Risk Management
Based on Stochastic Volatility

Ernst Eberlein
Jan Kallsen
Jörn Kristen

Institute for Mathematical Stochastics
and
Freiburg Center for Data Analysis and Modelling

University of Freiburg
Eckerstraße 1
D-79104 Freiburg im Breisgau

eberlein@stochastik.uni-freiburg.de
Tel. ++ 49/ 761/ 203-5660
FAX ++ 49/ 761/ 203-5661

Risk Management Based on Stochastic Volatility

Ernst Eberlein, Jan Kallsen, Jörn Kristen*
University of Freiburg

Abstract

Risk management approaches that do not incorporate randomly changing volatility tend to under- or overestimate the risk depending on current market conditions. We show how some popular stochastic volatility models in combination with the hyperbolic model introduced in Eberlein and Keller (1995) can be applied quite easily for risk management purposes. Moreover, we compare their relative performance on the basis of German stock index data.

Keywords: stochastic volatility, hyperbolic model, backtesting, risk management

1 Introduction

Risk in the sense of the possibility of losses is an inherent ingredient of financial markets. To measure and monitor risk as accurately as possible has become a competitive factor for financial institutions. How can we quantify risk? Theoretically, there are a number of possibilities, such as standard deviation, quantiles, interquartile range, lower partial moments, or shortfall measures. *Value-at-risk* or *VaR*, a quantile measure, has become the preferred tool in the financial industry. Although it is a rather natural concept from a probabilistic point of view, it became popular only as a consequence of the proposals of the Basel Committee on Banking Supervision for the internal model approach to manage market risk. The VaR value of a portfolio depends on the underlying model which is used. Three basic approaches are currently applied in practice to measure market risk: historical simulation, the variance-covariance approach and Monte Carlo simulation (cf. Jorion (1997)). Note that the variance-covariance approach as well as Monte Carlo simulation depend on the assumption of normality of returns.

The purpose of the present paper is to improve on these approaches. The method we investigate is based on two pillars: the *hyperbolic model* and *stochastic volatility*. Hyperbolic distributions were introduced to finance in Eberlein and Keller (1995). These and the

*Institut für Mathematische Stochastik, Universität Freiburg, Eckerstraße 1, D-79104 Freiburg, Germany, (e-mail: eberlein@stochastik.uni-freiburg.de, kallsen@stochastik.uni-freiburg.de)

wider class of generalized hyperbolic distributions constitute a very flexible family of distributions which is tailor-made for fitting empirical return data. In the same paper a new dynamic model, the hyperbolic model, was introduced. It is defined as the (ordinary) exponential of the *Lévy process* which is generated by any hyperbolic distribution. This model is not of diffusion type because it is driven by a pure-jump Lévy process. Due to its greater flexibility it improves considerably on the classical geometric Brownian motion. In a series of papers (Eberlein et al. (1998), Keller (1997), Barndorff-Nielsen (1998), Eberlein and Prause (1998), Eberlein (2001)) the initial model was pushed further. It can be handled now on the level of generalized hyperbolic Lévy motions. We note that many of the standard distributions used in statistics, such as Student t, Cauchy, variance gamma, normal inverse Gaussian, and the normal distribution are either special or limiting cases of the underlying class of distributions.

The second pillar is to model volatility as a stochastic process. Volatility can be considered as the temperature of the market and as such it can change rapidly. Risk management approaches which do not take these fluctuations into account tend to under- or overestimate risk depending on the current market situation. Because of the strong empirical evidence for stochastic volatility in financial time series, an impressive number of different approaches has been studied in the literature. Let us just mention the variety of ARCH and GARCH models introduced by Engle (1982) and Bollerslev (1986), the diffusion model of Hull and White (1987), as well as the models investigated in Chesney and Scott (1989), Stein and Stein (1991), and Heston (1993). The key idea using stochastic volatility in risk management is to devolatilize the observed return series and to revolatilize with an appropriate forecast value. This idea has been applied in several recent papers (Hull and White (1998), Barone-Adesi et al. (1998, 1999), McNeil and Frey (2001)).

Since we have always a portfolio view – any portfolio as complex as it may be is considered as a security of its own – we study here a data set consisting of the daily closing DAX values from 1992 to 1999. The DAX represents a portfolio of 30 German blue chip stocks and reflects the behaviour of the German stock market. One of the reasons to consider this time series is the availability of the volatility index VDAX, which will be considered as one of the various models for volatility.

The paper is organized as follows: We start with an exploratory view on stock index data in Section 2. This qualitative examination serves as a motivation for the stochastic volatility models that are introduced in the subsequent section. Afterwards, we discuss how to estimate model parameters and in particular current volatility. Ideally, a stochastic volatility model can be used to transform stock returns into independent, identically distributed (i.i.d.) random variables (*devolatilization*). This aspect is tested empirically in Section 5. Subsequently, we investigate the performance of stochastic volatility models as far as risk management is concerned. Note that for the sake of simplicity we restrict ourselves to hyperbolic distributions. With some additional computational effort, generalized hyperbolic distributions (cf. Eberlein and Prause (1998)) could be used along the same lines. The results we obtained using the smaller class are already quite convincing. Section 7 concludes.

2 Exploratory analysis of stochastic volatility

It has often been reported that stock return volatility changes randomly over time. We want to get a qualitative idea of these fluctuations before turning to particular models in the next section. To this end, we take a look at daily stock index data, namely the *Dow Jones Industrial Average* from May 26, 1896 to January 4, 2001. By X_t we denote the logarithm or *return process* of the index. As a general starting point, we assume that this process is of the form

$$\Delta X_t = \sigma_t \Delta L_t,$$

where $\Delta X_t := X_t - X_{t-1}$, the variable $\sigma_t > 0$ stands for the randomly changing volatility, and $(\Delta L_t)_{t=1,2,\ldots}$ denotes a sequence of identically distributed random variables with $\mathrm{Var}(\Delta L_t) = 1$ such that ΔL_t is independent of $(\Delta L_s)_{s=1,\ldots,t-1}, (\sigma_s)_{s=1,\ldots,t}$. In order to assess the unobservable volatility fluctuations it is convenient to consider the logarithmic squared daily returns

$$D_t := \log(\Delta X_t)^2 = \log \sigma_t^2 + \log(\Delta L_t)^2.$$

Note that $(\log(\Delta L_t)^2)_{t=1,2,\ldots}$ is a series of i.i.d. random variables. Hence, the time series $(D_t)_{t=1,2,\ldots}$ can be interpreted as a signal $(\log \sigma_t^2)_{t=1,2,\ldots}$ perturbed by an additive white noise with mean $E(\log(\Delta L_1)^2)$. Since we do not assume a particular model at this point, it makes sense to estimate the unobservable quantity $\log \sigma_t^2$ by applying a non-parametric smoother to the data $(D_t)_{t=1,2,\ldots}$. In Figure 1, the series $(D_t)_{t=1,\ldots,28565}$ is plotted for daily Dow Jones data. The wavy white curve corresponds to a cubic smoothing spline where the smoothing parameter is chosen by cross validation (cf. Hastie and Tibshirani (1990), Sections 2.10 and 3.4). Note that this estimate of $\log \sigma_t^2$ is biased by $E(\log(\Delta L_1)^2)$, which equals e.g. -1.27 in the case of standard normally distributed ΔL_t. One can see very clearly how the level of volatility changes randomly over time. Of course, one would not expect stock returns to behave in a stationary way over more than 100 years. The whole market structure has changed repeatedly during that period. Keeping this in mind, the evolution of the estimated volatility in Figure 1 as well as the residuals in Figure 2 look surprisingly stationary at first glance – maybe excepting the steep increase of volatility in the 30's. Note that large absolute price changes correspond to large values of D_t. The striped pattern for large negative values of D_t in Figure 1 is caused by the fact that the index moves on a discrete grid. Since it corresponds to tiny price changes, it is of no importance for our purposes.

Let us take a closer look at the residuals by comparing them to a simulated sample $(\log(\Delta L_t)^2 - E(\log(\Delta L_1)^2))_{t=1,\ldots,28565}$ in Figure 3, where $(\Delta L_t)_{t=1,\ldots,28565}$ are drawn from a standard normal distribution. Although this sample looks similar to Figure 2, one may observe two differences: Firstly, large values of $\log(\Delta L_t)^2 - E(\log(\Delta L_1)^2)$ occur much more frequently for real data than in the normal sample. This corresponds to the well-known fact that stock return data exhibits heavier tails than the normal distribution. Secondly, one can find some remaining clusters of extremely high values in Figure 2, which are absent in the simulated sample. They indicate that there may be a very short-lived stochastic volatility

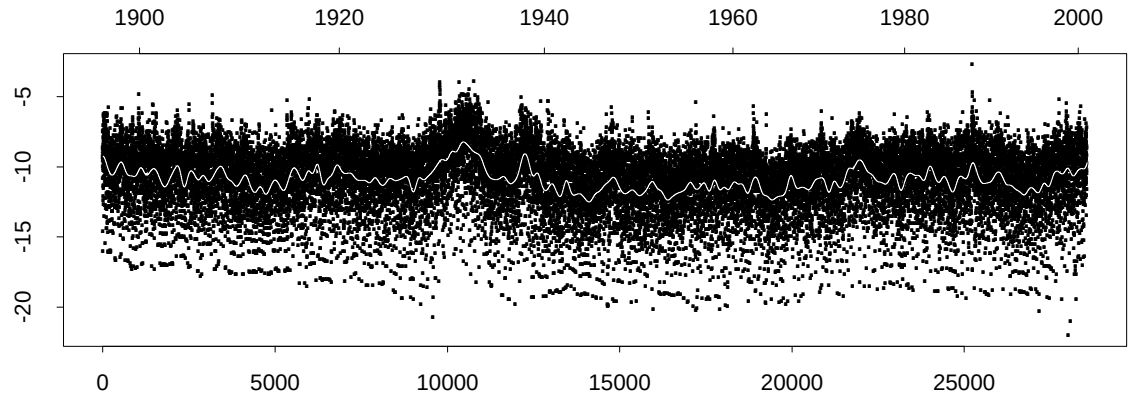

Figure 1: Logarithmic squared daily returns and estimated logarithmic squared volatility

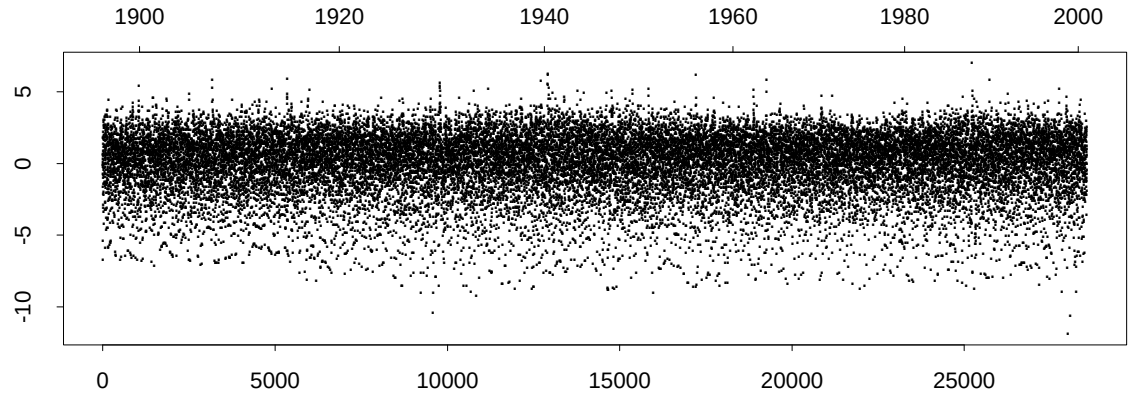

Figure 2: Logarithmic squared daily returns minus estimated logarithmic squared volatility

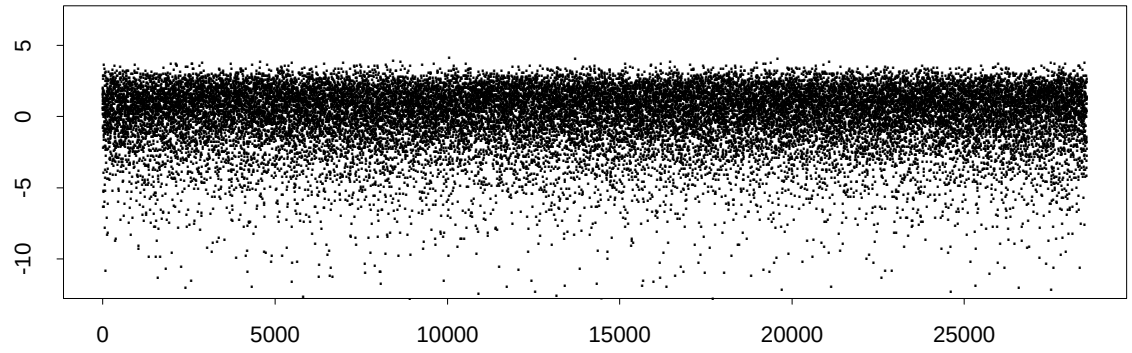

Figure 3: Simulated residuals in the case of normal daily returns

component in the data in addition to the comparatively slowly varying component that is detected by the smoothing spline.

3 Stochastic volatility models and risk management

The general model

We consider a univariate *price process* $S = (S_t)_{t\geq 0}$, typically a stock price or stock index. For simplicity, we suppose that this process is corrected for dividend payments etc., if there are any. By $X = (X_t)_{t\geq 0}$ we denote the *return process* of S, i.e.

$$S_t = S_0 \exp(X_t). \tag{3.1}$$

We assume that this return process is of the form

$$dX_t = \sigma_t dL_t, \tag{3.2}$$

where L is a process with stationary, independent increments starting at 0 (a *Lévy process*) and σ denotes a randomly changing predictable *volatility process*. Since we work with discrete-time data, we will henceforth replace Equation (3.2) with its discrete-time analogue

$$\Delta X_t = \sigma_t \Delta L_t \tag{3.3}$$

with $\Delta X_t := X_t - X_{t-1}$ and likewise for L. In fact, Equation (3.3) follows from (3.2) if σ is piecewise constant between integer time points. Note that although we use a discrete-time framework, the question continuous-time versus discrete-time is not an issue in this paper.

By *volatility* one usually refers to the standard deviation of the return series, or more precisely, to the standard deviation given all past observations. Therefore one should require that $\mathrm{Var}(\Delta L_1) = 1$. However, the statistical procedures below get more transparent if we do not insist on this normalization. In order to obtain properly standardized versions of σ and L, one simply has to multiply σ (and divide L) by the constant $\sqrt{\mathrm{Var}(\Delta L_1)}$.

In detail, we discuss the following set-ups:

Modelling the stochastic volatility σ

1. As a benchmark we consider deterministic, *constant volatility* σ. This case is studied extensively in the empirical and theoretical literature. One of our goals is to assess how much accuracy is gained by turning to more complex models.

2. The other extreme is to allow a basically *arbitrary stochastic volatility process* σ. However, in order for our estimation procedures to work, we have to assume that σ changes slowly over time as it is indicated by Figure 1. More precisely, we suppose that the persistence of volatility changes can be measured in weeks whereas stock price data is sampled daily.

3. A very popular stochastic volatility model is the *GARCH(1,1)-M process* (cf. e.g. Bollerslev et al. (1992), Shephard (1996), Gouriéroux (1997), Kallsen and Taqqu (1998), Shiryaev (1999), Section II.3). Here, we consider the case

$$\sigma_t^2 = c + a\sigma_{t-1}^2(\Delta L_{t-1} - m)^2 + b\sigma_{t-1}^2, \tag{3.4}$$

with parameters $\sigma_1 \geq 0$, $c \geq 0$, $a \geq 0$, $b \geq 0$, $m := E(\Delta L_1)$. The GARCH(1,1)-M model has been applied extensively to financial time series. From a theoretical viewpoint, it means that large volatility is *caused by* past returns of large absolute value.

4. Diffusion-type volatility models have been studied thoroughly in connection with option pricing and they attract increasing attention in the empirical literature (cf. e.g. Hull and White (1987), Barndorff-Nielsen and Shephard (1998)). In this paper we consider the particular case that $V_t := \log \sigma_t^2$ follows a *shifted Ornstein-Uhlenbeck process*

$$dV_t = -a(V_t - \overline{V})dt + bdW_t,$$

where V_0 is drawn from the stationary distribution $N(\overline{V}, \frac{b^2}{2a})$. Here, $a > 0$, $b, \overline{V}$ are parameters and W denotes a standard Wiener process that is independent of the Lévy process L. Viewed as a discrete-time process, $(V_t)_{t=0,1,2,\ldots}$ is a stationary AR(1)-time series with mean $\overline{V}$:

$$V_t = \varphi V_{t-1} + (1-\varphi)\overline{V} + \gamma\varepsilon_t, \quad \varepsilon_1, \varepsilon_2, \ldots \sim N(0,1) \text{ i.i.d.}, \tag{3.5}$$

where the parameters are given by $\varphi = e^{-a}$, $\gamma = b\sqrt{\frac{1-e^{-2a}}{2a}}$. This follows from a comparison of the covariance function of the two Gaussian processes (cf. Shiryaev (1999), II.2b and III.3a). From a theoretical point of view, diffusion-type volatilities differ substantially from ARCH-type models because volatility is driven by a *separate* random process W. Nevertheless, the two kinds of models behave statistically similarly, which can be explained by limit theorems (cf. Nelson (1990)).

5. In the previous section we observed that, in addition to a slowly varying stochastic volatility, there seems to exist a very short-lived volatility component which is reflected by clusters of excessively large positive residuals in Figure 2. Such clusters are typical of ARCH(1)-models. In order to build a stochastic volatility model which incorporates both the "slow" component visible in Figure 1 as well as the rapidly decaying clusters in Figure 2, we add an ARCH(1)-term to the above AR(1)-model. More specifically, we assume that

$$\log \sigma_t^2 = V_t + \log \widetilde{\sigma}_t^2,$$

where V is a AR(1)-time series as in Equation (3.5) and $\widetilde{\sigma}_t$ meets

$$\widetilde{\sigma}_t^2 = c + a\widetilde{\sigma}_{t-1}^2(\Delta L_{t-1} - m)^2$$

with parameters $\widetilde{\sigma}_1, c, a, m := E(\Delta L_1)$. As above, we assume that V and L are independent. Note that c could always be normalized to 1 by changing $\overline{V}, \widetilde{\sigma}_1$ accordingly. Since m is a parameter of the distribution of L_1 and $\widetilde{\sigma}_1$ affects only the very beginning of the time series, the additional ARCH(1)-component yields essentially only one additional degree of

freedom. The degeneracy caused by c has no effect on our risk management approach, so we stick to the above notation. Of course, we could consider even more complex stochastic volatility models. However, given the limitation of data we feel that we have reached a limit at this point.

6. In an idealized Black-Scholes world, the volatility of the underlying coincides with implied volatilities from options data. This relation ceases to hold in real markets where the stock price is not given by a geometric Brownian motion. Implied volatility typically changes across strike prices (smile effect) and time to maturity. Nevertheless, we may suspect that there exists a close relationship between the current stock price volatility σ_t in the sense of Equation (3.3) and a reasonably standardized implied volatility (cf. Christensen and Prabhala (1998)). Such a standardized volatility is available for German stock index data: the volatility index VDAX. Consequently, we assume as a sixth model that the volatility σ_t is given by the current level of the VDAX. Note that we consider stock index data in this paper. If one is interested in particular stocks, one could use the corresponding stock option implied volatilities instead.

The driving Lévy process L

In principle, the stochastic volatility models above can be combined with any reasonable process with stationary, independent increments. In this paper, we focus on two particular cases. As a benchmark process we consider *Brownian motion*, or more precisely, a process of the form

$$L_t = mt + \upsilon B_t,$$

where $m \in \mathbb{R}$, $\upsilon \in \mathbb{R}_+$, and B denotes standard Brownian motion. By multiplying σ and m appropriately, one can always choose $\upsilon = 1$. However, it will turn out to be convenient to work with a general υ.

It is well-known that empirical distributions from stock return data deviate substantially from the normal distribution. Generalized hyperbolic distributions (cf. Eberlein and Prause (1998), Eberlein (2001)) or certain subclasses such as the hyperbolic (cf. Eberlein and Keller (1995), Eberlein et al. (1998)) or the normal inverse Gaussian (cf. Barndorff-Nielsen (1998)) turned out to be tailor-made for financial time series. In this paper, we focus on the class of *hyperbolic distributions* which is flexible enough to fit empirical data well. More specifically, we assume that L is a Lévy process such that the density of the law of L_1 is

$$f_{(\alpha,\beta,\delta,\mu)}(x) = \frac{\sqrt{\alpha^2-\beta^2}}{2\alpha\delta K_1(\delta\sqrt{\alpha^2-\beta^2})} e^{-\alpha\sqrt{\delta^2+(x-\mu)^2}+\beta(x-\mu)}.$$

Here, K_1 denotes the modified Bessel function of the third kind with index 1 (cf. Abramowitz and Stegun (1968)) and $\alpha > 0$, $\beta \in (-\alpha, \alpha)$, $\delta > 0$, $\mu \in \mathbb{R}$ the parameters. As in the Brownian motion case we allow some degeneracy due to the fact that the variance $\mathrm{Var}(L_1)$ is not necessarily normalized to 1.

Risk management

In risk management one is primarily interested in the distribution of future prices based on past observations. Due to the predominant use of VaR-based regulations, quantiles corresponding to one- or ten-day horizons are of particular interest. To obtain a reasonable forecast for the return distribution, we proceed as follows.

The series of prices $(S_t)_{t=0,1,2,\dots}$ is transformed into a series of *daily returns* by computing $\Delta X_t := \log S_t - \log S_{t-1}$. Suppose that we are currently at time t. In a first step, we have to estimate the volatility σ_s for $s = 1, \dots, t$. While this is trivial if we believe in implied volatility and if the latter is available, appropriate estimation procedures have to be performed in particular in GARCH- and exponential AR-type stochastic volatility models. This is explained in detail in the subsequent section. Write $(\widehat{\sigma}_s)_{s=1,\dots,t}$ for the estimated values. In addition, we need a prediction of the volatility σ_{t+1} for the subsequent day.

In a second step, we divide past stock returns $(\Delta X_s)_{s=1,\dots,t}$ by the volatility estimates $(\widehat{\sigma_s})_{s=1,\dots,t}$. The resulting time series $\Delta\widehat{L}_s := \frac{\Delta X_s}{\widehat{\sigma}_s}$, $s = 1, \dots, t$ serves as an estimate for the unobservable Lévy process increments $(\Delta L_s)_{s=1,\dots,t}$. We call $(\Delta\widehat{L}_s)_{s=1,\dots,t}$ *devolatilized returns*. We estimate the parameters of the Lévy process L by applying standard maximum likelihood methods to the devolatilized return data $(\Delta\widehat{L}_s)_{s=1,\dots,t}$.

If Q denotes the distribution of ΔL_1 according to the estimated parameters from the previous step and if $\widehat{\sigma}_{t+1}$ is the predicted volatility for tomorrow, then Q serves as a predicted distribution of $\frac{\Delta X_{t+1}}{\widehat{\sigma}_{t+1}}$, i.e. $A \mapsto Q(\widehat{\sigma}_{t+1}A)$ is the forecast of the law of tomorrow's return ΔX_{t+1}.

The careful reader observes that we estimate the parameters of the devolatilized return process $(\Delta\widehat{L}_s)_{s=1,\dots,t}$ rather than the hidden driving Lévy process $(\Delta L_s)_{s=1,\dots,t}$. But note that the focus in this paper is not on estimation but on forecasting the distribution of e.g. ΔX_{t+1}. Since we write this next-day return as $\Delta X_{t+1} = \widehat{\sigma}_{t+1}\Delta\widehat{L}_{t+1}$ instead of $\Delta X_{t+1} = \sigma_{t+1}\Delta L_{t+1}$, we need the distribution of $\Delta\widehat{L}_{t+1}$ rather than ΔL_{t+1}.

Similarly, we base the estimation of σ_s above only on the returns up to time s although the data is available up to time t. Once more, we are not trying to obtain σ_s or ΔL_s as precisely as possible. For prediction purposes it is more important that the devolatilized returns are constructed in the same way for all $s \in \{1, \dots, t\}$. Indeed, this makes it more likely that $(\Delta\widehat{L}_t)_{t=1,2,\dots}$ comes close to an i.i.d. sequence.

If we wish to forecast an n-day return distribution, the law Q of one-day increments $\Delta L_{t+1} = L_{t+1} - L_t$ has to be replaced with the distribution of an n-day increment $L_{t+n} - L_t$. Strictly speaking, the predicted volatility $\widehat{\sigma}_{t+1}$ has to be adjusted as well. But for short time intervals it may be justified to work with the same value unless volatility is fluctuating very rapidly. For longer time horizons, predicting a fixed volatility is inadequate anyway. Indeed, the random fluctuations of volatility affect the variance of $X_{t+n} - X_t$, which should be taken into account for risk management. This, however, will usually necessitate the use of Monte-Carlo methods which is beyond the scope of this paper.

In general, even the quite general model (3.3) provides just an approximation to real data. Not only the current volatility but also the shape of the return distribution will change

as observation time increases. Therefore, we prefer to base the estimation of the hyperbolic parameters typically on the previous 500 data points, which corresponds to approximately two years.

4 Estimation of the volatility σ

In this section, we show how to estimate the unobservable volatility σ. The subsections correspond to the enumeration of stochastic volatility models above.

Constant volatility

Since a deterministic, constant scale parameter can always be incorporated in the Lévy process, we need not perform any estimation and devolatilization at this stage. This shows why it is convenient not to insist on a normalized variance for the Lévy process L.

Slowly fluctuating volatility

Let us assume that the volatility process in Equation (3.3) is independent of L. (This assumption can be relaxed.) Passing to the logarithm simplyfies the estimation problem:

$$\log(\Delta X_t)^2 = \log \sigma_t^2 + \log(\Delta L_t)^2.$$

$t \mapsto \log \sigma_t^2$ is a comparatively slowly fluctuating (random) function and $(\log(\Delta L_t)^2)_{t=1,2,\ldots}$ a sequence of i.i.d. random variables. Since our problem is to estimate the hidden volatility signal $\log \sigma_t^2$ from observations that are perturbed by an additive white noise $\log(\Delta L_t)^2$, it suggests itself to apply non-parametric smoothing (cf. Hastie and Tibshirani (1990)). For our study we use a *running-mean smoother*, i.e. we let

$$\log \widehat{\sigma}_t^2 := \frac{1}{k} \sum_{i=0}^{k-1} \log(\Delta X_{t-i})^2,$$

where k denotes an appropriately chosen smoothing parameter. For the reasons explained in the previous subsection, we work with a backward window even though a more balanced window would improve the estimation. The smoothing parameter k is chosen by cross-validation, i.e. one minimizes

$$CV(k) := \frac{1}{T} \sum_{t=1}^{T} (\log(\Delta X_t)^2 - (\log \widehat{\sigma}_t^2)^{-t}),$$

where T is the total number of data points and $(\log \widehat{\sigma}_t^2)^{-t} := \frac{1}{k} \sum_{i=1}^{k} \log(\Delta X_{t-i})^2$ denotes the running mean smoother based on the sample with missing observation at time t. Applied to German stock index data, the optimal window length (within the set $\{5, 10, 15, \ldots, 75, 80\}$) turns out to be 40, which corresponds to approximately 2 trading months.

Note that the logarithmic squared volatility estimate $\log \widehat{\sigma}_t^2$ is biased by the expectation of $\log(\Delta L_t)^2$. E.g. we have $E(\log(\Delta L_t)^2) \approx -1.27$ in the case of standard Brownian motion; for other Lévy processes we get different values. But since any multiplicative constant can be put into the Lévy process, there is no need to compensate for this bias.

Finally, we address the question how to construct a one-period predictor for σ_{t+1}. Since we have not assumed any particular model for σ, it seems most reasonable to choose the previous value $\widehat{\sigma}_t$.

GARCH(1,1)-M volatility

Parameter estimation in GARCH-type models is commonly performed by maximum likelihood (ML) methods. Suppose that $f(\cdot, \vartheta)$ is the density of ΔL_1, where ϑ denotes a possibly multivariate unknown parameter. Similarly, let $x_t \mapsto l_t(x_t, (x_s)_{s=1,\dots,t-1}, \sigma_1, c, a, b, m, \vartheta)$ be the logarithm of the density of ΔX_t given $(\Delta X_1, \dots, \Delta X_{t-1}) = (x_1, \dots, x_{t-1})$ if the parameters of the GARCH(1,1)-M model and of the Lévy process are $\sigma_1, c, a, b, m, \vartheta$. Then the log-likelihood function given the data $(\Delta X_1, \dots, \Delta X_T) = (x_1, \dots, x_T)$ can be written as

$$(\sigma_1, c, a, b, m, \vartheta) \mapsto \sum_{t=1}^{T} l_t(x_t, (x_s)_{s=1,\dots,t-1}, \sigma_1, c, a, b, m, \vartheta).$$

Note that σ_t and ΔX_t can be computed recursively from $\Delta L_1, \dots, \Delta L_t$ if the parameters σ_1, c, a, b, m are known. Conversely, ΔL_t and σ_t are deterministic functions of $\Delta X_1, \dots, \Delta X_t$. Since $\Delta X_t = \sigma_t \Delta L_t$, we have that $l_t(x_t, (x_s)_{s=1,\dots,t-1}, \sigma_1, c, a, b, m, \vartheta) = \log f(\frac{x_t}{\sigma_t}, \vartheta) - \log \sigma_t$, where σ_t is determined recursively from Equations (3.4) and (3.3). Therefore, the log-likelihood function given the data equals

$$(\sigma_1, c, a, b, m, \vartheta) \mapsto \sum_{t=1}^{T} \log f\Big(\frac{x_t}{\sigma_t}, \vartheta\Big) - \log \sigma_t.$$

In our study we work with the S-PLUS software package S+GARCH, which is based on the normal distribution. It estimates the unknown parameters using the BHHH algorithm (cf. Martin et al. (1996)) and it also returns the corresponding values for the unobserved volatility σ_t. If we assume non-normal increments ΔL_t, then this procedure does not correspond to a ML estimator. Nevertheless, this quasi-likelihood approach still provides reasonable values for σ_1, c, a, b, m as is shown in Gouriéroux (1997), Chapter 4. For this reason and since GARCH routines for the normal distribution are widely implemented, we use the above package also in the hyperbolic case.

To be more precise, we estimate the current volatility σ_t by applying the above procedure to the previous 500 data points $\Delta X_{t-499}, \dots, \Delta X_t$. This ensures that the algorithm leading to the estimated value $\widehat{\sigma}_t$ is the same for any t, relies on fairly actual data, but does not look furtively into the future.

For risk management, we also need a prediction of the volatility σ_{t+1} for the subsequent day given the previous data. Via Equations (3.3) and (3.4), such a prediction is immediately obtained from ΔX_t and the estimated values for σ_t, c, a, b, m.

Volatility of exponential AR(1)-type

Following Harvey et al. (1994), we apply a quasi-maximum likelihood method to estimate the parameters. The asymptotic and finite-sample properties of the corresponding estimator are studied in Ruiz (1994). For an overview of many alternative approaches to parameter estimation in this stochastic volatility model cf. Andersen et al. (1999).

Denote by $y_t \mapsto l_t(y_t, (y_s)_{s=1,\dots,t-1}, \overline{V}, \varphi, \gamma, \vartheta)$ the logarithm of the density of $\log(\Delta X_t)^2$ given the data $(\log(\Delta X_1)^2, \dots, \log(\Delta X_{t-1})^2) = (y_1, \dots, y_{t-1})$ if the underlying unknown parameters are $\overline{V}, \varphi, \gamma$ for the AR(1) model and ϑ (possibly multivariate) for the Lévy process L. Then the log-likelihood function given the data $(\log(\Delta X_1)^2, \dots, \log(\Delta X_T)^2) = (y_1, \dots, y_T)$ can be written as

$$(\overline{V}, \varphi, \gamma, \vartheta) \mapsto \sum_{t=1}^{T} l_t(y_t, (y_s)_{s=1,\dots,t-1}, \overline{V}, \varphi, \gamma, \vartheta).$$

In general it is hard to determine $l_t(y_t, (y_s)_{s=1,\dots,t-1}, \overline{V}, \varphi, \gamma, \vartheta)$ explicitly. A convenient way out has been proposed in Nelson (1988) and Harvey et al. (1994): Let us (wrongly) assume that $\log(\Delta L_t)^2$ is normally distributed with mean 0 and variance θ. Note that by multiplying L with an appropriate constant, we may suppose that $E(\log(\Delta L_t)^2) = 0$. Therefore, the simplifying assumption concerns only the shape of the distribution. Since $(V_t)_{t=1,2,\dots}$ is a Gaussian time series which is independent of the Gaussian series $(\log(\Delta L_t)^2)_{t=1,2,\dots}$, we have that the random variables $\log(\Delta X_1)^2, \dots, \log(\Delta X_t)^2$ are jointly normal as well. This implies that the conditional law of $\log(\Delta X_t)^2$ given $(\log(\Delta X_1)^2, \dots, \log(\Delta X_{t-1})^2) = (y_1, \dots, y_{t-1})$ is normal. Since $\log(\Delta X_t)^2 = V_t + \log(\Delta L_t)^2$, we have that

$$E_{(\overline{V},\varphi,\gamma,\vartheta)}(\log(\Delta X_t)^2 \,|(\log(\Delta X_1)^2, \dots, \log(\Delta X_{t-1})^2) = (y_1, \dots, y_{t-1})) = \widehat{V}_{t|t-1},$$

$$\mathrm{Var}_{(\overline{V},\varphi,\gamma,\vartheta)}(\log(\Delta X_t)^2 \,|\, \dots) = v_{t|t-1} + \theta,$$

where $\widehat{V}_{t|t-1} := E_{(\overline{V},\varphi,\gamma,\vartheta)}(V_t| \dots)$ denotes the Kalman-Bucy filter and $v_{t|t-1} := \mathrm{Var}_{(\overline{V},\varphi,\gamma,\vartheta)}$ $(V_t| \dots)$ its prediction error variance. Based on the normal assumption, they meet the recursive equations

$$\begin{aligned} \widehat{V}_{t+1|t} &= \varphi \widehat{V}_{t|t-1} + (1-\varphi)\overline{V} + \frac{\varphi v_{t|t-1}}{\theta + v_{t|t-1}}(y_t - \widehat{V}_{t|t-1}) \\ v_{t+1|t} &= \varphi^2 v_{t|t-1} + \gamma^2 - \frac{\varphi^2 v_{t|t-1}^2}{\theta + v_{t|t-1}} \end{aligned}$$

with $\widehat{V}_{1|0} = \overline{V}$ and $v_{1|0} = \frac{\gamma^2}{1-\varphi^2}$ (cf. Shiryaev (1995), Theorem VI.7.1 or Harvey (1989), Section 3.2). In particular, the prediction error variance does not depend on the observations (cf. Shiryaev (1995), Corollary VI.7.1).

Together, we have that the log-likelihood given the data $(\log(\Delta X_1)^2, \dots, \log(\Delta X_T)^2) = (y_1, \dots, y_T)$ equals

$$(\overline{V}, \varphi, \gamma, \vartheta) \mapsto -\frac{1}{2} \sum_{t=1}^{T} \left(\log\left(2\pi(v_{t|t-1} + \theta)\right) + \frac{(y_t - \widehat{V}_{t|t-1})^2}{v_{t|t-1} + \theta} \right),$$

	$\overline{V}$	φ	γ	θ
True parameter	0.986	0.120	-10.42	4.88
Median	0.980	0.130	-10.41	4.86
First quartile	0.969	0.104	-10.61	4.70
Third quartile	0.987	0.159	-10.23	5.02

Table 1: Results from 1000 simulations with normally distributed $\log(\Delta L_t)^2$

	$\overline{V}$	φ	γ	θ
True parameter	0.986	0.120	-10.42	4.93
Median	0.980	0.130	-10.40	4.87
First quartile	0.967	0.102	-10.58	4.61
Third quartile	0.987	0.164	-10.23	5.17

Table 2: Results from 1000 simulations with normally distributed ΔL_t

where $(\widehat{V}_{t|t-1})_{t=1,\ldots,T}$ and $(v_{t|t-1})_{t=1,\ldots,T}$ are given recursively as above. Hence, we will use the maximum of this function as a quasi-maximum likelihood estimator for $\overline{V}, \varphi, \gamma, \theta$. For a more thorough account of Kalman filtering and parameter estimation cf. Harvey (1989), Shiryaev (1995).

If we believe in an AR(1)-stochastic volatility, we need a lot of data for parameter estimation because the hidden AR(1) process is perturbed by heavy white noise which carries no information (cf. Figure 1). In our study we use a 3.5 year interval prior to the period that is considered for backtesting. In contrast to the GARCH case before, the estimated parameters are kept fixed.

For risk management, we need an estimate of the current volatility $\sigma_t = e^{\frac{1}{2}V_t}$ for the backtesting period. This is obtained by applying a Kalman filter to the corresponding data, i.e. we use $e^{\frac{1}{2}\widehat{V}_t}$ with

$$\begin{aligned}\widehat{V}_t &:= E\Big(V_t \Big| (\log(\Delta X_1)^2, \ldots, \log(\Delta X_t)^2) = (y_1, \ldots, y_t)\Big) \\ &= \widehat{V}_{t|t-1} + \frac{v_{t|t-1}}{v_{t|t-1} + \theta}(y_t - \widehat{V}_{t|t-1})\end{aligned}$$

(cf. Harvey (1989), Section 3.2). We also need one-period predictions for $\sigma_{t+1} = e^{\frac{1}{2}V_{t+1}}$ given the data up to t. These are provided by $e^{\frac{1}{2}\widehat{V}_{t+1|t}}$, where $\widehat{V}_{t+1|t}$ denotes once more the Kalman filter based on the normal assumption with parameters estimated from the initial 3.5 year interval.

To assess the reliability of the estimation, we run two simulation studies. Firstly, we consider normally distributed $\log(\Delta L_t)^2$, in which case the above procedure corresponds to a real maximum likelihood estimation. In a second study, we assume that ΔL_t is normally distributed with mean 0, which makes more sense and corresponds to model (3.3) for Brownian motion L. The parameters for the simulation are taken from an estimation on German stock index data (DAX) in the period from July 1, 1988 to December 30, 1991 (868 trading days). The results from 1000 simulations can be found in Tables 1 and 2.

Composite volatility

Statistically well-founded estimation in this complex model is beyond the scope of this paper. Therefore we suggest the following simple procedure: Guided by Figures 1 and 2 we expect the AR(1) term to be the dominating component. In a first step, we determine its parameters $\overline{V}, \varphi, \gamma, \theta$ as if no ARCH(1) component were present (cf. the previous subsection). We estimate the volatilities $e^{\frac{1}{2}V_t}$ as before by Kalman filtering and compute the corresponding partially devolatilized returns $\Delta\check{L}_t := \frac{\Delta X_t}{e^{\frac{1}{2}\widehat{V}_t}}$. Secondly, we estimate the ARCH(1) parameters as explained above, but now from the time series $(\Delta\check{L}_t)_{t=1,2,\ldots}$ instead of $(\Delta X_t)_{t=1,2,\ldots}$ and under the restriction $b = 0$ in Equation (3.4). Both the parameters of the AR(1) and the ARCH(1) component are estimated from a 3.5 year interval prior to the backtesting period.

Given the parameters, we obtain estimated volatilities $e^{\frac{1}{2}\widehat{V}_t}$ and $\widehat{\widetilde{\sigma}}_t$ and corresponding devolatilized residuals $\Delta\widehat{L}_t := \frac{\Delta\check{L}_t}{\widehat{\widetilde{\sigma}}_t} = \frac{\Delta X_t}{e^{\frac{1}{2}\widehat{V}_t}\widehat{\widetilde{\sigma}}_t}$ for the backtesting period. For risk management we also need a prediction of the volatility $\sigma_{t+1} = e^{\frac{1}{2}V_{t+1}}\widetilde{\sigma}_{t+1}$ based on the data up to time t. It is obtained by multiplying the one-step predictions from the AR(1) part and the ARCH(1) component, respectively.

Implied volatility

If we assume that stock volatility coincides with the implied volatility index VDAX, we clearly choose the VDAX itself as estimator $\widehat{\sigma}$. As for slowly fluctuating volatility, we do not claim any particular model for the dynamics of σ. Therefore, we take $\widehat{\sigma}_t$ also as a one-period predictor for σ_{t+1}.

5 Are the devolatilized residuals independent?

In the previous section we discussed a number of methods to devolatilize observed stock index returns. For application in risk management one would like these devolatilized returns $(\Delta\widehat{L}_t)_{t=1,2,\ldots}$ to be independent, identically distributed (i.i.d.) random variables. We want to investigate to what extent this is justified. As mentioned in the introduction, the procedures below are performed on German stock index data (DAX) from January 2, 1992 to June 29, 1999. As one of the so-called stylized facts it has been repeatedly observed that squared stock returns are positively autocorrelated and hence not independent. In Figure 4, we plot the empirical autocorrelation function of the squared devolatilized returns $(\Delta\widehat{L}_t^2)_{t=1,\ldots,T}$ corresponding to the various models from Section 3. The dotted lines represent pointwise asymptotic 95% confidence bounds under the assumption that the series $(\Delta\widehat{L}^2)_{t=1,\ldots,T}$ is i.i.d. One can observe that any of the five non-trivial methods reduces the autocorrelation of squared returns substantially. A closer look reveals that the GARCH approach yields the best results, whereas nonparametric volatility performs not as well as the other models.

A formal test for the i.i.d. property of a time series $Y_1, \ldots, Y_T$, the so-called *BDS-test*,

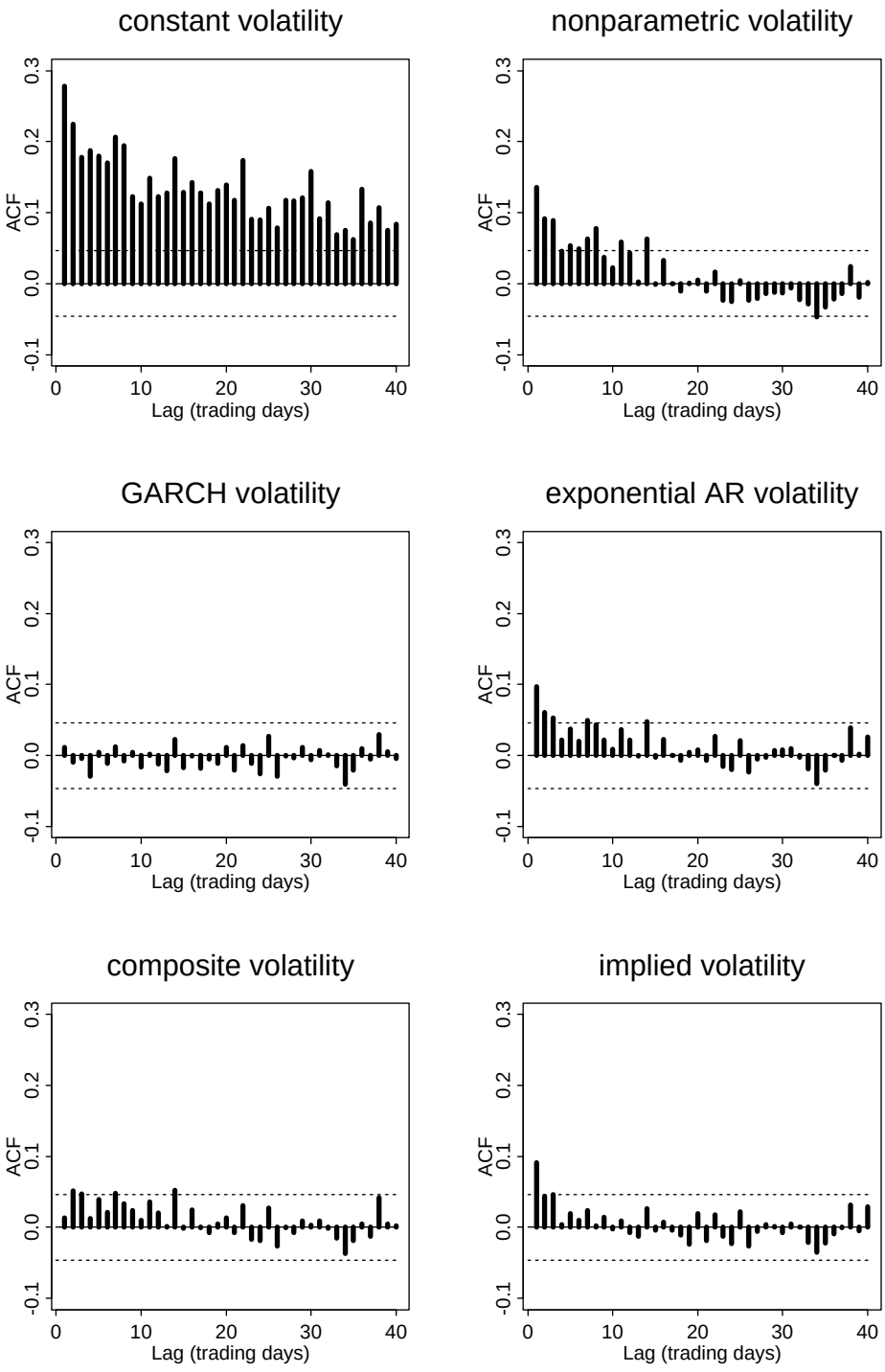

Figure 4: Autocorrelation functions from squared devolatilized returns

volatility model	BDS statistic	p-value	iid hypothesis
constant	15.20	0.00%	rejected
nonparametric	5.98	0.00%	rejected
GARCH	-0.78	43.47%	not rejected
exponential AR	2.66	0.79%	rejected
composite	0.54	58.58%	not rejected
implied	1.41	15.75%	not rejected

Table 3: BDS test on independent devolatilized returns

goes back to Brock et al. (1987). It reacts sensitively to accumulations of similar values anywhere in the time series. For fixed parameters $m \in \mathbb{N}$, $\varepsilon > 0$ the *BDS statistic* is defined as

$$W_{m,T}(\varepsilon) := \sqrt{T}\frac{C_{m,T}(\varepsilon) - (C_{1,T}(\varepsilon))^m}{\sigma_{m,T}(\varepsilon)},$$

where $T_m := T - m + 1$, moreover $1_\varepsilon(s,t) := 1_{[-\varepsilon,\varepsilon]}(\max_{i\in\{0,\dots,m-1\}} |Y_{t+i} - Y_{s+i}|)$,

$$C_{m,T}(\varepsilon) := \sum_{1\le t<s\le T_m} 1_\varepsilon(s,t)\frac{2}{T_m(T_m-1)}$$

(and accordingly for 1 instead of m),

$$K_T(\varepsilon) := \sum_{1\le t<s<r\le T_m} \frac{2(1_\varepsilon(t,s)1_\varepsilon(s,r) + 1_\varepsilon(t,r)1_\varepsilon(r,s) + 1_\varepsilon(s,t)1_\varepsilon(t,r))}{T_m(T_m-1)(T_m-2)},$$

$$\begin{aligned}\sigma^2_{m,T}(\varepsilon) \;:=\; & 4\Bigg(K_T(\varepsilon)^m + 2\sum_{j=1}^{m-1} K_T(\varepsilon)^{m-j}C_{1,T}(\varepsilon)^{2j} \\ & + (m-1)^2 C_{1,T}(\varepsilon)^{2m} - m^2 K_T(\varepsilon)C_{1,T}(\varepsilon)^{2m-2}\Bigg).\end{aligned}$$

Under the null hypothesis this statistic is asymptotically standard normally distributed. Following rules of thumb based on simulations in Brock et al. (1991), we choose the parameters $m = 4$ and $\varepsilon = 1.5\sqrt{s_L^2}$, where s_L^2 denotes the empirical variance of $(\Delta\widehat{L}_t)_{t=1,\dots,T}$. For explicit computations, a C program by B. LeBaron has been used. Table 3 summarizes the results of the BDS-test at a level of 5% applied to $(\Delta\widehat{L}_t)_{t=1,\dots,T}$. For GARCH, composite, and implied volatility the null hypothesis is not rejected. Once more, one can observe that the assumption of constant volatility produces inacceptable results.

In risk management applications we do not want to rely too heavily on the assumption that devolatilized residuals are identically distributed. Therefore we base parameter estimation for the corresponding hyperbolic (resp. normal) distribution only on the previous 500 data points. This compromises between stationarity of devolatilized returns and disposing of enough data for parameter estimation. From Figure 5 one can get an impression how much the shape of the estimated hyperbolic distribution changes through time. For comparison we

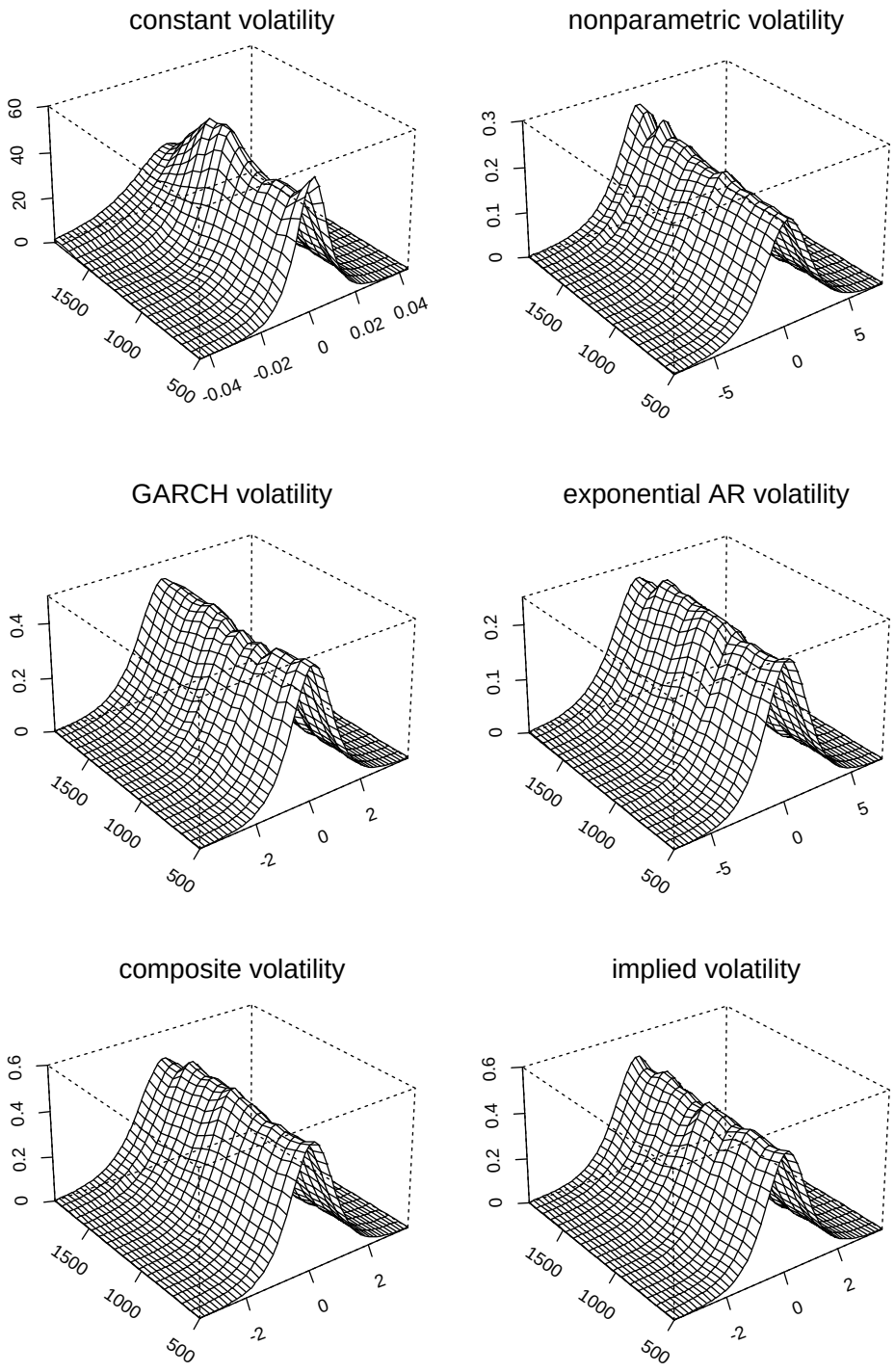

Figure 5: Sequential estimation of hyperbolic densities from devolatilized returns

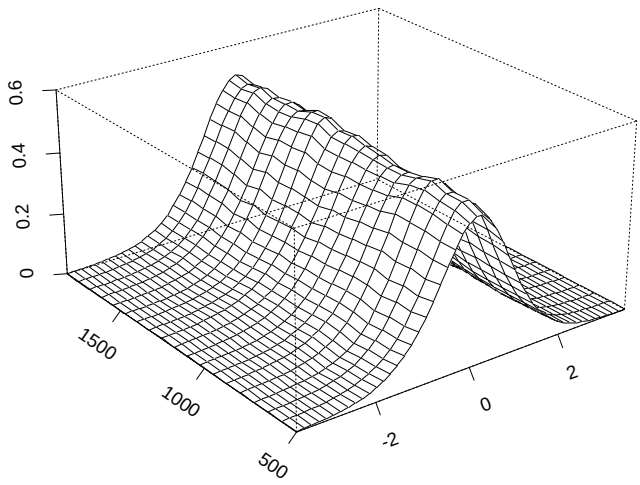

Figure 6: Sequential estimation of hyperbolic densities from a simulated hyperbolic sample

plot the estimated hyperbolic densities from a simulation of 1875 i.i.d. hyperbolic random variables in Figure 6. As in Figure 5, the parameter estimation is based only on the respective previous 500 data points. One clearly sees that devolatilization enhances the stationarity of the fitted hyperbolic distribution. Once more, nonparametric volatility is outperformed by the other approaches.

6 Backtesting

How well do stochastic volatility models perform in risk management applications? Any of the approaches in Section 3 provides a forecast of tomorrow's return distribution given the past return data $(\Delta X_1, \ldots, \Delta X_t)$. We will now assess the quality of this prediction in a number of ways, considering again the DAX data. Since the first 500 data points are only used for parameter estimation, the backtesting procedures are run on the remaining 1375 observations.

Because of the legal obligation to use VaR-based risk management, quantiles are of particular interest. In our study, we consider the 97.5% and 99% levels of one-day value-at-risk corresponding to an investment in the DAX. Ideally the *frequency of excessive losses*, i.e. of days where the loss exceeds the predicted VaR level, should be close to 2.5% resp. 1%. Following Kupiec (1995), we apply a likelihood ratio test (*FOEL test*) at a level of 5% to examine whether the observed frequency deviates substantially from the predicted level. Under the null hypothesis, the number of excessive losses follows a binomial distribution with parameters T and p_0, where $T = 1375$ denotes the total number of days in the backtesting period and $1 - p_0$ is the predicted level of VaR (i.e. 97.5% or 99%). Define

$$R(f, T, p_0) := -2\log((1-p_0)^{T-f} p_0^f) + 2\log\left(\left(1-\frac{f}{T}\right)^{T-f}\left(\frac{f}{T}\right)^{f}\right),$$

where f is the number of days in the sample where the loss exceeds the corresponding pre-

volatility	VaR level	FOEL	95% confidence bounds	R	p-value	VaR level hypothesis
constant	97.5%	4.7%	[3.6%, 5.9%]	22.27	0.00%	rejected
nonparametric	97.5%	3.4%	[2.4%, 4.3%]	3.65	5.60%	not rejected
GARCH	97.5%	3.5%	[2.5%, 4.5%]	4.94	2.62%	rejected
exponential AR	97.5%	4.1%	[3.0%, 5.1%]	11.76	0.06%	rejected
composite	97.5%	4.0%	[3.0%, 5.0%]	10.77	0.10%	rejected
implied	97.5%	2.9%	[2.0%, 3.8%]	0.90	34.34%	not rejected
constant	99%	3.1%	[2.2%, 4.1%]	40.19	0.00%	rejected
nonparametric	99%	2.2%	[1.4%, 3.0%]	14.50	0.01%	rejected
GARCH	99%	2.3%	[1.5%, 3.0%]	16.12	0.01%	rejected
exponential AR	99%	2.4%	[1.6%, 3.2%]	19.55	0.00%	rejected
composite	99%	2.3%	[1.5%, 3.1%]	17.81	0.00%	rejected
implied	99%	1.6%	[0.9%, 2.3%]	4.23	3.97%	rejected

Table 4: FOEL test if devolatilized returns are assumed to be normally distributed

volatility	VaR level	FOEL	95% confidence bounds	R	p-value	VaR level hypothesis
constant	97.5%	3.9%	[2.8%, 4.9%]	8.90	0.28%	rejected
nonparametric	97.5%	2.6%	[1.7%, 3.4%]	0.01	91.43%	not rejected
GARCH	97.5%	2.7%	[1.8%, 3.6%]	0.20	65.42%	not rejected
exponential AR	97.5%	3.0%	[2.1%, 3.9%]	1.23	26.65%	not rejected
composite	97.5%	2.8%	[2.0%, 3.7%]	0.61	43.40%	not rejected
implied	97.5%	2.2%	[1.4%, 3.0%]	0.60	44.00%	not rejected
constant	99%	2.2%	[1.4%, 3.0%]	14.50	0.01%	rejected
nonparametric	99%	1.5%	[0.8%, 2.1%]	2.52	11.27%	not rejected
GARCH	99%	1.2%	[0.7%, 1.8%]	0.72	39.56%	not rejected
exponential AR	99%	1.6%	[0.9%, 2.3%]	4.23	3.97%	rejected
composite	99%	1.8%	[1.1%, 2.5%]	7.49	0.62%	rejected
implied	99%	1.0%	[0.4%, 1.5%]	0.04	83.75%	not rejected

Table 5: FOEL test if devolatilized returns are assumed to be hyperbolically distributed

dicted quantile. Under the null hypothesis, this statistic is asymptotically $\chi^2(1)$-distributed. Tables 4 and 5 summarize the results for the stochastic volatility models from Section 3 combined with the normal resp. hyperbolic distribution for devolatilized returns.

On the whole, the normal distribution fails to provide acceptable results, especially on the 99% level. Combined with the hyperbolic distribution, the devolatilization methods produce mostly reasonable results – in particular compared to constant volatility. Among the stochastic volatility models , the implied, GARCH, and nonparametric approaches perform better than exponential AR and composite volatility.

Let us examine more closely how the predicted value-at-risk evolves through time. Figures 7 and 8 show the predicted VaR, the actual losses (if a loss occured at all), and the times of excessive losses for the 99% level of daily VaR. Because of its superiority, we focus on the hyperbolic case. One can observe that the predicted value-at-risk fluctuates rapidly in the GARCH case. This may create problems for risk management since the portfolio has to be rebalanced more frequently. Note further that the VaR changes substantially even in the case of constant volatility, which seems to contradict the underlying assumption of i.i.d. returns. This phenomenon stems from the fact that the estimation of the Lévy process parameters is based only on the previous 500 data points. Without this precaution, the constant volatility model would fail even more severely. On the other hand, this effect will be less pronounced in calmer periods.

So far, we have concentrated on particular extreme quantiles. But it is also possible to test the distribution as a whole. The idea is to consider the time series $(U_t)_{t=1,\dots,T} := (\widehat{F}_t(\Delta X_t))_{t=1,\dots,T}$, where $\widehat{F}_t$ denotes the predicted return distribution function for day t based on the observations up to time $t-1$. Under the null hypothesis that $\widehat{F}_t$ coincides with the law of ΔX_t given the past observations, we have that $(U_t)_{t=1,\dots,T}$ is an i.i.d. sequence of random variables that are uniformly distributed on $[0,1]$. Following Crncovic and Drachman (1996), we compute the *Kuiper statistic* to test this hypothesis. It is defined as

$$K := \max_{x\in[0,1]}(F_{\text{emp}}(x) - x) + \max_{x\in[0,1]}(x - F_{\text{emp}}(x)),$$

where F_{emp} denotes the empirical distribution function of $(U_t)_{t=1,\dots,T}$. The p-value of the *Kuiper test* based on this statistic is given asymptotically by

$$2\sum_{j=1}^{\infty}(4j^2\lambda^2 - 1)\exp(-2j^2\lambda^2) \text{ with } \lambda := \left(\sqrt{T} + 0.155 + \frac{0.24}{\sqrt{T}}\right)K$$

(cf. Press et al. (1992), Stephens (1970)). Similarly as the Kolmogorov-Smirnov test, this test assesses whether the empirical distribution function deviates significantly from a given (here: uniform) law. We use the Kuiper test because it is more sensitive to the tails that are particularly important for risk management. The test results on a 5% level can be found in Table 6. As for the FOEL test, constant volatility and the normal distribution are mostly rejected, whereas any of the real stochastic volatility models seems to perform well in conjunction with the hyperbolic distribution.

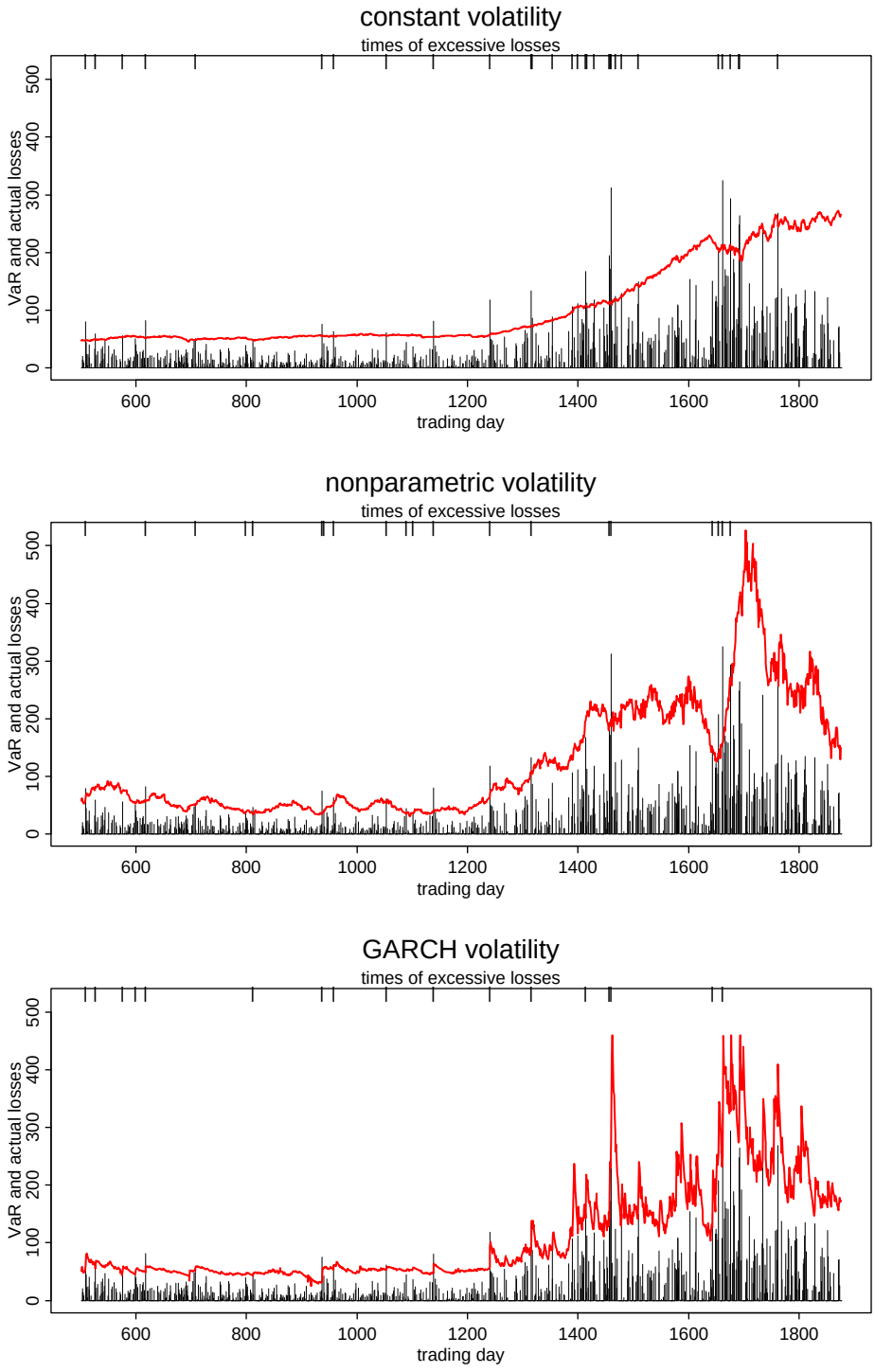

Figure 7: VaR predictions and actually occured losses

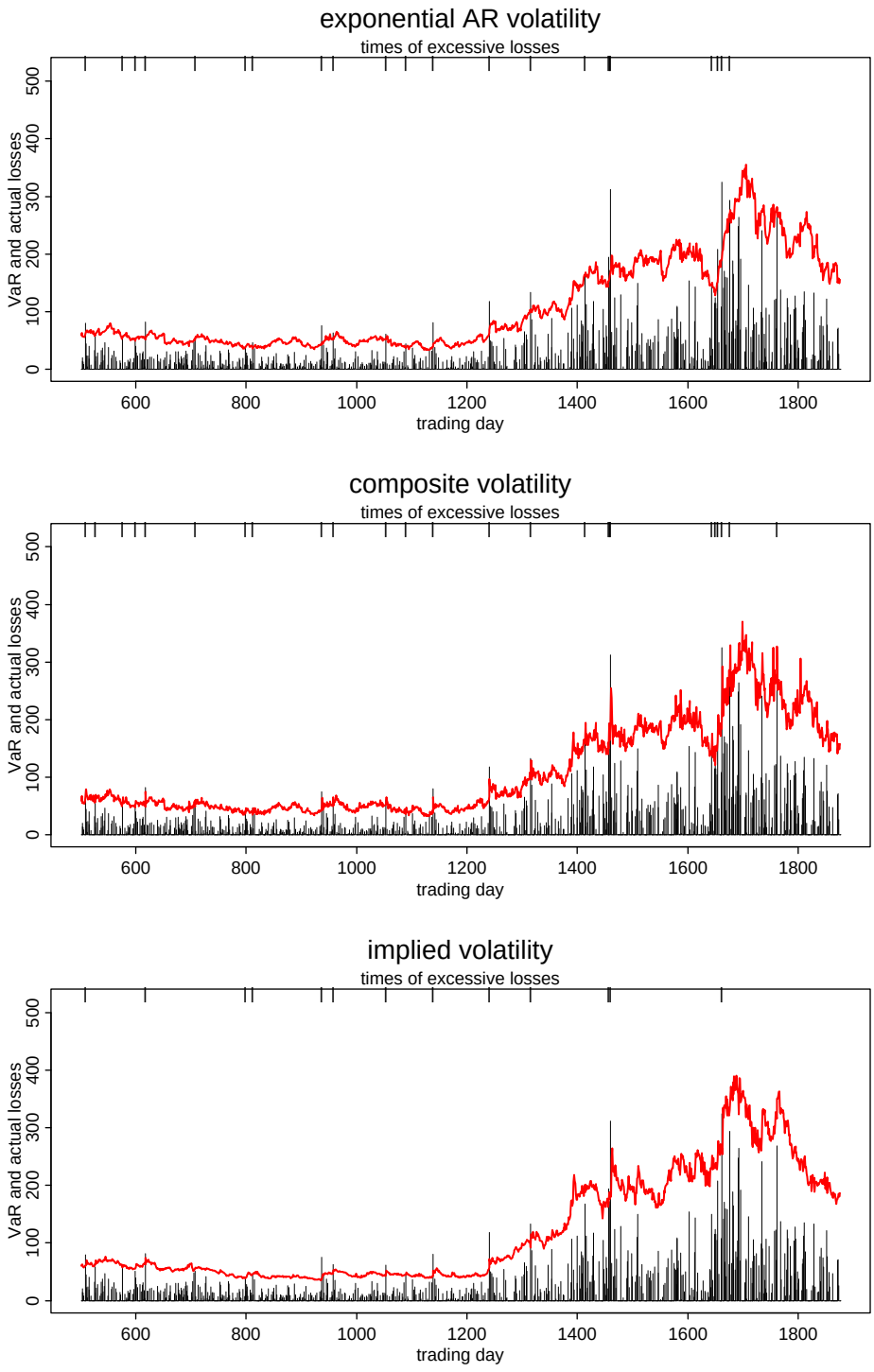

Figure 8: VaR predictions and actually occured losses

normal devolatilized returns			
volatility model	Kuiper statistic	p-value	null hypothesis
constant	0.0563	0.50%	rejected
nonparametric	0.0744	0.00%	rejected
GARCH	0.0468	5.11%	not rejected
exponential AR	0.0479	4.06%	rejected
composite	0.0486	3.45%	rejected
implied	0.0563	0.47%	rejected
hyperbolic devolatilized returns			
volatility model	Kuiper statistic	p-value	null hypothesis
constant	0.0670	0.02%	rejected
nonparametric	0.0285	74.14%	not rejected
GARCH	0.0318	55.81%	not rejected
exponential AR	0.0405	17.24%	not rejected
composite	0.0362	32.96%	not rejected
implied	0.0290	71.70%	not rejected

Table 6: Kuiper test of the predicted return distribution

normal devolatilized returns			
volatility model	BDS statistic	p-value	iid hypothesis
constant	3.87	0.01%	rejected
nonparametric	1.23	21.99%	not rejected
GARCH	-1.24	21.67%	not rejected
exponential AR	-0.42	67.58%	not rejected
composite	-1.79	7.29%	not rejected
implied	-0.82	41.48%	not rejected
hyperbolic devolatilized returns			
volatility model	BDS statistic	p-value	iid hypothesis
constant	3.69	0.02%	rejected
nonparametric	0.33	73.81%	not rejected
GARCH	-1.70	8.88%	not rejected
exponential AR	-0.84	40.00%	not rejected
composite	-2.19	2.88%	rejected
implied	-1.14	25.40%	not rejected

Table 7: BDS test on independent transformed returns

	normal		hyperbolic	
volatility model	excessive losses	zone	excessive losses	zone
constant	43	red	30	red
nonparametric	30	red	20	yellow
GARCH	31	red	17	green
exponential AR	33	red	22	yellow
composite	32	red	25	yellow
implied	22	yellow	13	green

Table 8: Classification according to the Basel rules

So far we have focused on the distribution of returns. But note that the transformed returns $(U_t)_{t=1,\ldots,T}$ are also independent under the null hypothesis. We use the BDS statistic to test this hypothesis (cf. Section 5). The results of the BDS test at a level of 5% are listed in Table 7. Apart from constant volatility, only the composite model fails in conjunction with hyperbolic returns.

For a financial institution the ultimate touchstone of a model is its approval by the supervising authorities. On the other hand approval is not the only point. According to the traffic light concept of the Basel Committee on Banking Supervision, internal models are classified. This classification into the green, yellow, or red zone depends on how often the actual losses exceed the daily VaR predictions on the 99% level over a period of T trading days. Depending on this classification, the necessary capital reserves are assigned. *Green* means that the minimum factor of 3 is applied to the VaR value, *yellow* results in a higher (*add-on*) factor between 3 and 4, whereas *red* normally means rejection of the model. Our backtesting period covers $T = 1375$ trading days, which exceeds the 250 days that are typically used in practice. The results of a hypothetical classification are listed in Table 8. Once more, the combination of stochastic volatility with hyperbolic devolatilized returns yields the most reliable setup. Among the various models, implied and GARCH volatility perform superior to the other devolatilization approaches.

7 Conclusion

Our study shows that randomly fluctuating volatility can and should be considered for risk management in practice. By applying adequate (quasi-)likelihood resp. non-parametric methods or by using implied volatility from option data, only moderate computing power is needed to predict the short-term risk profile according to a number of models. It turns out that both stochastic volatility and more flexible return distributions have to be taken into account in order to produce accurate predictions. Neglecting one of the two factors leads to a substantial loss of precision. Among the stochastic volatility models, GARCH and implied volatility seem to perform better then the nonparametric, exponential AR, or the composite approach. In spite of erratic market movements in the backtesting period, the models under

consideration produced excellent forecasts in combination with the hyperbolic distribution. The availability of option prices, the computational effort, and the day-to-day variability of value-at-risk predictions may ultimately decide which devolatilization procedure to choose.

References

Abramowitz, M. and I. Stegun (1968). *Handbook of Mathematical Functions*. New York: Dover.

Andersen, T., H. Chung, and B. Sørensen (1999). Efficient method of moments estimation of a stochastic volatility model: A Monte Carlo study. *Journal of Econometrics 91*, 61–87.

Barndorff-Nielsen, O. (1998). Processes of normal inverse Gaussian type. *Finance & Stochastics 2*, 41–68.

Barndorff-Nielsen, O. and N. Shephard (1998). Aggregation and model construction for volatility models. Technical Report 10, Centre for Analytical Finance, Århus Universitet.

Barone-Adesi, G., F. Bourgoin, and K. Giannopoulos (1998). Don't look back. *Risk 11*, 100–104.

Barone-Adesi, G., K. Giannopoulos, and L. Vosper (1999). VaR without correlations for non-linear portfolios. *Journal of Futures Markets 19*, 583–602.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics 31*, 307–327.

Bollerslev, T., R. Chou, and K. Kroner (1992). ARCH modeling in finance; a review of the theory and empirical evidence. *Journal of Econometrics 52*, 5–59.

Brock, W., W. Dechert, and J. Scheinkman (1987). A test for independence based on the correlation dimension. Working Paper University of Wisconsin at Madison.

Brock, W., D. Hsieh, and B. LeBaron (1991). *Nonlinear Dynamics, Chaos, and Instability: Statistical Theory and Economic Evidence*. Cambridge: MIT Press.

Chesney, M. and L. Scott (1989). Pricing European currency options: A comparison of the modified Black-Scholes model and a random variance model. *Journal of Financial and Quantitative Analysis 24*, 267–284.

Christensen, B. and N. Prabhala (1998). The relation between implied and realized volatility. Technical Report 14, Centre for Analytical Finance, Århus Universitet.

Crncovic, C. and J. Drachman (1996). Quality control. *Risk 9*, 138ff.

Eberlein, E. (2001). Application of generalized hyperbolic Lévy motions to finance. In O. Barndorff-Nielsen, T. Mikosch, and S. Resnick (Eds.), *Lévy Processes: Theory and Applications*. Basel: Birkhäuser.

Eberlein, E. and U. Keller (1995). Hyperbolic distributions in finance. *Bernoulli 1*, 281–299.

Eberlein, E., U. Keller, and K. Prause (1998). New insights into smile, mispricing and value at risk: The hyperbolic model. *Journal of Business 71*, 371–406.

Eberlein, E. and K. Prause (1998). The generalized hyperbolic model: Financial derivatives and risk measures. FDM Preprint 56, University of Freiburg.

Engle, R. (1982). Autoregressive conditional heteroskedasticity with estimates of the variance of U.K. inflation. *Econometrica 50*, 987–1008.

Gouriéroux, C. (1997). *ARCH Models and Financial Applications*. New York: Springer.

Harvey, A. (1989). *Forecasting, Structural Time Series Models and the Kalman Filter*. Cambridge: Cambridge University Press.

Harvey, A., E. Ruiz, and N. Shephard (1994). Multivariate stochastic variance models. *Review of Economic Studies 61*, 247–264.

Hastie, T. and R. Tibshirani (1990). *Generalized Additive Models*. London: Chapman & Hall.

Heston, S. (1993). A closed-form solution for options with stochastic volatilities with applications to bond and currency options. *The Review of Financial Studies 6*, 327–343.

Hull, J. and A. White (1987). The pricing of options on assets with stochastic volatilities. *Journal of Finance 42*, 281–300.

Hull, J. and A. White (1998). Incorporating volatility updating into the historical simulation method for value-at-risk. *The Journal of Risk 1*, 5–19.

Jorion, P. (1997). *Value at Risk*. New York: McGraw-Hill.

Kallsen, J. and M. Taqqu (1998). Option pricing in ARCH-type models. *Mathematical Finance 8*, 13–26.

Keller, U. (1997). *Realistic Modelling of Financial Derivatives*. Dissertation Universität Freiburg i. Br.

Kupiec, P. (1995). Techniques for verifying the accuracy of risk management models. *The Journal of Derivatives 3*, 73–84.

Martin, R., H.-Y. Gao, Y. Zhan, and Z. Ding (1996). *S+GARCH User's Manual*. Seattle: MathSoft Inc.

McNeil, A. and R. Frey (2001). Estimation of tail-related risk measures for heteroscedastic financial time series: An extreme value approach. *Journal of Empirical Finance*, forthcoming.

Nelson, D. (1988). *The Time Series Behavior of Stock Market Volatility and Returns*. Cambridge: Ph.D. dissertation (Massachusetts Institute of Technology).

Nelson, D. (1990). ARCH models as diffusion approximations. *Journal of Econometrics 45*, 7–38.

Press, W., S. Teukolsky, W. Vetterling, and B. Flannery (1992). *Numerical Recipes in C* (second ed.). Cambridge: Cambridge University Press.

Ruiz, E. (1994). Quasi-maximum likelihood estimation of stochastic volatility models. *Journal of Econometrics 63*, 289–306.

Shephard, N. (1996). Statistical aspects of ARCH and stochastic volatility. In D. Cox, D. Hinkley, and O. Barndorff-Nielsen (Eds.), *Time Series Models*, pp. 1–67. London: Chapman and Hall.

Shiryaev, A. (1995). *Probability* (second ed.). New York: Springer.

Shiryaev, A. (1999). *Essentials of Stochastic Finance*. Singapore: World Scientific.

Stein, E. and J. Stein (1991). Stock price distributions with stochastic volatility: An analytic approach. *The Review of Financial Studies 4*, 727–752.

Stephens, M. (1970). Use of the Kolmogorov-Smirnov, Cramér-von-Mises and related statistics without extensive tables. *Journal of the Royal Statistical Society, Series B 32*, 115–122.