For details of the Markov chain approximation, the boundary transition probabilities and determination of the normalizing constant  $\tilde{Q} = h^2/\delta$ , including adjusting  $\tilde{Q}$  to speed computation, see Fitzpatrick & Fleming (1991).

A grid size of  $N = 1000$  was used, with state variable  $\tilde{W}$  taking values of from 0 through 20.00, in steps of  $h = 0.02$ . As noted in other work (Fitzpatrick & Fleming 1991), values at the top and bottom of the grid are inaccurate, while values in the middle are accurate.

The solution method was coded by the author, in `C`, and run on a 166MHz Digital AlphaStation 200 4/166, with 64M of memory operating under OpenVMS v7.0. Solving the model (going from terminal age of 110 back to 30) took up to 3 hours.

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RCS revision number: 1.3  
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