Conversion Factors, Delivery Option and Hedge Efficiency of a Multi-Issuer Bond Future

Wolfgang Bühler, Klaus Düllmann *

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Secondly, we propose three conversion-factor systems to reduce the danger of a short squeeze in the spot market. The conversion factors simultaneously account for differences in coupon, time to maturity, and the credit risk of deliverable bonds.

Thirdly, in the empirical part of the paper we analyse the impact of these conversion-factor systems on the hedge efficiency of the corresponding futures contracts. Based on the value of the delivery option and the results for the hedge efficiency, an appropriate conversion-factor system is recommended, having a prospective European sovereign bond future in mind.

Keywords: multi-issuer future, delivery option, conversion factors, credit spreads, affine model

JEL classification: G13, G15

*Address: Prof. Dr. Wolfgang Bühler: Chair of Finance, University of Mannheim, D-68131 Mannheim, Germany, Phone: +49 621-181-1526, Fax: +49 621-181-1519, e-mail: w.buehler@uni-mannheim.de, and Klaus Düllmann: Deutsche Bundesbank (The analysis and conclusions of this paper are those of the authors and do not indicate concurrence by Bundesbank or by other members of its research staff.)

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1 Introduction

This paper addresses two theoretical and one empirical problem: The first problem deals with the valuation of the delivery option which consists of the traditional quality option and an additional *issuer option* component. The latter provides the seller of a bond futures contract, hereafter named the *short*, with the right to choose for delivery between bonds of different issuers with, possibly, different credit risk. To our knowledge this is the first study which examines the value of this option in detail. This is carried out by analysing the interdependency between the issuer option and the *quality option*, that derives from differences in coupon and maturity and has been a subject of an extensive literature. Our analysis of the delivery option is based on a two-factor affine credit-risk model as developed by Duffie and Singleton (1999).

Including credit-risky bonds in the delivery basket increases its heterogeneity. To control for this effect we replace the traditional conversion factor system that accounts only for maturity and coupon differences by *issuer-dependent* conversion factor systems. These new price factors explicitly account also for differences in credit risk. Note, that the futures contract itself is assumed to be default-free as it is daily marked to market. The construction of these new conversion factor systems and the analysis of their relationship with the value of the delivery option is our second theoretical contribution.

If the value of the delivery option increases in the presence of credit risk because of a more heterogenuous delivery basket this may adversely affect the hedge efficiency of the futures contract which is a well-known key success factor. Therefore, in the empirical part of the paper we compare the hedge efficiency of single and multi-issuer futures with traditional and issuer-dependent conversion-factor systems. As multi-issuer futures endowed with an issuer-dependent conversion-factor system are not traded yet, we determine theoretical futures prices from the OTC-prices of those German and Italian government bonds that constitute the delivery basket.

Besides these theoretical and empirical contributions the paper is of high relevance for the development of the European derivatives markets. The traded volume of the Bund Future has increased tremendously, especially after the introduction of a single European currency and contrary to the outstanding volume of the German government bonds that are eligible for delivery. A European sovereign bond future whose delivery basket includes bonds of different sovereign issuers offers a straightforward solution to mitigate the increasing danger of short squeezes. By analysing the relationship between the issuer-dependent conversion factors and key success factors of a futures contract we can draw important conclusions for the design of a prospective European sovereign bond future. We suggest eight requirements that a conversion-factor system should meet to advance the success of a futures contract in the market. Based on these requirements a recommendation is given which of the three issuer-dependent conversion-factor systems will be best suited for such a contract.

Three strands of literature are closely related to our study. The valuation of a multi-issuer future requires a multi-factor model that accounts for interest rate as well as for default risk in the deliverable bonds. Duffie et al. (1996) have shown for so-called *intensity-based models* that the value of a defaultable claim can be calculated under fairly general assumptions by discounting the promised payments at a default-risk adjusted discount rate. Their results justify using an affine model framework in the spirit of Cox et al. (1985).

Equally important as the literature on credit risk models is the previous research on the valuation of quality options. Among these are the theoretically oriented studies of Ritchken and Sankarasubramanian (1992), Berendes and Bühler (1994), Cherubini and Esposito (1995) and Bick (1997) who value the quality option in the context of arbitrage-free interest rate models. Among others Lin and Paxson (1995) and Yu (1997) present notable empirical results on the quality option of the Bund Future and a Japanese government bond future.

Finally, this paper relies on two preparatory empirical studies by Düllmann and Windfuhr (2000) and Bühler et al. (2001). In these studies the parameters of the two-factor affine model are estimated from the term structures of German and Italian government bonds. The parameter estimates from Bühler et al. (2001) are used in the comparative-static analysis of the delivery option. Additionally the prices of the government bonds from this study are used to value the delivery option and to determine the futures prices in the analysis of the hedge efficiency.

This paper is structured as follows: Section 2 recalls stylized facts about the delivery option and the construction principles of conversion-factor systems. Three new conversion-factor systems for credit-risky deliverable bonds are developed in section 3. In section 4 the valuation model for the multi-issuer future and its delivery option is presented. Section 5 contains comparative-static results for the value of the delivery option. In section 6 multi-issuer futures prices for three selected conversion-factor systems are determined and the hedge efficiency of the corresponding contracts is analysed. The results of section 5 and 6 are used in section 7 to determine a ranking of the conversion-factor systems in terms of their suitability for a prospective multi-issuer futures contract. Section 8 summarizes and concludes.

2 Delivery Options in Bond Futures [1]

The high number of about 75 % of exchange-traded derivatives that were delisted because of an insufficient trading volume[2] indicates that the design of a futures contract is a

[1] Readers who are familiar with the principles of conversion-factor systems may skip this section.

[2] See Johnston and McConnell (1989), p. 2.

delicate issue. Arguably the most damaging factor to the success of a futures contract is the danger of a *short squeeze*. In a short squeeze a manipulator exploits the delivery mechanism of the future by establishing a portfolio that combines a long position in a futures contract with a high concentration of ownership in the deliverable supply.[3]

In order to secure a sufficient volume of deliverable bonds and to forestall a short squeeze the short is conceded three types of delivery options. We differentiate in the following[4] between the *quality option* that derives from different coupon sizes and maturities of the deliverable assets and depends on the term structure of interest rates and the *issuer option* in the presence of differences in credit risk.

To account for different maturities, coupons and credit risk we develop conversion-factor systems, often called *price factor* systems, which adjust the invoice amount of a deliverable bond relative to the value of the synthetic underlying. For a *traditional* bond future, endowed with a quality but without an issuer option, these conversion factors account for differences in coupon size and maturity of the deliverable assets.

However the price adjustment by conversion factors is generally not perfect and the bond with the lowest adjusted price is called *cheapest-to-deliver* or briefly *ctd-bond*. This asset is regularly used for the *cash-and-carry arbitrage* which garanties the link between the bond and the futures market. Therefore, the existence of a ctd-bond accompanied by a positive value of the delivery option is principally favored by market participants. The relationship between the bond and futures price at maturity, henceforth named the *terminal* prices, and the conversion factors is as follows. Assume that n bonds with different coupon size c_j, maturities M_j, conversion factors κ_j and prices $P(T, M_j, c_j)$ can be delivered into a futures contract at its maturity T. Then the arbitrage-free futures price $F(T, T, \{1 \ldots, n\})$ at maturity is given by[5]

$$F(T, T, \{1 \ldots, n\}) = \min_{j \leq n} \left\{ \frac{P(T, M_j, c_j)}{\kappa_j} \right\}. \tag{1}$$

The ratio of the bond price and the conversion factor is called its *futures-equivalent* price and the bond with the lowest futures-equivalent price is the ctd-bond.

The futures-equivalent prices determine a monotonicly increasing ranking of the deliverable bonds beginning with the ctd-bond. The price differences between them restrict the up-movement of the futures price in the case of a short squeeze. Therefore, conversion factors aim at reducing the differences between the futures-equivalent prices. Note that theoretically an *ideal* conversion-factor system can be constructed that eliminates the differences between the future-equivalent prices. The delivery option becomes worthless

[3]See Manaster (1992), p. 143.

[4]As a third type there exist *timing options* which offer the short a choice when to deliver the assets (see Gay and Manaster (1984), pp. 41–51, 68–71.). Timing options have been studied by Boyle (1989) and more recently by Cohen (1995).

[5]Compare e. g. Duffie (1989), p. 327.

and a manipulator would not only have to squeeze the ctd-bond but all deliverable bonds simultaneously. Such ideal conversion factors κ_j^* are determined as the ratio of the terminal market price of the bond and the market price of the synthetic underlying (with coupon c_0 and maturity M_0):

$$\kappa_j^* = \frac{P(T, M_j, c_j)}{P(T, M_0, c_0)}. \tag{2}$$

In this case the futures price equals the market value of the notional bond. However, this solution is not practically feasible because the notional bond price $P(T, M_0, c_0)$ is not observable.

Current price factors adopt the idea of a price ratio of the deliverable asset and the notional bond but they use proxies instead of market prices. These proxies are determined by assuming that the yield-to-maturity of each deliverable bond equals the coupon of the notional bond so that the denominator of the price factor is standardized to the face value of the notional bond. The invoice amount received by the short equals the futures price multiplied by a bond-specific conversion factor. This conversion factor specifies the amount of the notional bond in nominal terms which matches one unit nominal value of the deliverable asset. Note that the quality of a conversion factor system now depends on the term structure of interest rates at maturity.

Summarizing, we pose three theoretical requirements for a conversion-factor system. Firstly, the price factors should reduce the differences between the futures-equivalent bond prices. Secondly, they should increase the uncertainty of the terminal ctd-bond because then the manipulator doesn't know which bond he has to accumulate in order to squeeze the futures price. Thirdly, the conversion factors should bring the terminal futures price close to the theoretical price of its synthetic underlying. The difference between these two prices determines the value of the so-called *synthetic delivery option* that has been studied by Berendes and Bühler (1994) for the German Bund Future. Note, that in contrast to the common option definition the value of the synthetic option is not necessarily non-negative. In the following section, based on these two requirements, we develop new conversion-factor systems which account for differences in credit risk of the deliverable assets.

3 Conversion Factors for Multi-Issuer Contracts

The three issuer-dependent conversion-factor systems which are presented in this section are constructed after the design of the Bund Future and differ from this contract in so far as bonds of one default-free and one credit-risky issuer are eligible for delivery. Note, that although a generalization for more than one credit-risky issuer is straightforward we pose this restriction here for expository purposes. The notional bond is defined as a default-risk free bond with a coupon size of 6 %. Eligible for delivery are fixed coupon bonds, without any option features and maturities between 8.5 and 10.5 years at future maturity. Two of the three new issuer-conversion factors are *rating based*. They are described in sections 3.1 and 3.2. The third conversion-factor system, presented in section 3.3, minimizes the potential losses from delivering a default-free instead of a credit-risky bond.

The key idea of all three issuer-dependent conversion factors is to modify the traditional price factor system by adding a credit-risk adjustment π_j to the discount rate c_0 of the deliverable bond. If bond j has m outstanding cash flows $C_{j,k}$ until maturity $M_j = t_m$ with

$$C_{j,k} = \begin{cases} c_j & : \quad 1 \leq k \leq m-1 \\ 100 + c_j & : \quad k = m \end{cases} \tag{3}$$

then its issuer-dependent price factor κ'_j is defined by

$$\kappa'_j(\pi_j) = \frac{\sum_{k=1}^{m}(1 + c_0 + \pi_j)^{-t_k} C_{i,k}}{100}. \tag{4}$$

The issuer-dependent price factors differ with respect to the calculation of the issuer-specific yield adjustment π_j which may or may not depend additionally on the maturity of the bond.

3.1 Intensity-Based Conversion Factors

The first conversion-factor system needs as input only the issuer-dependent term structure of cumulative default probabilities. It is derived from a credit risk model that is static in the sense that the cumulative default probabilities that are observed when the trading in the contract begins are assumed still to hold at maturity T of the future. In this model the default of the bond issuer is treated as a singular jump event. The time to default τ is modeled as a continuous random variable with a distribution Φ which describes the probability $q(t)$ of default in a time interval $(T; T+t]$, namely $q(t) = \Phi(t)$.[6] Assuming that the corresponding probability density function ϕ exists, the conditional probability $p(T+t < \tau \leq T+t+\Delta t | \tau > T+t)$ that a credit-risky asset that has not defaulted up to time $T+t$ enters default in the time span Δt can be determined as follows:

$$p(T+t < \tau \leq T+t+\Delta t | \tau > T+t) = \frac{\phi(T+t)}{1 - \Phi(T+t)} \cdot \Delta t. \tag{5}$$

The first term of the product on the right hand side of (5) is called the *default intensity* $h(t)$. From (5) it follows immediately for the default probability for the time interval of length t:

$$\Phi(t) = 1 - \exp\left(-\int_{T}^{T+t} h(s)\, ds\right). \tag{6}$$

[6]The future maturity T is selected as reference point in time but because of the static nature of the model it can be chosen arbitrarily. Therefore this parameter is dropped for notational convenience whenever possible.

From the issuer-dependent cumulative default probabilities $q(t_m)$ a term structure of hazard rates $h_j(t_j)$ can iteratively be determined by exploiting the following relationship that holds for discrete default intensities:

$$q(t_m) = 1 - \exp\left(-\sum_{j=1}^{m} h_j(t_j)\cdot\Delta t\right). \tag{7}$$

In addition to the cumulative default probabilities a *recovery fraction* φ is needed to quantify the credit risk and this again is assumed to be given exogenuously. Under the recovery-of-market value assumption the bond price $\hat{V}(T, T+t_m, c_j)$ can be determined as follows by discounting the outstanding promised cash flows with a discount rate $r'(t)$. This rate $r'(t)$ consists of two components, namely a risk-free rate, that equals the notional bond coupon c_0 after transformation into a continuous rate, and the credit-risk adjustment.

$$r'(t_k) = \sum_{i=1}^{k} \left(\log(1+c_0) + (1-\varphi)\, h_i(t_i)\right)\Delta t_i \tag{8}$$

$$\hat{V}(T, T+t_m, c_j) = \sum_{k=1}^{m} e^{-r'(t_k)}\, C_{j,k}. \tag{9}$$

Note that from (9) by requiring

$$\hat{V}(T, T+t_m, c_j) = \sum_{i=1}^{m} \frac{C_{j,i}}{(1+c_0+\pi_j)^{t_i}} \tag{10}$$

a credit spread π_j can be calculated in order to achieve consistency with the construction principle laid out in (4). However, this last step is not mandatory because the nominator of the conversion factor κ_j' in (4) can already be determined by $\hat{V}(T, T+t_m, c_j)$.

Arriving at $\hat{V}(T, T+t_m, c_j)$ by discounting at a default-risk adjusted rate in (9) reveals a great similarity with the valuation of credit-risky bonds in the framework of Duffie and Singleton (1999). Whereas the Duffie/Singleton model offers a much richer structure and belongs to the class of continuous time models here the term structure of credit spreads is static and the model structure much coarser. It is theoretically possible but for two reasons it is not practically feasible to apply conversion factors that are fully consistent with a Duffie/Singleton model as it is presented in section 4. Firstly, employing a continuous time model implies constantly changing conversion factors until maturity of the futures contract which severely complicates the arbitrage between the bond and the futures market. Secondly, market participants seem to prefer transparent conversion factors and reject complex price factor systems as being easy to manipulate. The intensity based conversion factors therefore are constructed to take a middle-ground between a complex price factor system based on a continuous time credit risk model and the rather crude traditional price factor system.

The rating-specific cumulative default frequencies differ from the risk-neutral probabilities if investors are risk averse. Employing these probabilities for bond valuation in (9) assumes that investors are risk neutral. This unrealistic assumption leads to credit risk adjustments that are too small and it is dictated purely by the requirement to avoid complexity and by the empirical problems attached to an explicit assessment of this risk premium. The second conversion factor system that is introduced in the following section is subject to the same critique whereas the third circumvents this assumption by relying on term structures of interest rates and credit spreads that already include a premium for risk aversion.

3.2 Certainty-Equivalent Conversion Factors

The second conversion-factor system uses the same rating-specific default probabilities and recovery rates as the intensity-based one. The key idea is to determine a theoretical price of the deliverable bond by replacing its promised cash flows by their certainty-equivalent values and discounting at the risk-free interest rate.

$$\hat{V}(T, T+t_m, c_j) = \sum_{i=1}^{m} \frac{(1-q(t_i))\, C_{j,i} + (100+c_j)\, \varphi_j\, (q(t_i)-q(t_{i-1}))}{(1+c_0)^{t_i}}. \qquad (11)$$

$\hat{V}(T, T+t_m, c_j)$, determined by (11), can either be used directly as nominator of the price factor or a credit-risk premium π_j that is needed in (4) can be determined in the same way as for the intensity based price factors.

Note that the calculation of the conversion factors in (9) and (11) is rather similar but differs in two ways. Firstly, the certainty-equivalent price factor system possesses less model structure and poses no assumptions about the default event. It is posed in discrete time whereas the intensity based price factors, although static in nature, too, are posed in continuous time. Secondly, both conversion factor systems differ because the recovery-of-market value assumption in (9) is replaced by a recovery-of-nominal value assumption in (11).

To ensure transparency of the conversion factor system we rely for the issuer-dependent default probabilities and the recovery fraction on publicly available data from international rating agencies. The historical default probabilities are obtained from the rating agency Moody's and depend on the rating category of the issuer.[7] We assume that the credit-risky issuer is rated Aa. This is in line with the actual rating of Italian sovereign bonds which are used for estimating the model parameters for the comparative-static analysis. It is more difficult to justify a recovery value for sovereign debt because if a government bond fails it is not clear how the lender can enforce his claim by legal means. Therefore we assume that in case of a sovereign default the claimant receives no compensation at all, i. e. the recovery value is zero.[8]

[7] See Moody's (1999).

[8] In a recent paper Gibson and Sundaresan show how a sovereign default can be modeled as a strategic decision depending on the reputation costs, which concern the future access to credit markets, and the

3.3 Delivery Loss minimizing Conversion Factors

The first two rating-based conversion-factor systems rest on the assumption of risk-neutrality which is not expected to hold in a market with supposedly risk averse investors. Therefore, the credit risk premium π_j will typically be too small. The third conversion-factor system solves this problem by treating the default-free term structure and the credit spread term structure as exogenuous and determining π_j directly from term structures of credit spreads.

The key idea of this price-factor system is to determine the default spread in (4) by minimizing the delivery loss that is incurred if a default-free instead of a credit-risky bond with the same coupon size and maturity is delivered. Note that the default-free bond with identical coupon and maturity characteristics serves only as a benchmark to determine the delivery loss and does not necessarily belong to the delivery basket. To account for different cash flows of the deliverable bonds we minimize the *average delivery loss* of n deliverable credit-risky bonds with prices $V(T, M_j, c_j)$ at future maturity T. $P(T, M_j, c_j)$ denotes the price of a corresponding default-free bond and $ADL(\pi)$ the average delivery loss, depending on the default-yield adjustment π. Note that π depends only on the issuer but not on other bond characteristics. $ADL(\pi)$ is determined as follows:[9]

$$ADL(\pi) = \frac{1}{n} \sum_{j=1}^{n} |\kappa_j \cdot \frac{V(T, M_j, c_j)}{\kappa_j'(\pi)} - P(T, M_j, c_j)|. \tag{12}$$

Clearly, the bond prices $V(T, M_j, c_j)$ and $P(T, M_j, c_j)$ are unknown before future maturity. Therefore we consider a broad range of realistic interest rate and credit spread scenarios and determine the credit-risk premium π_j by minimizing the average delivery loss over all scenarios.

To specify the term structures of interest rates and credit spreads, we need a (static) term-structure model. As our focus is on robustness and not on the best fit of a specific term structure we use the parsimonious term-structure model of Nelson and Siegel (1987). Discount factors $\delta(M; \beta_0, \ldots, \beta_3)$ in this model depend only on their maturities M and on four parameters $\beta_0, \ldots, \beta_3$:

$$\delta(M; \beta_0, ..., \beta_3) = \exp\left[-M\left(\beta_0 + (\beta_1 + \beta_2)\left(1 - \exp\left(-\tfrac{M}{\beta_3}\right)\right) \times \tfrac{\beta_3}{M} - \beta_2 \exp\left(-\tfrac{M}{\beta_3}\right)\right)\right]. \tag{13}$$

The parameter β_0 of this model is the limit of the interest rate or credit spread for $M \to \infty$. The sum $\beta_0 + \beta_1$ can be interpreted as the instantaneous short rate or short spread. The

potential for retaliatory actions, e. g. trade sanctions. See Gibson and Sundaresan (2000) with references to earlier studies.

[9] An alternative to minimizing ADL is to minimize the difference between the futures-equivalent price of the default-free and the corresponding credit-risky bond. This results in almost identical conversion factors.

Table 1:

Parameters of the Nelson-Siegel Term Structure Model in Percent

	interest rate term structure		credit spread term structure	
	β_0	β_1	β_0'	β_1'
interval	J_1	J_2	J_3	J_4
Minimum	4.00	-2.00	0.06	-0.06
Maximum	10.40	2.00	0.30	0.12
reference points	9	5	9	5

other parameters β_2 and β_3 determine the position and the magnitude of a hump in the term structure. These two parameters are treated as fixed and their values are taken from a former empirical study where the credit spreads between two sovereign borrowers are fitted applying as well the Nelson-Siegel model.[10]

The range of the parameters β_0 and β_1 which determine the level and the slope of both term structures are given in table 1. The range of the interest rate parameters is chosen to reflect weekly observed term structures of the German government bond market from 1970 to 1997. The minimum and maximum of the credit spread parameters are derived from the observed credit spreads between German and Italian government bonds as reported in Bühler et al. (2001). The extreme values of β_0' and β_1' are in line with a range between zero and 42 bp for the long run limit of the credit spreads. We refer to this study as the parameter estimates used to value the multi-issuer future in section 5 are also taken from this study.

Referring to the delivery basket we consider four credit-risky bonds with coupons of 4 % and 8 % and maturities of 8.5 years and 10.5 years, the minimum and maximum value permitted for delivery.

The default-yield adjustment π is determined by minimizing the following objective function $G(\pi)$ where the average delivery loss $ADL(.)$ is given by (12):

$$G(\pi) = \frac{1}{|J_1| \cdots |J_4|} \sum_{\beta_0 \in J_1} \sum_{\beta_1 \in J_2} \sum_{\beta_0' \in J_3} \sum_{\beta_1' \in J_4} ADL(\pi; \beta_0, \beta_1, \beta_0', \beta_1') \qquad (14)$$

Here $|J_i|, i \in \{1, \ldots, 4\}$ denotes the number of parameter values as described in table 1. The default-yield adjustment which minimizes $G(\pi)$ is 20 bp. This value is assigned to π_j in (4) in order to determine a loss minimizing conversion factor.

The fact that for the delivery-loss minimizing conversion factors the credit risk premium π does not depend explicitly on the maturity of the bonds contrasts with the first two

[10] See Bühler et al. (2001).

conversion factor systems and gives rise to the following concern. If the future-equivalent prices are raised with the same premium π and, therefore, for a similar amount, then the probability to become the terminal ctd-bond may concentrate on a single bond. However, this problem is only of secondary nature for the following reasons. Firstly, the conversion factor still depends on the maturity of the bond and, therefore, the increase of the future-equivalent prices differ considerably in absolute terms. Secondly, re-running the optimization with a restricted basket that includes only the two credit-risky bonds with a maturity of 8.5 years yields the same optimal value of π. This result underlines that the difference between 10.5 and 8.5 years is too small to observe a perceptible change in π. Thirdly, the credit risk premium π primarily aims at accounting for the difference in credit risk between the obligors and not between the bonds of one single issuer. In the comparative static analysis in section 5 it is exactly the possibility of a ctd-change between bonds of different obligors that has the most important impact on the value of the delivery option.

For notational convenience the following abbreviations are used for the three conversion-factor systems, IS for the *intensity*-based, CE for the *certainty equivalent* and LM for the delivery *loss minimizing* price-factor systems.

4 Valuation of the Delivery Option in Multi-Issuer Futures

The value of the delivery option depends on the interest-rate and credit risk. From the many possibilities to model these two risk components we rely on an intensity-based affine two-factor model for the following reasons:

- It is difficult to model sovereign risk by *structural models* as the notion of a firm value is not easily transferable to the government bond sector. In addition, the classical model of Merton (1974) and his immediate successors cannot explain empirically observed credit spreads.[11] Contrary to the observation of considerable credit spreads for short-term government bonds, in structural models this spread converges to zero as the time-to-maturity approaches zero.[12]
- A positive credit spread of short-term debt is explained by models that treat the default event as a consequence of a strategic decision of stock holders.[13] In principle this type of models could be applied to value credit-risky government bonds even if the strategic players in this game are not easily identified. However, this type of models is hard to implement empirically.

[11] See Jones et al. (1984) or Kim et al. (1993).

[12] Duffie and Lando (2001) explain this observation as well in a firm value model by asymmetric information. However this solution does not transfer easily to the case of sovereign debt.

[13] See e. g. Leland (1998), Mella-Barral and Perraudin (1997) or Anderson and Sundaresan (1996).

- The most promising group of credit-risk models for government bonds are the *reduced-form* or *intensity*-based models.[14] This model class offers flexible and tractable models which can explain positive credit spreads even immediately before the maturity of a defaultable claim. Their key property, that is to treat the bankruptcy process as exogenuous, makes them well suited for the valuation of public bonds.

From a variety of intensity-based models we have selected the approach developed in Duffie et al. (1996) and Duffie and Singleton (1999) who show that under fairly general assumptions a credit-risky claim can be valued by adding to the instantaneous interest rate a factor that reflects the credit risk of the issuer.[15]

Additionally this model framework becomes very tractable by offering a straightforward application of affine models which possess analytic solutions for many standard derivatives. Furthermore, this affine framework has already been applied successfully in various empirical articles, e. g. by Duffie and Singleton (1997) in the US swap market, Duffee (1999) in the US corporate bond market and by Collin-Dufresne and Solnik (1999) in the European swap market. Most notably an affine intensity based model has been used successfully in a recent study by Duffie et al. (2000) for the valuation of Russian public debt.

The stochastic model of Duffie and Singleton (1999) poses the usual assumptions of arbitrage-free valuation theory. Therefore we assume a frictionless market with continuous trading. The filtration $\mathcal{F} = \{F_t : t \geq 0\}$ describes the arrival of information in the market over time. The process of the short rate r is adapted to the filtration. Furthermore, we assume the existence of an equivalent martingale measure Q.

Then, for a claim with a promised cash payment of C at time T the value of this claim is given under the risk neutral measure by its expected value and discounting under a default-risk adjusted rate that depends on the instantaneous short-term rate r, the default intensity h and the recovery fraction φ.[16]

$$V(t,T) = E_t^Q \left[\exp\left(- \int_t^T \left(r_s + h_s(1-\varphi_s) \right) \, ds \right) \, C \right] \tag{15}$$

The default risk adjustment $h_t(1-\varphi_t)$ is modeled by a single state variable s_t which has a natural interpretation as the instantaneous credit spread or simply the *short spread.* We further assume that the two state variables $x_1 = r$ and $x_2 = s$ follow a mean reverting

[14] See e. g. Jarrow and Turnbull (1995), Das and Tufano (1996) and Duffie and Singleton (1999).

[15] In fact these models make one assumption which contrasts with conventional legal norm. They define the recovery value as a fraction of the market value of the claim immediately before default instead of the nominal value. However, according to their own calculations the difference is negligible (see Duffie and Singleton (1999), pp. 3–4).

[16] See Duffie and Singleton (1999).

square root process. This process is defined under the physical measure as follows:

$$dx_i(t) = \alpha_i(\theta_i - x_i(t))\,dt + \sigma_i\sqrt{x_i(t)}dW_i(t) \ \ , i \in \{1,2\} \tag{16}$$

We assume that the Wiener processes are uncorrelated. The market price of risk is defined as $\lambda_i \frac{\sqrt{x_i(t)}}{\sigma_i}$.

The prices of pure discount bonds are given in closed form.[17] The prices of coupon bonds can be determined as a portfolio of pure discount bonds.

Having specified the default risk model we turn to the valuation of the futures contract. The key feature that distinguishes future and forward contracts is the marking-to-market which ensures that the buyer of the future can be unconcerned about a default of the contracting party. In the general framework studied in Duffie and Stanton (1992), the futures price can be determined as the expectation value under the risk-neutral measure of the value of the claim at maturity. In the case of a basket of n credit-risky bonds with associated conversion factors $\kappa_1, \ldots, \kappa_n$ the futures price can be determined as follows:

$$F(t,T,\{1,\ldots,n\}) = E_t^Q\left[\min_{j=1,\ldots,n} \frac{V(T,M_j,c_j)}{\kappa_j}\right]. \tag{17}$$

In special cases (17) has an analytic solution. In the model framework of Heath et al. (1988) a closed form solution is derived by Ritchken and Sankarasubramanian (1992) in the case of a one-factor Gaussian interest rate model and a delivery basket of pure discount bonds. They remark that an extension of their closed form solution for coupon bonds is feasible, however, much more complex.[18] In a more general two-factor model they employ numerical methods for the determination of futures prices.[19] Cherubini and Esposito (1995) derive closed form solutions for a bond future with deliverable coupon bonds in a one-factor interest rate model of Cox et al. (1985). In a more recent paper Bick (1997) derives a closed form solution for a future on multiple coupon bonds in the context of a one-factor Vasicek model[20].

Our case differs from the ones discussed in the literature because we have to consider multiple deliverable coupon bonds in a *two-factor* affine model. Therefore, (17) has to be solved numerically. This is achieved by solving the following integral numerically:[21]

$$F(t,T,\{1,\ldots,n\}) = \int_0^\infty \int_0^\infty \min_{j=1,\ldots,n} \frac{V(T,M_j,c_j;\ r,s)\ \phi_1(t,T,r)\ \phi_2(t,T,s)}{\kappa_j}\,dr\,ds. \tag{18}$$

The value of a futures contract whose delivery basket is reduced to a single bond, namely the ctd-asset j^* is given by

$$F(t,T,\{j^*\}) = \min_{j=1,\ldots,n} E_t^Q\left[\frac{V(T,M_j,c_j)}{\kappa_j}\right]. \tag{19}$$

[17] See Cox et al. (1985), p. 393.

[18] See Ritchken and Sankarasubramanian (1992), p. 208.

[19] See Ritchken and Sankarasubramanian (1995).

[20] See Vasicek (1977).

[21] The numerical integration is conducted with a modified adaptive algorithm developed in Genz and Malik (1980), that is implemented in Mathematica.

In this special case, when the delivery option disappears,there exists an analytic solution, given in Cox et al. (1981).[22] The value of the delivery option $DO(t, T, \{1, \ldots, n\})$ is the difference between the value of a future written on the ctd-bond and the value of the multi-issuer future:

$$DO(t, T, \{1, \ldots, n\}) = F(t, T, \{j^*\}) - F(t, T, \{1, \ldots, n\}). \tag{20}$$

The transformation of (20) into (21) reveals the two drivers of the value of the delivery option $DO(t, T)$. The lower integration bound is (21) is determined by the fact that zero forms a reflecting barrier in the CIR model.

$$\begin{aligned} DO(t,T) = \int_0^\infty \int_0^\infty \frac{V(T, M_{j^*}, c_{j^*})}{\kappa_{j^*}} \, \phi(t,T,r) \, \phi(t,T,s) \, dr \, ds - \\ \int_0^\infty \int_0^\infty \min_{j=1,\ldots,n} \left\{ \frac{V(T, M_j, c_j)}{\kappa_j} \right\} \phi(t,T,r) \, \phi(t,T,s) \, dr \, ds \\ = \sum_{j=1}^n \int_0^\infty \int_0^\infty \left(\frac{V(T, M_{j^*}, c_{j^*})}{\kappa_{j^*}} - \frac{V(T, M_j, c_j)}{\kappa_j} \right) 1_{\{(r,s) \in \mathcal{G}_j\}} \, \phi(t,T,r) \, \phi(t,T,s) \, dr \, ds \end{aligned} \tag{21}$$

$$\mathcal{G}_j = \left\{ (r,s) | \frac{V(T, M_j, c_j)}{\kappa_j} \leq \frac{V(T, M_l, c_l)}{\kappa_l} \; \forall l \in \{1, \ldots, n\} \backslash \{j\} \; \wedge (r,s) \notin \bigcup_{l<j} \mathcal{G}_l \right\}^{23}.$$

The last transformation in (21) shows how the option value depends on two factors,

- the difference of the future-equivalent prices between the current ctd-bond j^* and the ctd-bond at future maturity and, because these price differences only contribute to the option value in the region $\mathcal{G}_j$ where j is the terminal ctd-bond,
- the probability of a change in the terminal ctd-bond that depends as well on the conditional density functions $\phi(t, T, r)$ and $\phi(t, T, s)$ of the state variables r and s.

If the credit-risky bonds have coupons and maturities different from those of the default-free issues it is unclear to what extent a higher value of the delivery option must be attributed to different default-risk premiums or to additional coupon size and maturity characteristics. However, in the following comparative-static analysis this fundamental problem can be solved by a careful construction of the delivery basket. For every credit-risky bond we add a corresponding default-free bond with the same coupon size and maturity and vice versa. This allows to determine the value of the issuer option as a residual, that is the difference between the values of the delivery option for the whole basket and the delivery option for a subset of the basket that consists only of the default-free assets.

[22] For completeness this formula is given as well in appendix A.

[23] The second condition for (r, s) ensures that the subsets $\mathcal{G}_j$ are pairwise disjoint and build a partition of the integration domain.

The parameter values of the two-factor model, given by (16), are estimated in Bühler et al. (2001) from the term structures of interest rates of German and Italian government bonds under the assumption that the German government bonds are default-free.[24]. The estimation period extends from May 1998 when the member states of the EMU have been determined until February 2000. Therefore, we observe the rare case of two liquid government bond markets trading without currency risk among them. The use of empirically estimated parameter values improves on the validity of the comparative-static analysis presented in the following section.

5 Comparative-Static Analysis of the Issuer Option

In the following comparative-static analysis we focus on the sensitivity of the value of the delivery option with respect to three groups of determinants. The first group includes the maturity of the futures contract and the composition of the *delivery basket*. The relevant basket parameters are the number of bonds, their coupon size and time to maturity. These determinants are analysed first. The second group refers to the level of the term structures of interest rates and credit spreads. These levels depend on the instantaneous short rate, the instantaneous short spread and the long term mean θ_i of the two stochastic processes. The third group of parameters α_i, σ_i and λ_i determine the volatility of interest rates and credit spreads. Note that the mean reversion parameter α_i influences also the shape of the term structures. The parameter estimates from Bühler et al. (2001) which are given in table 13 are selected as *reference values* in the following analysis. The reference values of the instantaneous short rate $r = 3.5\%$ and of the instantaneous short spread $s = 5.4$ bp are determined as averages of the time series of state variables implied by the model in Bühler et al. (2001). The reference value for the maturity of the futures contract is three months. This length is selected because it covers the period when the future is the most liquid contract.

5.1 Dependency on the Delivery Basket and Time to Maturity

In the first step we examine the delivery option for two different delivery baskets. The first is given in table 2 and consists of four default-free and four credit-risky bonds with coupons of 4 % and 8 % and time to maturities of 8.5 and 10.5 years respectively. The basket is *symmetric* in the sense that for every credit-risky bond it contains a default-free bond with the same maturity and coupon size. Numerical results from Cherubini and Esposito and own calculations suggest, that the value of the delivery option increases strongly with the difference in maturities so that this is set as wide apart as is allowed by

[24]The parameter estimates are given in appendix B.

the contract design.[25] For consistency the same basket is used for the determination of the LM-conversion factors in section 3.3.

Table 2:
Composition of the Delivery Basket of Eight Bonds

bond characteristics	1	2	3	4	5	6	7	8
credit-risky	no	no	no	no	yes	yes	yes	yes
Coupon	8 %	4 %	8 %	4 %	8 %	4 %	8 %	4 %
time to maturity	8.5	8.5	8.5	8.5	10.5	10.5	10.5	10.5

Table 3 shows the value of the delivery option for the delivery basket from table 2 and a subgroup of those four bonds that are default-free. For the basket of eight bonds traditional, CE-, IS- and LM-conversion factors are employed. To put the values of

Table 3:
Value of the Delivery Option for Basket of Eight Bonds in Percent

bond characterictics		conversion-factor	value of the
default-free	credit-risky	system	delivery option
4	0	traditional	0.03
4	4	traditional	0.22
4	4	CE	0,24
4	4	IS	0.23
4	4	LM	0.44

the delivery option under perspective it should be noted that under the traditional conversion-factor system the value of the quality option, calculated for the default-free subsample of the delivery basket, has a value of 3 bp. Under the same conversion-factor system the delivery option of the full basket leads to a value of 22 bp so that the value of the issuer option, calculated as the difference between both option values, is 18 bp. This value is six times higher than the value of the quality option.

It is striking that the differences in the value of the delivery option between the traditional, the IS- and the CE-conversion-factor systems are less than 2 bp and, therefore, almost negligible. However, under the LM-price factors the option value doubles to a value of 44 bp. In the case of a nominal contract value of 100.000 Euro this transfers into 440 Euro which is an economically significant amount. In order to explain these differences we focus on two key factors which determine the value of the delivery option according to (21): the probability that another bond becomes cheapest to deliver at future maturity and the delivery loss that is incurred by delivering the current ctd-bond instead.

[25]See Cherubini and Esposito (1995), pp. 8–9.

Panels 1 to 3 in figure 1 demonstrate the contribution of the two value drivers of the delivery option for the traditional und the issuer-dependent conversion factor systems. The case of CE-conversion factors is omitted because it is not distinguishable from the case of applying IS-conversion factors. Each panel shows the future-equivalent price differences between the current ctd-bond 7 and the other deliverable bonds conditional on a reference value of 3.55 % of the instantaneous short rate at maturity. This reference value is the expected value of the short rate under the risk-neutral measure conditional on its current value of 3.5 %. For this interest rate scenario the 5 %-quantiles of the instantaneous short spread s are given on the x-axis.

For the traditional conversion factor system in panel 1 of figure 1 we observe no changes in the future-equivalent price differences if the short rate is below 1 bp which occurs with a probability of 50 %. The default-free bonds 1 to 4 are never cheapest to deliver because this conversion factor system does not account for default risk. Note that this result depends on the symmetric composition of the delivery basket, in which the default-free and credit-risky bonds have the same coupon size and maturity. For credit spreads s beyond 20 bp which belong to the 90 %-quantile of s the credit-risky bond 8 replaces bond 7 as the terminal ctd-bond.

If the traditional price factor system is replaced by IS-conversion factors the situation presented in panel 2 arises. The IS-price factors reduce the future-equivalent price difference between the default-free and the credit-risky bonds at maturity. For the default-free bond 4 under a traditional conversion factor system the reduction is 40 bp for $s \leq 0.1$ bp. However, this reduction of heterogeneity has no impact on the delivery option, at least in this interest rate scenario, because a default-free bond never becomes cheapest to deliver. For low short spreads the current ctd-bond 7 is replaced by bond 5 but the future-equivalent price difference is only 6 bp and has only a negligible impact on the value of the delivery option.

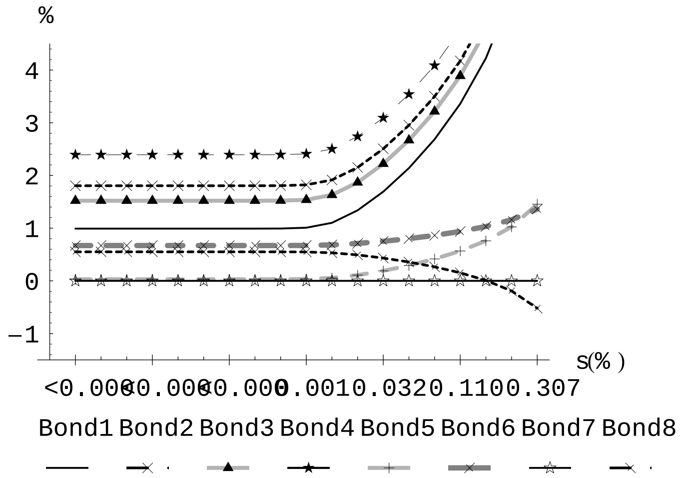
Whereas the IS-conversion factor system does not change the value of the delivery option in an economically significant way the application of LM- instead of traditional price factors leads to an economically relevant increase of 22 bp according to table 3. The reason for this is revealed by panel 3 of figure 1. For a credit spread s of 3 bp and below the default-free bond 1 becomes the terminal ctd-bond. In these cases the future-equivalent price difference between bond 1 and the ctd-bond is 54 bp and economically very relevant. Combined with a high probability of occurance of 65 % this contributes to the strong increase of the delivery option value relative to the traditional, the IS- and the CE-conversion factors.

In order to understand why a default-free bond can become cheapest to deliver we focus on the future-equivalent prices $V(t,T,c)/\kappa'$ and $P(t,T,c)/\kappa$ of a credit-risky and a default-free bond with same coupons c and maturity T. The adjustment of 20 bp in the LM-conversion

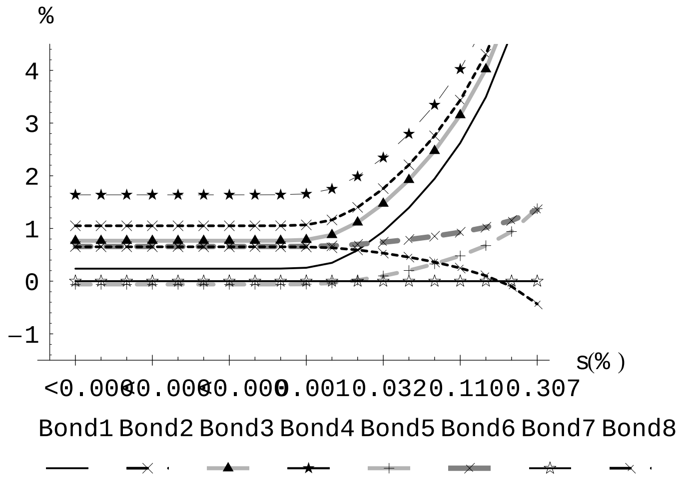
Figure 1:

Future-Equivalent Price Differences from the Ctd-Bond at Maturity

Panel 1: Traditional Conversion Factor System



Panel 2: IS-Conversion Factor System



Panel 3: LM-Conversion Factor System

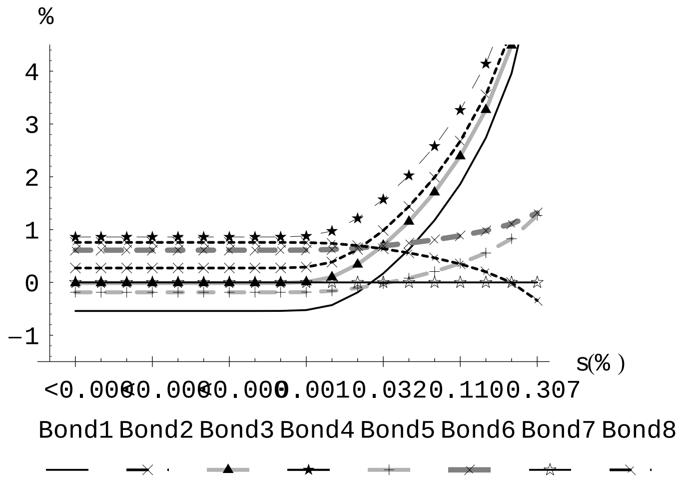


Table 4:

Probabilities of Becoming the Cheapest-to-Deliver at Maturity for Eight Deliverable Bonds

conversion factor system	default-free bonds coupon/ time to maturity in years (y)			
	8 %, 8.5 y	4 %, 8.5 y	8 %, 10.5 y	4 %, 10.5 y
Traditional	0.0	0.0	0.0	0.0
CE	0.0	0.0	0.0	0.0
IS	0.0	0.0	0.0	0.0
LM	55.4	0.0	5.5	0.0
conversion factor system	**credit-risky bonds** coupon/ time to maturity in years (y)			
	8 %, 8.5 y	4 %, 8.5 y	8 %, 10.5 y	4 %, 10.5 y
Traditional	32.7	0.0	47.6	19.7
CE	41.3	0.0	42.1	16.6
IS	41.0	0.0	42.4	16.6
LM	7.5	0.0	20.5	11.1

factor κ' decreases the denominator of this ratio for the credit-risky bond more than the credit spreads for an instantaneous short spread of 3 bp decrease the bond price $V(t,T,c)$ in the nominator. Therefore the future-equivalent price of the credit-risky bond is higher than that of the corresponding default-free bond and the latter is cheaper to deliver.

In order to verify if this change of the ctd-bond causes the strong increase in the value of the delivery option, we determine the probabilities of all deliverable bonds to become the terminal ctd-bond over all interest rate scenarios. The results are given in table 4. For the traditional, the IS- and the CE-conversion-factor system the probability of a default-free bond becoming cheapest to deliver is zero. When applying LM-factors instead, the two default-free 8 %-bonds will become the terminal ctd-bonds with probabilities of 55.4 % and 5.5 % which confirms the results from figure 1.

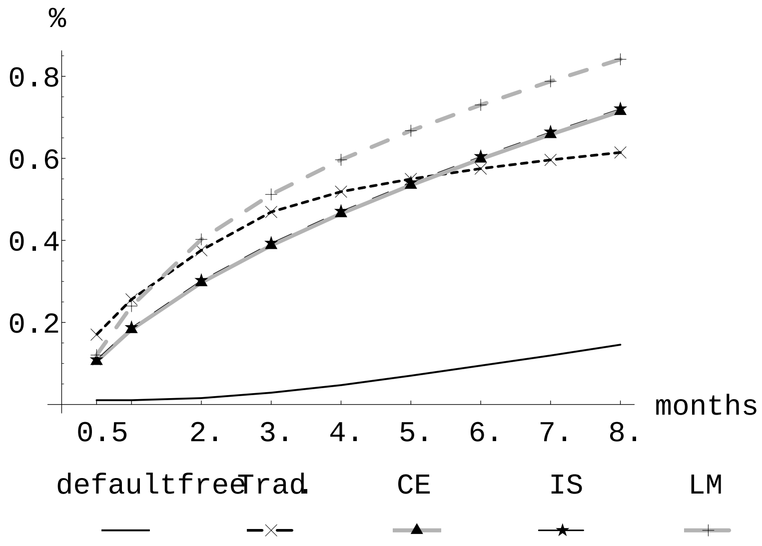
Summarizing the preliminary results from the analysis of futures contracts we conclude that

- although the application of issuer-dependent conversion factor systems narrows the future-equivalent price differences between the default-free and the credit-risky bonds or rather the ctd-bond this increases the value of the delivery option.
- A high probability that default-free bonds become cheapest to deliver at maturity is the key factor increasing the value of the delivery option relative to a future with a traditional price factor system and this happens only for the LM-price factor system.

A smaller delivery basket of a subset of four bonds retains qualitatively the results that are obtained for the basket of 8 bonds. The smaller basket contains only the two bonds of both issuers which have either the longest maturity/lowest coupon or the shortest maturity/highest coupon. These are the bonds 1, 4, 5, and 8 in table 2. Because of its parsimonious structure the reduced basket is used as reference in the following analysis of the dependency of the option values on time to maturity and again in the comparative-static analysis in the following section.

Figure 2:

Dependency of the Value of the Delivery Option on Time to Maturity



In order to study the delivery option value dependent on the maturity of the future figure 2 shows the value of this option for a time to maturity that lies between 0.5 and 8 months. For the traditional ("Trad.") and the three issuer-specific conversion-factor systems the option values decrease when the expiration date draws nearer. This is to be expected because the continuously arriving information reduces the uncertainty which bond will become the terminal ctd-bond. However, with only two weeks until delivery the option is between 10 and 18 bp and therefore still very valuable. The order of the conversions factor systems in terms of the option value changes with decreasing time to maturity. Whereas at eight months to maturity the delivery option has the highest value for LM-conversion factors and the lowest for traditional price factors, this order is reversed at two weeks to maturity.

5.2 Dependency on the Level of the Term Structures of Interest Rates and Credit Spreads

In the following section the sensitivity of the delivery option value against changes in the level of the term structures of interest rates and credit spreads are analysed. For there are no noticeable differences between the results of the CE- and the IS-conversion factor system only the latter are reported in this and the following subsection. The change in level is achieved by simultaneously varying the two state variables, the instantaneous short rate and the instantaneous short spread, and the long term means of the stochastic processes (θ_1 or θ_2) in a range given in table 5. The delivery option values in the six

Table 5:
Parameter Values for the Analysis of Level-Dependency of the Delivery Option

short interest rate	2.5	3.5	4.5	5.5	6.5	7.5	
long term mean θ_1	5.0	6.0	7.0	8.0	9.0	10.0	
short credit spread	0.03	0.05	0.07	0.10	0.15	0.25	0.55
long term mean θ_2	0.00	0.20	0.40	0.70	1.20	2.20	5.20

interest rate scenarios and seven credit spread scenarios given by table 5 are presented in figure 3. The option values of the credit spread scenarios are collected as a group of bars for every interest rate scenario. The three panels in this figure show the option values for traditional, IS- and LM-conversion factors.

The first panel in figure 3 presents the result for a traditional conversion-factor system conditional on each of the interest rate scenarios. The option values depend strongly on the interest rate scenario. For a low level of the term structure (r=2.5 %, θ_1=5 % and r=3.5 %, $\theta_1 = 6$ %) the option is relatively valuable with a maximum value of 85 bp. In these cases the option value reacts sensitively to changes in the level of the credit spreads. Especially for the lowest short rate (r=2.5 %, $\theta_1 = 5$ %) the option value varies over a wide range between 4 and 85 bp. Contrary, for a relatively high level of the term-structure of interest rates ($r \geq 4.5$ %, $\theta_1 \geq 7$ %) the delivery option is essentially worthless and does not depend on the level of the credit spreads.

Comparing the three panels in figure 3, which belong to different price factor systems, we observe that surprisingly the values of the delivery option for the traditional and the IS-price factors are almost identical. However, applying the LM-conversion method leads by far to higher option values in those interest rate scenarios when the option is already valuable under a traditional price factor system. In the three scenarios with relatively high interest rates ($r \geq 4.5$ %, $\theta_1 \geq 7$ %) and apart from the lowest scenario for the credit spreads ($s = 3$ bp, $\theta_2 = 0$ bp) the delivery option is essentially worthless as it is for the

Figure 3:
Dependency of the Delivery Option on Simultaneous Changes in the Long-Term Mean θ_i and the State Variable x_i

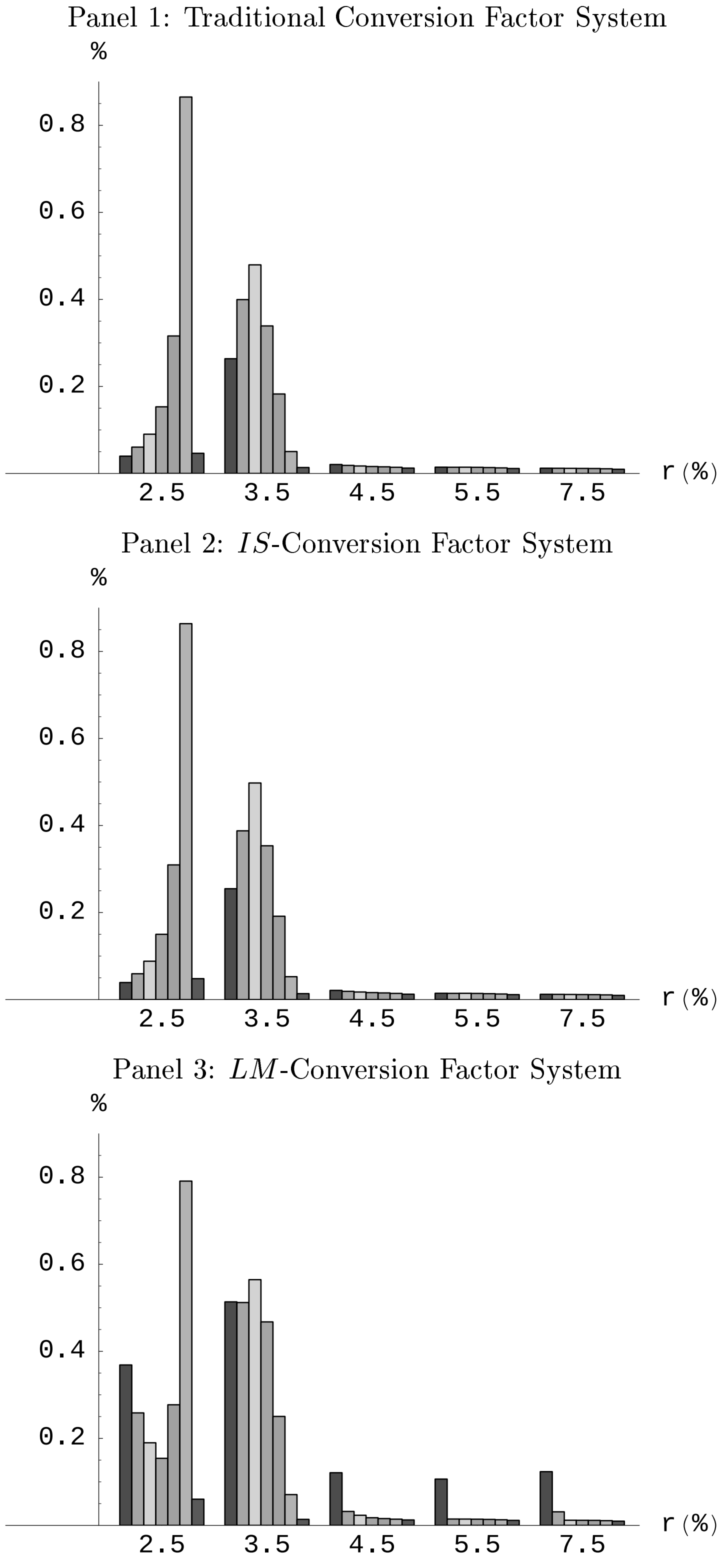
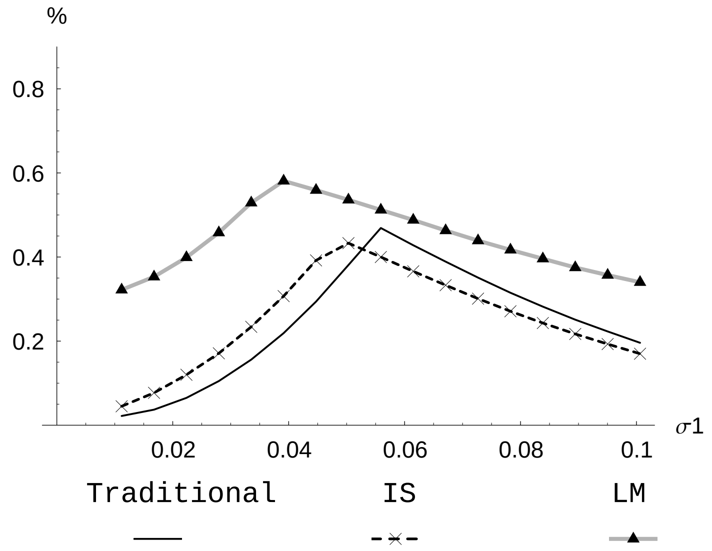


Figure 4:
Dependency of the Value of the Delivery Option on σ_1



other price factor systems.

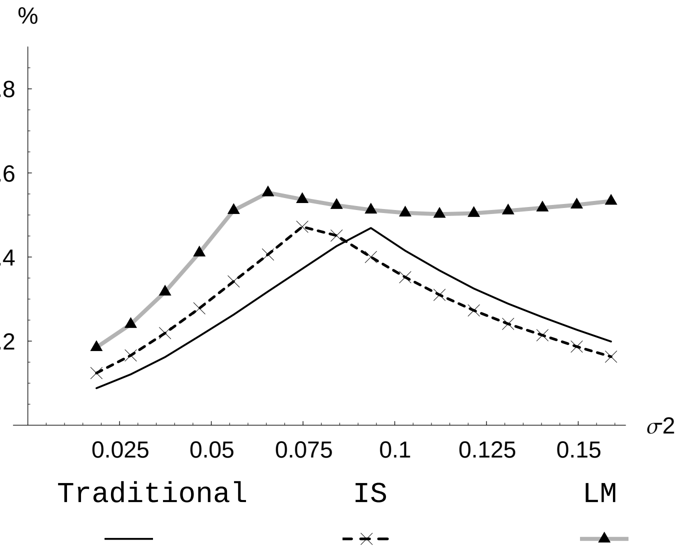
The traditional price-factor system provides no credit-risk adjustment, in contrast to the IS- and the LM-factors which provide the highest yield adjustment. This ascending order but not the relative size of the yield adjustment for credit risk is reflected by the values of the delivery option. Figure 3 shows that LM-conversions factors lead by far to the highest option values because of a high probability that a German bond becomes cheapest to deliver at maturity. This observation confirms the results from section 5.1.

5.3 Dependency on the Volatiliy of Interest Rates and Credit Spreads

In the following section the sensitivity of the issuer option against changes in the volatility parameters is analysed. Apart from σ_1 and σ_2 the volatility of the interest rates and credit spreads is affected by the mean reversion parameters α_1 and α_2.

Figure 4 presents the value of the delivery option depending on the volatility parameter σ_1 of the short rate process for selected conversion-factor systems. For all price-factor systems except of the LM-conversion factors the delivery option reaches its maximum at or close to $\sigma_1 = 0.06$. For the LM-price factors the maximum is at $\sigma_1 = 0.04$ but a more prominent difference are the relatively high option values for low parameter values (between $\sigma_1 = 0.01$ and $\sigma_1 = 0.04$). If σ_1 decreases relative to the location of the maximum of the option value, the ex ante ctd-asset switches for all but the LM-price factors from the credit-risky 8 %-bond to the credit-risky 4 %-bond. A switch back becomes less likely

Figure 5:
Dependency of the Value of the Delivery Option on σ_2



given the lower volatility parameter and therefore, the option values diminishes for the lowest value of $\sigma_1 = 0.01$. For the LM-conversion factors, however, the option value stays above 30 bp even for low values of σ_1. For this price-factor system the option value derives from a positive probability that the default-free 8 %-bond becomes cheapest to deliver which is already 60 % for the reference value of σ_1 and stays close to this level when σ_1 decreases.

Figure 5 shows the values of the delivery option dependent on σ_2. For values of σ_2 higher than 0.09 the option value stays on a high level around 55 bp if the LM-conversion method is applied. In this range of σ_2 the option values for the other conversion-factor systems decrease monotonicly. The reason is again that only in the case of the LM-factors there exists a high probability of a default-free asset to become cheapest to deliver at maturity and this probability increases to 80 % for $\sigma_2 = 0.16$.

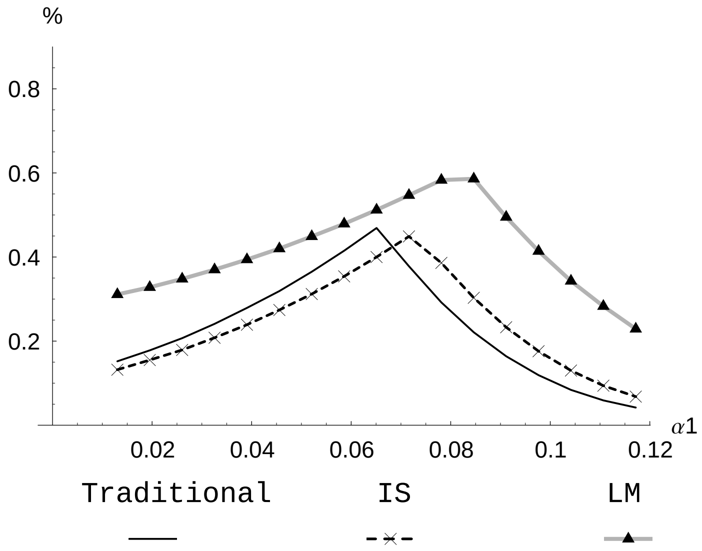
The mean reversion parameters α_1 and α_2 influence the futures price and, therefore, the delivery option in two ways: firstly, via their effect on the level and slope of the term structure and, secondly, via the volatility $v_i(T-t)$ of the interest rates and credit spreads for a time to maturity of $T-t$. $v_i(T-t)$ depends on α_i in the following way when x_i denotes the state variable:[26]

$$v_i(T-t) = \frac{\sigma_i \sqrt{x_i}}{(T-t)} B_i(T-t;\, \alpha_i, \sigma_i, \lambda_i). \tag{22}$$

In this analysis we assign α_1 and α_2 to the volatility parameters because the absolute

[26] $B_i(T-t;\, \alpha_i, \sigma_i, \lambda_i)$ is abbreviated by $B_i(T-t)$ in (31) in appendix A.1.

Figure 6:
Dependency of the Value of the Delivery Option on α_1



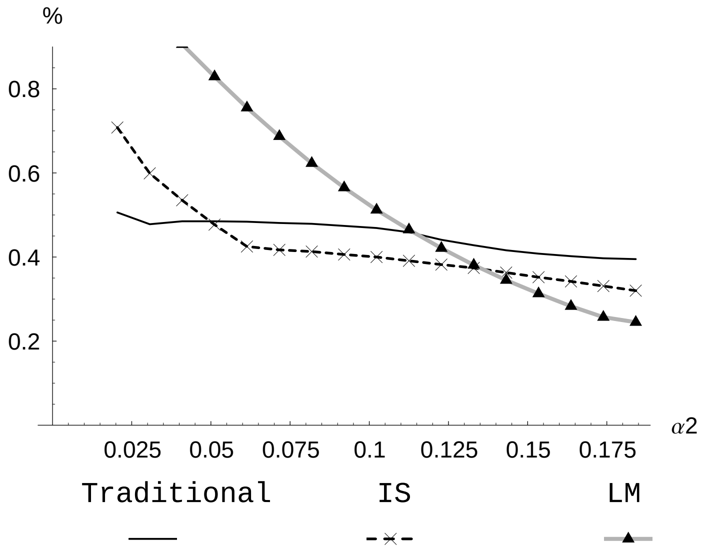
values of their estimates are rather small and we expect that the impact on the volatility of the term structures dominates the one on the level.

The dependency of the value of the delivery option on α_1 is shown in figure 6. The option values peak where the uncertainty about which of the two credit-risky bonds will become the ctd-bond at maturity is highest. For the LM-conversion factor system this is the case when $\alpha_1 = 0.08$.

Figure 7 shows that if α_2 is smaller than its reference value the delivery option increases for all issuer-dependent conversion-factor systems. This result differs from the situation in figure 6. The key difference is that whereas a change of α_1 affects all bonds in the delivery basket in the same way, a change of α_2 affects only the credit-risky assets. With decreasing α_2 the volatility of the spread $v_2(T-t)$ increases and the probability of a default-free bond to become cheapest to deliver accordingly increases for all issuer-dependent conversion factor systems together with the value of the delivery option. For the lowest value of $\alpha_2 = 0.02$ the probability that the default-free 8 %-bond becomes the terminal ctd-asset are 65 % for the LM- and 60 % for the IS-price factors. Only the traditional conversion-factor system ensures that the ctd-asset will always be credit-risky and hence the value of the delivery option stays overall unaffected by changes in α_2.

Comparing figures 5 and 7 we observe for the LM-price factors, that the value of the delivery option peaks for high values of σ_2 but low values of α_2. The reason for this observation is, that the volatility of the credit spreads, given by (22) increases with σ_2

Figure 7:
Dependency of the Value of the Delivery Option on α_2



but decreases with α_2. Therefore, for relatively high values of σ_2 and low values of α_2 extremely high and low values of the short rate at expiration are more likely. Whereas the former have a negligible impact on the value of the delivery option, the latter imply a strong increase of the option value because the probability increases that default-free bonds become cheapest to deliver.

Summarizing the results of the comparative-static analysis the following conclusions can be drawn:

- For a high level of the term structure the ctd-bond can be anticipated with a high probability and the influence of the credit spread on the value of the delivery option is negligible.[27]
- The value of the delivery option reacts sensitively to changes in the parameters of the two stochastic processes. Under a LM-conversion factor system the value of the delivery option is in most cases considerably higher than under the other three price-factor systems.
- The adjustment for credit risk in the nominator of the issuer-dependent conversion factors necessarily decreases the differences between the futures-equivalent prices of default-free and credit-risky bonds. This does not necessarily lower the value

[27]There exists one exception to this rule when the spreads are very low and the LM-conversion factors are applied.

of the delivery option. Although the issuer-dependet conversion factors reduce the heterogeneity the probability of a change of the ctd-bond between different issuers increases exactly because the future-equivalent are closer to each other than under a traditional price-factor system. This causes almost in any case the highest value of the delivery option under the LM-price factor system.

The use of empirically estimated model parameters in the comparative-static analysis contributes to its validity and justifies comparing our results with findings of earlier empirical papers on the quality option. Berendes and Bühler (1994), Yu (1997) and Lin and Paxson (1995) qualify as benchmark because they base their valuation of the delivery option as well on consistent term structure models[28] and examine with the Bund Future a futures contract that is similar to the multi-issuer contracts in this paper.

Berendes and Bühler (1994) decompose the value of the delivery option into a *synthetic delivery option* and a *flexibility* option. The former is defined as the price difference between delivering the notional and the ctd-bond.[29] Contrary to a common option definition the value of the synthetic delivery option can be negative any time when the notional bond is cheaper to deliver than the ctd-bond. The flexibility option corresponds with the definition of the delivery option usually found in the literature and in this paper. For a selected observation day Berendes and Bühler determine a value of the flexibility option of the Bund Future that is 8 bp for 8 months before expiration and only 3 bp for 5 months.[30]

Yu (1997) analyses the quality option of Japanese government bond futures with a 6 %-notional bond of 10 years to maturity. Government bonds are eligible if their maturity is between 7 and 11 years. This means that the maturity window is wider than for the Bund Future and a higher value of the delivery option is to be expected from the findings of Cherubini and Esposito[31]. Three months prior to expiration the quality option has a value of 12 bp which the author rates as "valuable but not significant".[32]

Lin and Paxson (1995) determine for the Bund Future a quality option value of 9 bp three months before maturity which equals approximately the value of the *new issue option*. The name of the latter derives from the option conceded to the short to deliver a bond that has not been issued at the time of the evaluation but is going to be issued and enter the delivery basket before the the futures contract expires.

[28] There are earlier studies from Boyle (1989) and Hemler (1990) which evaluate the delivery option as a *switching option* in the spirit of Margrabe (1978). However, the assumption that the price processes follow geometric-brownian motions are fraud with theoretical problems, in particular with respect to their strong dependency on the correlation estimates.

[29] See Berendes and Bühler (1994), p. 1000-1001.

[30] See Berendes and Bühler (1994), p. 1017.

[31] See Cherubini and Esposito (1995), pp. 8–9.

[32] See Yu (1997), pp. 606–607.

To compare these earlier results with the findings of this study we consider first the delivery option values of the futures contracts with delivery baskets of default-free subsets of the multi-issuer future presented in table 3. For these two contracts the quality option values are 9 bp in the case of four and 3 bp in the case of two deliverable bonds when the traditional conversion-factor system is applied. This means that the empirical results from earlier papers concerning the valuation of the quality option are remarkably in line with the results in this study.

Note that the issuer option, calculated as the difference between the value of the delivery option and the quality option, is in most scenarios more than four times higher than the quality option and, therefore, economically much more important.

6 Hedge-Efficiency of Selected Conversion Factor Systems

An important result of the previous section is the increase in the value of the delivery option if issuer-dependent price factors are applied, especially in the case of the LM-conversion factor system. In the following we explore the effect of this result on the hedge efficiency of futures contracts. The hedging of bond positions has been the primary purpose of inventing bond futures contracts in the first place. Empirical work by Black (1986) and the Bank for International Settlements (1996) reveals the importance of the hedge efficiency for the success of a futures contract in the market. Their results underline the importance of analysing the hedge efficiency of a prospective multi-issuer contract under different conversion factor systems.

The following analysis involves specifying four design aspects that are the *hedge instrument* together with its delivery basket, the *hedge strategy*, the composition of the *long position in the bond market* that is to be hedged and the statistical method of *measuring the hedge efficiency.*

6.1 Hedge Instrument

The contract characteristics of the futures that are used as hedging instruments are defined as in section 5. Because these contracts are not traded at this time, prices are generated from the two-factor model presented in section 4. The parameters of this model are estimated in Bühler et al. (2001).[33] To obtain futures prices consistent with the observed market prices of the deliverable assets a calibration to the term structures of interest rates and credit spreads is carried out. The calibration method has already been applied successfully by Vetzal (1998) and is based on a proposition from Dybvig (1999). The key idea of this method is to add to the two state variables a deterministic factor that depends

[33]See appendix B.

Table 6:

Delivery Baskets of the Futures Contracts between May 1998 and February 2000

deliverable bonds			maturity of the future (month/year)							
issuer	coupon	maturity	6/98	9/98	12/99	3/99	6/99	9/99	12/99	3/00
GER	6.000 %	4.1.2007	×							
GER	6.000 %	7.7.2007	×	×	×					
GER	5.250 %	4.1.2008		×	×	×	×	×	×	×
GER	4.750 %	4.7.2008			×	×	×	×	×	×
GER	4.125 %	4.7.2008				×	×	×	×	×
GER	3.750 %	4.1.2009					×	×	×	×
GER	4.000 %	4.7.2009						×	×	×
GER	4.500 %	4.7.2009							×	×
GER	5.375 %	4.1.2010								×
ITL	5.750 %	10.7.2007	×	×	×	×	×	×	×	×

only on calendar time. The latter is determined so that the mean squared model errors of the prices of deliverable bonds are minimized.

The hedge analysis is conducted with weekly prices for the time period from May 1, 1998 to February 23, 2000. From the available futures contracts always the *next-by future* is known to be the most liquid one.[34] Assuming the same maturity dates as those of the Euro Bund Future there would have been nine next-by contracts starting with a contract maturing in June 98 and ending with a contract maturing in March 00. Table 6 presents the composition of the delivery baskets for these nine futures contracts. Because we restrict ourselves to OTC-traded bonds, there exists only a single Italian government bond maturing inside the relevant range of time to maturity that spans 8.5 to 10.5 years. Focusing on the multi-issuer character of the future this time window for delivery is slightly modified so that this bond stays in the delivery basket even when its time to maturity is less than 8.5 years. In order to avoid a distortion of the results by different time windows for the two issuers, German Bunds with a time to maturity between the Italien public bond and the lower limit of 8.5 years are also included.

Note that because there is only a single credit-risky bond eligible for delivery, the calibration of the default-model matches its market prices exactly whereas this does not hold for the German government bonds.

[34]See Ederington (1979), pp. 164–165.

6.2 Hedge Strategy

In order to assess to what extent the future qualifies as a hedging tool we assume *risk minimization* as the primary hedging purpose.[35] To this purpose two different hedging strategies are conducted. The first is a delta-hedge strategy against interest rate and credit risk. In the framework of the two-factor model the multi-issuer future serves as hedging instrument against credit risk. A short position in the youngest and therefore most liquid German government bond of the delivery basket is selected as the hedging instrument against interest rate risk.

The hedge ratios $\nu_{F,i}$ and $\nu_{P,i}$ are used for the multi-issuer future and the German bond as second hedging instrument. They are defined for this *delta-hedge* strategy as follows, when V, P and F denote the prices of the credit-risky bond to be hedged, the default-free bond and the multi-issuer future respectively:[36]

$$\nu_F = -\frac{\partial V/\partial s}{\partial F/\partial s} \tag{23}$$

$$\nu_P = -\frac{\partial V/\partial r + \nu_F \frac{\partial F}{\partial r}}{\partial P/\partial r} \tag{24}$$

However, the affine model does not explain the observed bond prices exactly. Therefore, we cannot expect a *perfect hedge*, in which case the returns of the long and the short position would cancel each other out.

Although the delta hedge is the best hedging strategy from a theoretical perspective, a *duration hedge* is still the pre-dominant strategy pursued by institutional investors to protect their bond positions. Furthermore, it has performed well in empirical studies like Brennan and Schwartz (1983) who compare it with a hedge strategy based on a two-factor equilibrium model of the term structure of interest rates.[37] In order to assess the future prospects of a multi-issuer future it is important how well such a contract performs in a hedging strategy that is prefered by practitioners. Therefore, we apply a traditional duration hedge as a second hedging strategy. The hedge ratio is defined as the ratio between the modified duration[38] of the bond acquired in the cash market and the modified duration of the ctd-bond times the conversion factor. Note that in this case only the interest rate risk is accounted for in the hedge ratio, so that a less efficient hedge is to be expected.

[35] See Ederington (1979), pp. 159–162, for references of other approaches with a two-dimensional risk-return consideration.

[36] The values of the futures contract is determined numerically and, therefore, the partial derivatives $\partial Fu/\partial r$ and $\partial Fu/\partial s$ have to be approximated as well numerically in the usual way by central differences.

[37] See Brennan and Schwartz (1983), pp. 17–28.

[38] See Sundaresan (1997), p. 132.

Table 7:
Long Position in the Bond Market to be Hedged

issuer	maturity	designation of the bond position			
		ITL04	ITL11	GERITL05	GERITL11
GER	second-youngest			×	×
ITL	2005	×		×	
ITL	2007				
ITL	2011		×		×

6.3 Long Position of Bonds to be Hedged

The long position in the bond market, that is to be hedged, consists, firstly and for expositional simplicity, of one single Italian government bond and, secondly, of portfolios of one German and one Italian government bond. The five bond positions that are considered in the analysis are listed in table 7. We assume that a new long position is established in weekly intervals by investing a nominal amount of Euro 100 and liquidating the position in the following week. If the long position consists of only one asset this amount is invested fully in the Italian bond and in case of a portfolio the amount is split in half between the German and the Italian government bond. The *hedge portfolio* of a delta hedge strategy consists of a long position K_i in the bonds to be hedged and short positions in the future with price F and a Bundesanleihe with price P. w_i denotes the portfolio weights of the long position in the bonds. The hedge returns of the hedge portfolio $\Delta H(t) = H(t+\Delta t) - H(t)$ for the time interval Δt of one week are calculated as follows when $\nu_{i,F}(t)$ and $\nu_{i,P}(t)$ denote the respective hedge ratios.

$$\Delta H(t) = \Delta K(t) - \sum_{i=1}^{2} w_i \cdot \nu_{i,F}(t) \cdot \Delta F(t, T, \{1, \ldots, m\}) - \sum_{i=1}^{2} w_i \cdot \nu_{i,P}(t) \cdot \Delta P(t, M, c) \quad (25)$$

$$\Delta K(t) = \sum_{i=1}^{2} w_i \left(K_i(t + \Delta t) - K_i(t) \right) . \quad (26)$$

Usually investors roll over their position into the next futures contract at the end of the last month before maturity and the liquidity of the next-by contract decreases afterwards. This procedure is adopted in this analysis and the roll-over date is in the last week of the month before future expiration.

6.4 Assessing the Hedge Efficiency

In previous studies of the hedge efficiency and in practitioners' handbooks the *Johnson*-statistic[39] VRM serves as the predominant measure for the hedge quality. This statistic is defined as the ratio between the variance reduction of the returns achieved by the hedged portfolio and the variance of the returns of the unhedged long position.[40] N denotes the number of weekly hedge intervals and $\overline{\Delta K}$ and $\overline{\Delta H}$ the average of the respective time series.

$$VRM = 1 - \frac{\sum_{t=1}^{N}(\Delta H(t) - \overline{\Delta H})^2}{\sum_{t=1}^{N}(\Delta K(t) - \overline{\Delta K})^2}. \tag{27}$$

We suggest that a hedging statistic should consider instead the squared differences from the *ideal hedge return* $\Delta H^*(t)$ instead of the time series averages $\overline{\Delta K}$ and $\overline{\Delta H}$. The ideal hedge return $\Delta H^*(t)$ is defined as the return of the hedge portfolio if the multi-issuer future is replaced as hedging instrument by a futures contract, written on the bond that is to be hedged. Contrary to the multi-issuer contract this new future for bond i with price $F^*(t, T, \{i\})$ is not endowed with a delivery option and hence its hedge quality is not distorted by this option. The ideal hedge return $\Delta H^*(t)$ of a hedge portfolio is defined as follows with $\Delta K(t)$ defined in (26):

$$\Delta H^*(t) = \Delta K(t) - \sum_{i=1}^{2} w_i \cdot \nu_{i,F^*}(t) \cdot \Delta F^*(t, T, \{i\}). \tag{28}$$

The variance reduction relative to the ideal hedge returns VRI is defined accordingly:

$$VRI = 1 - \frac{\sum_{i=1}^{N}(\Delta H(t) - \Delta H^*(t))^2}{\sum_{i=1}^{N}(\Delta K(t) - \Delta H^*(t))^2}. \tag{29}$$

The hedge statistic VRM is used additional to VRI in the following analysis in order to facilitate comparing the results with previous studies.

6.5 Analysis of Hedge Results

The hedge analysis is carried out in three steps: In the first step the hedge efficiency is determined for the LM-conversion-factor system applying two hedge strategies and it is measured by two different hedge statistics. This step aims at exploring the robustness of the results with respect to these two criteria and to compare the achieved hedge efficiency with earlier studies. In the second step the value of the delivery option is analysed for multi-issuer futures with different conversion-factor systems. The third step focuses on the impact of the conversion-factor system on the hedge efficiency.

[39] See Johnson (1960), pp. 142–144.

[40] See Ederington (1979) and Figlewski (1996) among others.

Table 8:

Variance Reduction in Percent Relative to an Ideal Hedge for LM-Conversion Factors

hedge statistic	hedge-strategy	bond position to be hedged ITL04	ITL11	GERITL05	GERITL11
VRI	duration	92.4	94.3	87.2	96.2
	delta	88.1	89.1	84.7	95.3
VRM	duration	12.9	67.5	77.4	50.0
	delta	12.3	50.6	93.6	50.9

6.5.1 Dependency on the Hedge Strategy and on the Hedge Statistic

The LM-price-factor system is applied in the first step because it leads to the highest value of the delivery option in the comparative-static analysis. Therefore it offers the widest scope for improvement of the hedge efficiency. If, instead, the current ctd-bond almost surely stays cheapest to deliver until expiration, then the conversion factors embedded in the hedge ratio and the futures price formula cancel each other out in (25) and do not effect the hedge result.

Table 8 presents the results for the hedge efficiency in the case of LM-conversion factors when both hedge strategies, delta hedge and duration hedge, are applied. The hedge efficiency is measured by the variance reduction relative to the ideal hedge returns VRI and the Johnson statistic VRM. The composition of the long positions in bonds is given in table 7.

Focusing first on the VRI-statistic the variance reduction for the duration hedge is between 87.2 % and 96.2 % and between one and five bp higher than in the case of a delta hedge. This result can be attributed to the dependency of the delta hedge on the returns of the second hedge instrument which is a short position in the German government bond. Because there are more than one German bond in the deliverable basket their observed prices are not fully consistent with the calibrated model. Therefore, the bond returns are not fully captured by the model which reduces the hedge quality.

Comparing the results in table 8 with respect to the two different hedge statistics VRM and VRI the hedge efficiency is overall higher if the VRI-statistic is applied. Exceptions are GERITL07 for both hedge strategies and GERITL05 if a delta-hedge is applied. The reason for these differences is the following. The time-homogenuous model cannot explain the returns of the long position perfectly. This 'model error' affects the return $\Delta H^*(t)$ of an ideal hedge in the same way as the long position in bonds, because at the beginning and at the end of every hedge interval the futures prices that are used for the calculation of $\Delta H^*(t)$ are calibrated to the term structures of interest rates and credit spreads. Therefore, this model error roughly cancels out in the terms of the sum in the nominator and the denominator and does not affect the VRI-statistic. However, it affects the VRM-statistic because it is not captured by the averages $\overline{\Delta H}$ and $\overline{\Delta K}$.

Next we compare our results with three previous papers concerning the hedge efficiency of a bond futures contract and which employ similar hedge strategies. Toevs and Jacob (1986) analyse the implications of hedging US government and GMAC bonds with 30 year US Treasury-bond futures, applying among others a duration hedge strategy. They hedge a long position in a US government bond with a maturity similar to the ctd-bond.[41] The VRM-statistic shows a variance reduction of 92 %. Note that this result refers to hedging on a daily basis and taking 10-day moving averages in order to remove purely random basis risk.[42] For a maturity mismatched hedge (8 years to maturity) and for a long position in a AA-rated GMAC bond the VRM-statistics drop to 78 % and 80 %. These results suggest that a mismatch in maturity harms the hedge quality considerably and even more than the difference in default risk does. The effect of a maturity mismatch is observed in table 8, too, in which we find a very different hedge quality when the same future is used to hedge bonds of the same issuer but with different maturities.

Meyer (1994) uses market prices of the Bund Future in order to hedge in two-week intervals portfolios consisting of Bundesanleihen and other German public bonds. According to the VRM-statistic the variance reduction is 80 % and still 70 % after including corporate bonds in his portfolio.[43] To obtain a comparable hedge scenario, we hedge a long position in the second youngest Bundesanleihe from the delivery baskets given in table 6 and use for a duration hedge the multi-issuer future, equipped with a traditional conversion-factor system. The variance reduction according to the VRM-statistic is 78 % and, therefore, close to the result in Meyer (1994).

Lin and Paxson (1995) analyse the hedge efficiency of the Bund Future for portfolios of Bundesanleihen. Similar to our study they measure the hedge quality for bonds of several maturities (two, five and ten years) separately and employ two hedge strategies, the first uses the conversion factor as hedge ratio and the second is a delta hedge. They find that the future provides a low reduction in variance, measured by VRM, in the case of a cross-hedge with a Bundesanleihe maturing in two years. In three out of six time periods of half a year there is no variance reduction at all and for the other three periods it varies between 4 % and 60 %. The low hedge efficiency, especially for bonds with shorter maturities, is in line with the results in table 8 for the Italian government bond maturing in 2005.

Summarizing the results of the first step we conclude that replacing the common Johnson-statistic by the new hedge statistic VRI gives different and more reliable results. The

[41] See Toevs and Jacob (1986), pp. 66–68.
[42] See Toevs and Jacob (1986), p. 69.
[43] See Meyer (1994), pp. 264–266.

hedge efficiency reacts sensitively against a change in the long position of the bonds, especially with respect to differences in their maturities. These results are in line with earlier analyses of the hedge efficiency of bond futures with similar hedge strategies, namely Toevs and Jacob (1986), Meyer (1994) and Lin and Paxson (1995).

6.5.2 Hedge Efficiency and the Value of the Delivery Option

In the second step the value of the delivery option is analysed. Panels 1 to 3 in figure 8 show the value of this option calculated for weekly observations. The vertical dashed lines indicate the roll-over days when the next-by future is replaced by the next contract. Whereas in the first panel the option value under traditional conversion almost vanishes most of the time, the time series of the option values change erraticly in the last panel when LM-price factors are applied. For the IS-conversion factors the option values in the second panel of figure 8 resemble those when traditional conversion factors are applied but they increase stronger towards the end of the observation period.

The delivery basket for the hedge analysis differs from the one used in the comparative-static analysis, especially because only a single credit-risky bond is available for delivery. Therefore, the delivery option cannot derive its value from future changes of the ctd-bond between different credit-risky bonds of the delivery basket but only from the possibility that a default-free asset will become cheapest to deliver. Instead, if the delivery basket is built symmetrically of default-free and -risky assets, as it is in the comparative-static analysis in section 5, then for a traditional conversion-factor system the short can always select a credit-risky bond that is cheaper to deliver. In this case and if there is only a single credit-risky bond eligible for delivery the delivery option would be worthless. However, the delivery baskets in the hedge analysis are *not* symmetrically built and it is possible that a default-free bond with other coupon and maturity characteristics will become the ctd-asset in certain interest rate scenarios even under a traditional conversion factor system. This happens when the future maturing in February 2000 is the next-by contract and the delivery option has a value of up to 10 bp as is demonstrated in panel 1 of figure 8.

Considering that the delivery basket contains only one single credit-risky bond, the values of the delivery option for the LM-price factors are relatively high compared with our results in the comparative-static analysis. Table 9 provides descriptive statistics of the time series of the delivery option values for the three selected conversion-factor systems. The high differences in relative terms between the mean and the median option value which surface for the LM-conversion factors indicate the presence of relatively few observations with relatively high option values which is confirmed by the third panel in figure 8. In how far the price factor-dependent differences in the option value are related to differences in the hedge efficiency is explored in the third step of this analysis.

Figure 8:

Value of the Delivery Option from May 1998 until February 2000 for Multi-Issuer Futures

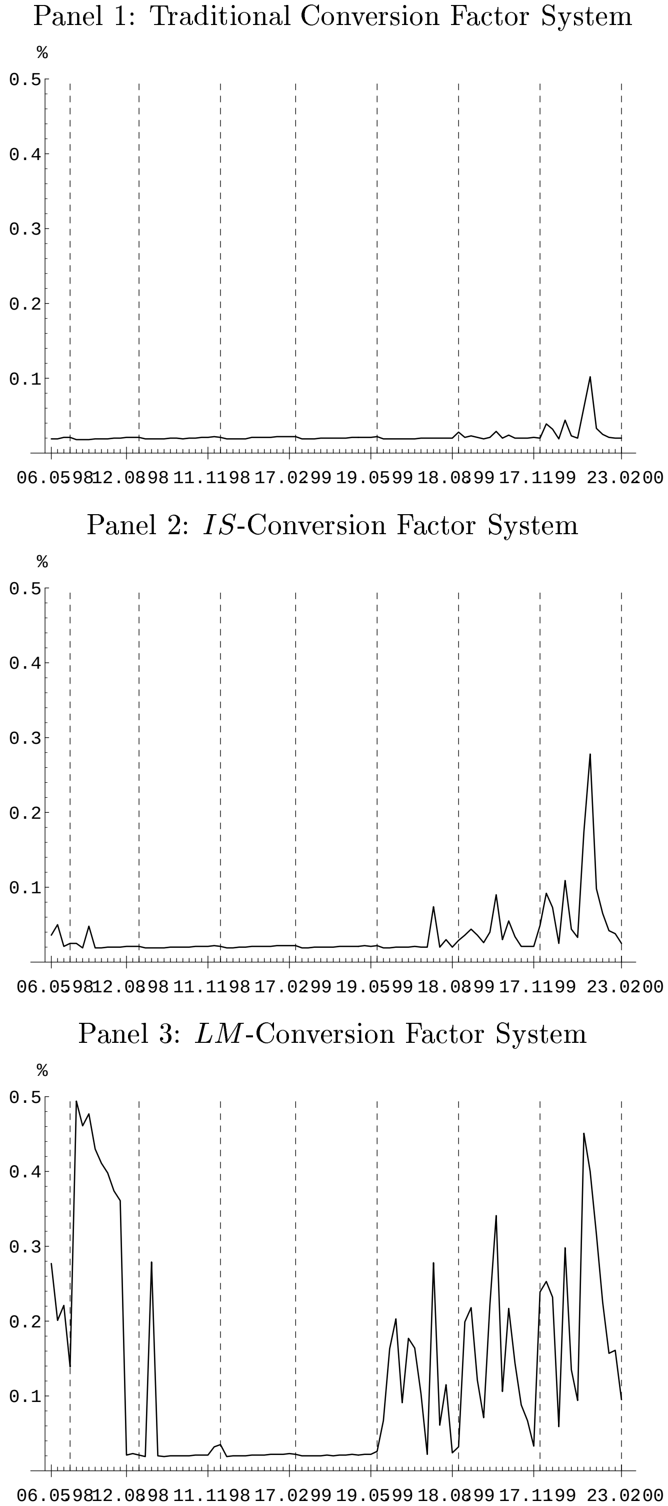


Table 9:

Descriptive Statistics of the Value of the Delivery Option for Selected Conversion Factor Systems in Basispoints

Conversion factor	Traditional	IS	LM
Maximum	10	28	49
Mean	2	3	13
Median	2	2	6
Standard Deviation	1	4	14

Table 10:

Variance Reduction in Percent Relative to an Ideal Hedge with a Delta-Hedge Strategy

conversion-	bond position to be hedged			
factors	ITL04	ITL11	GERITL05	GERITL11
Traditional	89.7	90.2	84.6	95.0
IS	89.6	90.2	84.6	95.2
LM	88.1	89.1	84.7	95.3

Table 11:

Variance Reduction in Percent Relative to the Time Series Averages (Johnson-Statistic) with a Delta-Hedge Strategy

conversion-	bond position to be hedged			
factors	ITL04	ITL11	GERITL05	GERITL11
Traditional	12.3	51.6	93.7	50.9
IS	12.3	50.8	93.6	50.9
LM	12.3	50.6	93.6	50.9

6.5.3 Impact of the Conversion Factor System on the Hedge Efficiency

Tables 10 and 11 present the hedge efficiency measured by VRI and VRM for the selected traditional, IS- and LM-conversion factors. The differences are negligible for both hedge statistics. We conclude that although the delivery option values differ considerably this does not transfer into differences in hedge quality. Note that even under a LM-conversion factor system, under which the highest option values are observed, the hedge efficiency is effectively still the same as under a traditional price factor system. The reason for this observation is the result that the hedge efficiency is not harmed by an increase of the value of the delivery option in the first place. Therefore, there is not enough room for improvement in the hedge quality to bring about a noticeably higher hedge quality.

The empirical result, that a valuable delivery option does not harm the hedge efficiency of a bond future is consistent with results in Lin and Paxson (1995).[44] In their analysis of the quality option they determine model prices of the Bund Future, firstly endowed with a delivery option and then, secondly, without. Comparing the hedge efficiency of both cases reveals no worse hedge quality in the presence of a delivery option.

7 Ranking of the Conversion Factor Systems

Based on the quantitative results from section 5 and 6 we compare the four selected conversion factor systems concerning their suitability for a prospective multi-issuer bond futures contract, e. g. for European government bonds. To this purpose we propose four *theoretical requirements* as the first of two parts of a requirement catalogue.[45]

1. The first theoretical requirement is a *fair settlement price.* This means that the invoice amount for the short has to reflect accurately the differences between the physically delivered asset and the notional bond which influence the asset price, notably the coupon size, the time to maturity and the credit risk.

2. The conversion-factor system should *reduce the heterogeneity*, measured as the potential delivery losses from delivering an asset different from the ctd-bond. The lower these delivery losses are the lesser becomes the potential gain of a short squeeze, at least as long as the manipulator cannot squeeze the whole delivery basket.

3. Another aspect that helps to prevent a squeeze is a *high uncertainty* concerning the terminal ctd-bond. Ideally the probabilities to become cheapest to deliver at maturity of the future contract should be the same for all assets in the basket. This and the previous requirement are the key factors which determine the value of the delivery option.

[44]See Lin and Paxson (1995), pp. 117–123

[45]Note that the first three requirements restate conclusions from section 2.

Table 12:

Ranking of Conversion Factor Systems

requirement	traditional	*CE*-	IS-	LM-
	conversion-factor systems			
1. fair settlement price	4	2	2	1
2. minimize potential delivery loss	4	2	2	1
3. high uncertainty about delivery	4	2	2	1
4. high hedge efficiency	1	1	1	1
1. transparency	1	2	3	4
2. easy accessibility	1	1	1	1
3. robust against manipulation	1	3	3	1
4. offering of speculation profit	4	2	2	1

4. The conversion-factor system should *improve the hedge efficiency* of the futures contract.

Additional to the four theoretical requirements, we propose four *practical requirements* that a conversion-factor system is expected to meet.

1. Comments from practitioners suggest that traders will be uncomfortable with a conversion-factor system that is too complex and *lacks transparency*

2. Another requirement is a *full-time accessibility.* The easiest way to ensure this is by conversion factors that are determined only once.

3. The conversion-factor system should be *robust against manipulation attempts*, for instance by manipulating the market prices that are used for the determination of the conversion factors.

4. Any uncertainty about the ctd-bond offers traders a *chance of a speculation profit.* From this point of view the existence of a valuable delivery option may even contribute to the success of the future in the market.

In order to rate the four proposed conversion-factor systems we apply a ranking with respect to these eight requirements. We assign the number '1' to the conversion factors that meet the requirement best and the number '4' to the conversion factors that do so worst. The ranking is presented in table 12. With respect to the theoretical requirements the LM-conversion factors perform best and the traditional price factors perform worst. Ranked in between are the CE- and IS-conversion factors for whom we obtain very similar results.

The ranking changes when the trading requirements are considered. The traditional conversion-factor system meets best the requirements of transparency, accessibility and robustness against manipulation. However it does not account for a more heterogenuous delivery basket when credit risky assets are included and, therefore, facilitates a ctd-squeeze. Additionally, if a credit-risky asset is always cheaper to deliver, this reduces the prospect of speculation profits. The CE- and IS-conversion factors are better suited than a traditional price-factor system considering the third and fourth requirement because they account for differences in credit risk. All conversion factors are calculated only once. However, because of their reliance on rating information the IS- and CE-price factors are not as transparent as the traditional conversion method.

The LM-conversion factors must be considered as least transparent because they are determined by solving a complex optimization problem. However, they perform best with respect to the third and fourth practical requirement. They correct best for differences in default risk and therefore increase in most cases the delivery option value which offers a better chance of speculation profit.

Summarizing the ranking results, the LM-conversion factors are best suited for a prospective multi-issuer futures contract as to the theoretical requirements. Concerning the practical requirements the results are mixed but in three out of four it performs best again and as to one of the other three criteria all price factor systems do not not differ at all. Therefore, we recommend this price-factor system for a prospective European government future.

8 Conclusion

In this paper we address two theoretical and one empirical problem. The first problem deals with the construction of appropriate conversion factors in the presence of default risk, the second with the value of the delivery option in these cases and the third deals with the impact of the conversion-factor system and the delivery option on the hedge efficiency of a multi-issuer contract.

We propose three conversion-factor systems which extend the construction principle of traditional price factors by introducing an issuer-dependent premium for credit risk. The data requirements are parsimonious: Two of the new price factor systems only need cumulative default frequencies that are readily available from rating agencies. The third conversion-factor system requires a set of term structures of interest and credit spreads from which the credit risk premium is determined by minimizing potential delivery losses. All three conversion factor systems reduce the difference in the future-equivalent prices between default-free and credit-risky deliverable bonds. This reduces the heterogeneity of the delivery basket and renders a short squeeze more difficult.

In the theoretical part of the paper the value of the delivery option is analysed with and without credit-risky bonds in the delivery basket and for different conversion factor systems. This study is extended to an extensive sensitivity analysis by varying the parameters of the delivery basket and of the valuation model. The value of the default-risk dependent issuer option is determined as a residual by subtracting the value of the interest rate dependent quality option from the total value of the delivery option. We find that for the traditional conversion-factor system the value of the issuer option is roughly more than four times higher than the value of the quality option. The application of the new issuer-dependent conversion-factor systems increases the value of the delivery option. The most notable increases are caused by an increase of the probability that a default-free bond becomes cheapest to deliver.

An important empirical question is the impact of the delivery option on the hedge efficiency of multi-issuer futures. Although we observe under different conversion factor systems quite different values of the delivery option, there is no evidence that this harms the hedge efficiency of the futures contract. This result is verified to be robust against changes in the hedge strategy, the composition of the delivery baskets and against different methods how to measure the hedge efficiency.

Additional to these theoretical and empirical contributions the appropriateness of the selected conversion-factor systems for a multi-issuer contract, e. g. a prospective European government bond future is discussed. To this purpose a catalogue of eight requirements is compiled. Four theoretical requirements concern the settlement procedure, the heterogeneity of the delivery basket, the uncertainty of the terminal ctd-bond and the hedge efficiency. Added to this are four practical requirements concerning the transparency, accessibility, robustness against manipulation attacks and the prospect of speculation profits. Based on these eight requirements and the quantitative results from the previous analysis a ranking is proposed of the traditional and three selected issuer-dependent conversion-factor systems. This ranking enables us to recommend the LM-price-factor system as most suitable for a prospective multi-issuer contract.

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A Valuation of Default-Risky Pure Discount Bonds and Futures Contracts in a CIR-Model

A.1 Valuation of a Default-Risky Pure Discount Bond

$$V\left(t,T;r_t,s_t\right)=A_1\left(T-t\right)A_2\left(T-t\right)e^{-B_1(T-t)r_t-B_2(T-t)s_t} \tag{30}$$

where for $i\in\{1,2\}$:

$$B_i\left(T-t\right)=\frac{2\left(e^{\gamma_i(T-t)}-1\right)}{\left(\gamma_i+\alpha_i+\lambda_i\right)\left(e^{\gamma_i(T-t)}-1\right)+2\gamma_i} \tag{31}$$

$$A_i\left(T-t\right)=\left(\frac{2\gamma_i e^{(\gamma_i+\alpha_i+\lambda_i)(T-t)/2}}{\left(\gamma_i+\alpha_i+\lambda_i\right)\left(e^{\gamma_i(T-t)}-1\right)+2\gamma_i}\right)^{2\alpha_i\theta_i/\sigma_i^2} \tag{32}$$

$$\gamma_i=\sqrt{\left(\alpha_i+\lambda_i\right)^2+2\sigma_i^2}. \tag{33}$$

A.2 Valuation of a Future on a Pure Discount Bond

$$\begin{aligned}F(t,T,\{1\})=\prod_{i=1}^{2}A_i(M-T)&\left(\frac{\eta_i}{B_i(M-T)+\eta_i}\right)^{2\alpha_i\theta_i/\sigma_i^2}\\ \times\exp&\left[-x_i\ \left(\frac{\eta_i\,B_i(M-T)\,e^{-(\alpha_i+\lambda_i)(T-t)}}{B_i(M-T)+\eta_i}\right)\right]\end{aligned} \tag{34}$$

where for $i\in\{1,2\}$:

$$\eta_i=\frac{2\left(\alpha_i+\lambda_i\right)}{\sigma_i^2\left(1-e^{-(\alpha_i+\lambda_i)(T-t)}\right)}$$

and $A_i(M-T)$, $B_i(M-T)$ and γ_i are given by (32), (31) and (33).

B Parameter Estimates of the Term Structures of Interest Rates and Credit Spreads

Table 13:

Maximum Likelihood Parameter Estimates of the German Term Structure

This table summarizes the parameter estimates of the CIR one-factor model for their German term structure of interest rates and the credit spread structure between Italian and German government bonds. The estimates are from Bühler et al. (2001). The numbers in parentheses are the asymptotic standard errors. The sample period includes 92 weekly observations and extends from May 1998 to February 2000.

Estimates (Standard Errors)				
Parameter	German Term Structure		Credit Spreads	
α_i	0.0651	(1.4638)	0.1023	(2.2430)
θ_i	0.0598	(1.3439)	0.0018	(0.0394)
σ_i	0.0558	(0.0006)	0.0935	(0.0022)
λ_i	-0.0340	(1.4740)	-0.3460	(2.2436)
$\alpha_i + \lambda_i$	0.0311	(0.0107)	-0.2437	(0.0046)
$\frac{\theta_i \alpha_i}{\alpha_i + \lambda_i}$	0.1249	(0.0414)	-0.0008	(0.0000)