THE TERM STRUCTURE OF INTEREST RATES: BOUNDED OR FALLING?

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Abstract. This short paper resolves an apparent contradiction between Feldman's (1989) and Riedel's (2000) equilibrium models of the term structure of interest rates under incomplete information. Feldman (1989) showed that in an incomplete information version of Cox, Ingersoll, and Ross (1985), where the *stochastic* productivity factors are unobservable, equilibrium term structures are "interior" and bounded. Interestingly, Riedel (2000) showed that an incomplete information version of Lucas (1978), with an unobservable *constant* growth rate, induces a "corner" unbounded equilibrium term structure: it decreases to negative infinity. This paper defines constant and stochastic asymptotic moments, clarifies the apparent conflict between Feldman's and Riedel's equilibria, and discusses implications. Because productivity and growth rates are not directly observable in the real world, the question we answer is of particular relevance.

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Key Words: term structure of interest rates, incomplete information, equilibrium asset pricing, constant/stochastic asymptotic moments

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1 Introduction

This short paper resolves an apparent contradiction between Feldman's (1989) and Riedel's (2000) equilibrium models of the term structure of interest rates under incomplete information.

Cox, Ingersoll, and Ross (1985), (CIR), showed that in a complete information multiperiod production and exchange economy, the equilibrium term structure of interest rates is a deterministic function of the *stochastic* economic productivity factors (or growth rates, or security expected returns). Thus, mean reverting stochastic productivity factors induce bounded term structures of interest rates that, for long terms, become flat. A Lucas (1978) exchange economy, with a *constant* growth rate, induces a flat term structure.[1] Feldman (1989) investigated an incomplete information version of the CIR economy.[2] He assumed productivity factors that are unobservable and showed that the equilibrium term structure is set in terms of the conditional moments (the estimates of the unobservable factors/moments and the dynamic precision of these estimates). In Feldman's incomplete information economy, as in CIR's complete information one, mean reverting stochastic productivity factors induce "interior" bounded term structures of interest rates. Riedel (2000) investigated an incomplete information version of the Lucas (1978) exchange economy with an unobservable, *constant* growth rate. Interestingly, in contrast to both Lucas' complete and Feldman's incomplete information equilibrium term structures, Riedel's equilibrium term structure is "corner" and unbounded: as the term to maturity grows, it decreases to negative infinity.[3]

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[1] See also Dybvig, Ingersoll, and Ross (1996).

[2] The first general equilibrium works under incomplete information in this context were Feldman (1983), Dothan and Feldman (1986), Detemple (1986), and Gennotte (1986).

[3] Riedel (2000) shows that the yield curve is decreasing under non-normal priors as well. In the latter case, however, the yield curve, though "corner" and decreasing, is bounded if the range of possible growth rate values is bounded.

The apparent contradiction between Feldman's equilibrium (FE) and Riedel's equilibrium (RE) suggests the following comparisons. 1) In FE an unobservable *stochastic* productivity process induces a bounded term structure in a CIR-type production and exchange economy, while an unobservable *constant* productivity rate induces an unbounded one in a Lucas-type exchange only economy. 2) In FE the dynamic quality of information, or estimation error, increases or decreases to a non-negative steady state as time passes, while in RE the estimation error always decreases asymptotically to zero.[4] Thus, in RE, the unobservable constant growth rate always becomes asymptotically observable, and the original incomplete information economy always asymptotically turns into a complete information one. Furthermore, these patterns of evolution of the quality of information are *ex-ante* common knowledge to investors. The equilibrium term structure is set to reflect this information. 3) As we demonstrate in the next section, the state equations that define RE seem to be special cases of those that define FE.

All the above comparisons position FE as more general and complex than RE. How is it possible, then, that an equilibrium term structure in a relatively "complex" economy is bounded, thus leading to a "stable" equilibrium, while the corresponding equilibrium term structure in a relatively "simple" economy induces an unbounded term structure, leading the equilibrium to "explode," especially when one could possibly perceive the "simple" economy as a special case of the "complex" one? What causes the drastic difference between the equilibria? This paper defines constant and stochastic asymptotic moments, clarifies the apparent conflict between FE and RE, and discusses implications. Because productivity and growth rates are not directly observable in the

[4] In FE, the steady state value of the estimation error is usually positive, but could be zero for certain parameter values. In RE, however, because the steady state value of the estimation error is always zero, a zero initial condition of the estimation error implies a complete information economy. Thus, in RE, the estimation error must start at a positive value and asymptotically decrease to zero.

real world, the question we answer is of particular relevance to theoretical and empirical asset pricing.

Section 2 describes FE and RE; Section 3 brings closed form solutions to FE and RE; Section 4 introduces the definition of constant and stochastic asymptotic moments and characterizes FE and RE; Section 5 resolves the question that the paper poses; and Section 6 concludes.

2 The Economies

For simplicity and brevity, we will present the incomplete information structures of FE and RE in a parallel way using one set of notation. In addition, we will simplify FE from a multiple production technologies economy into a single production technology economy in order to match the single output of RE. Under these conditions, this section presents FE, taken from Feldman (1989), and RE, taken from Riedel (2000).

We begin by introducing an FE and an RE that are described by unobservable state variables and, thus, are non-Markovian and do not allow state vector solutions. We then present σ-algebra equivalent complete information Markovian economies that are based on the (observable) conditional moments of the unobservable variables of the original incomplete information economies. These conditional moments are the outcome of consumers' Bayesian inference process. Within the σ-algebra equivalent complete information economies, we can solve for the equilibrium using stochastic optimal control methods [as in Merton (1971) and, later, CIR] or Martingale methods [as in Duffie and Huang (1985) or Pliska (1986)]. Demands, prices, and consumption in the σ-algebra equivalent complete information economies are identical to those in the original incomplete information economies. Thus the two economies are observationally equivalent [see Feldman (1992)].

Both the FE and the RE are multiperiod, competitive, and frictionless. Both have a representative consumer that maximizes time additive utility of consumption by choosing optimal portfolio and consumption. Both have a consumable output that evolves stochastically, having an unobservable productivity/growth rate.

<u>Feldman's Incomplete Information Economy</u>

Individuals observe the realized output ξ_{Ft} of a single consumption/production numeraire good. This constant stochastic returns-to-scale output evolves as the Itô diffusion

$$\frac{d\xi_{Ft}}{\xi_{Ft}} = (A_0 + A_1\theta_{Ft})dt + B_1 dW_{Ft}\,, \tag{F1}$$

with initial condition ξ_{F0}. A $[2\times 1]$ vector of unobservable independent Wiener processes, $W_F = \{W_{Ft}, F_{Ft}\}$, $W_{F0} = 0$, describes the underlying uncertainty in the economy. W_F is defined over a complete probability space (Ω_F, F_F, P_F) with a non-decreasing right continuous family of sub-σ-algebras $\{F_{Ft}, 0 \le t \le T\}$. A_0 and A_1 are known constants; B_1 is a known $[1\times 2]$ vector of constants; and $B \triangleq B_1 B_1'$—the instantaneous variance of the realized output—is positive.

Individuals, however, do not observe the realization of the stochastic productivity factor θ_{Ft}, which evolves as

$$d\theta_{Ft} = (a_0 + a_1\theta_{Ft})dt + b_1' dW_{Ft}\,. \tag{F2}$$

The distribution of the initial condition θ_{F0}, given ξ_{F0}, is Gaussian with mean m_{F0} and variance γ_{F0}; b_1 is a known $[2\times 1]$ vector of constants; $b \triangleq b_1' b_1$ is positive; and a_0 and a_1 are known constants. To ensure stability of θ_{Ft}, in other words, to guarantee that its values do not diverge to infinity or negative infinity, we assume that $a_0 > 0$ and $a_1 < 0$.

These parameter values induce a mean reversion of θ_{Ft} to a positive asymptotic mean, -a_0/a_1.

Riedel's Incomplete Information Economy

For brevity and to avoid redundancy, we will describe only the features by which RE differs from FE and are relevant to our analysis. The stochastic differential equation that governs the evolution of the realized output in RE is

$$\frac{d\xi_{Rt}}{\xi_{Rt}} = \theta_R dt + dW_{Rt}, \tag{R1}$$

with initial condition ξ_{R0}. W_R describes the underlying uncertainty in the economy. It is a scalar unobservable Wiener process $W_R = \{W_{Rt}, F_{Rt}\}$, $W_{R0} = 0$. W_R is defined over a complete probability space (Ω_R, F_R, P_R) with a non-decreasing right continuous family of sub-σ-algebras $\{F_{Rt}, 0 \le t \le T\}$. An unobservable constant productivity/growth rate θ_R has a Gaussian prior distribution with mean m_{R0} and variance γ_{R0}.[5] [See Riedel (2000), pp. 54-55.]

Feldman's σ-Algebra Equivalent Complete Information Economy

The posterior distribution (filter) of the unobservable productivity/growth rate θ_{Ft}, given the observations $\xi_{Fs}, 0 \le s \le t$, is[6]

$$dm_{Ft} = (a_0 + a_1 m_{Ft})dt + (D + A_1\gamma_{Ft})B^{-1/2}d\overline{W}_{Ft}, \tag{F3}$$

$$d\gamma_{Ft} = [b + 2a_1\gamma_{Ft} - B^{-1}(D + A_1\gamma_{Ft})^2]dt, \tag{F4}$$

$$d\overline{W}_{Ft} = B^{-1/2}[\frac{d\xi_{Ft}}{\xi_{Ft}} - (A_0 + A_1 m_t)dt], \tag{F5}$$

[5] Riedel (2000) examined also the case of non-normal priors and got qualitatively similar results.
[6] See Liptser and Shiryayev (1978) Theorems 12.1 through 12.8, pp. 21-35.

where $m_{F0} \triangleq \mathrm{E}[\theta_{F0} \mid \xi_{F0}]$, $\gamma_{F0} \triangleq \mathrm{E}[(\theta_{F0} - m_{F0})^2 \mid \xi_{F0}]$, and E is the expectation operator. The variable $m_{Ft} \triangleq \mathrm{E}[\theta_{Ft} \mid F_{Ft}^{\xi}]$ is the mean and $\gamma_{Ft} \triangleq \mathrm{E}[(\theta_{Ft} - m_{Ft})^2 \mid F_t^{\xi}]$ is the variance of the conditional distribution; and $\overline{W}_t$ is the innovation process, endogenously determined to be a Wiener process. The innovation process describes the deviations of the observations from their expected values. $D \triangleq B_1 b_1$ is the instantaneous covariance between realized returns and the unobservable factor.[7] The conditional mean, the estimate of the unobservable factor, is a sufficient statistic to the posterior distribution and is updated recursively.

Riedel's σ-Algebra Equivalent Complete Information Economy

The posterior distribution of the unobservable growth rate θ_R, given the observations $\xi_{Rs}, 0 \le s \le t$, is

$$dm_{Rt} = \gamma_{Rt} d\overline{W}_{Rt}, \tag{R3}$$

$$d\gamma_{Rt} = -\gamma_{Rt}{}^2 dt, \tag{R4}$$

$$d\overline{W}_{Rt} = \frac{d\xi_{Rt}}{\xi_{Rt}} - m_{Rt} dt, \tag{R5}$$

where $m_{R0} \triangleq \mathrm{E}[\theta_{R0} \mid \xi_{R0}]$ and $\gamma_{R0} \triangleq \mathrm{E}[(\theta_{R0} - m_{R0})^2 \mid \xi_{R0}]$. The variable $m_{Rt} \triangleq \mathrm{E}[\theta_{Rt} \mid F_{Rt}^{\xi}]$ is the mean and $\gamma_{Rt} \triangleq \mathrm{E}[(\theta_{Rt} - m_{Rt})^2 \mid F_{Rt}^{\xi}]$ is the variance of the conditional distribution; and $\overline{W}_{Rt}$ is the innovation process, endogenously determined to be an independent Wiener process. [See Riedel (2000), pp. 56-57.]

[7] Liptser and Shiryayev (1978 Theorems 12.5 and 12.7) prove that if the conditional distribution of θ_{F0} given ξ_{F0} is Gaussian with mean m_0 and variance γ_{F0}, then $\overline{W}_{Ft}$, m_t, and γ_{Ft} are the unique, continuous, measurable solutions of the system of filter equations. Moreover, the innovation process $\overline{W}_{Ft}$ generates the economy; that is, the σ-algebras $F_t^{\xi_{F0}, \overline{W}_F}$ and F_t^{ξ} are equivalent.

3 The Equilibria

Examining the two state equations of FE, Equations (F1) and (F2), and the single state equation of RE, Equation (R1), it seems that the former could degenerate into the latter. For example, if we let $A_0 = 0,\ A_1 = 1,$ in Equation (F1) and allow the distribution of the process θ_{Ft} to degenerate to realize the same value at every date—in other words, to degenerate into a constant—it might seem that FE becomes RE.[8] Thus, it appears that RE is a special case of FE. Feldman (1989) showed that the equilibrium term structure in FE is bounded, and Riedel (2001) showed that the equilibrium term structure in RE is unbounded. How is it possible, then, that a special case of an economy with a bounded equilibrium term structure has an unbounded equilibrium term structure?

We will solve the puzzle by examining the σ-algebra equivalent complete information economies, Equations (F3) – (F5) for FE and Equations (R3) – (R5) for RE.

<u>The Solution to Feldman's Equilibrium</u>

The solution of the stochastic differential Equation (F3) is

$$m_{Ft} = -\frac{a_0}{a_1}\left(1 - e^{a_1 t}\right) + m_{F0}e^{a_1 t} + \int_0^t f(\gamma_{Fs})e^{a_1(t-s)}d\overline{W}_{Fs}\,, \tag{F6}$$

where $f(\gamma_{Ft}) \triangleq (D + A_1\gamma_{Ft})B^{-1/2}$, and γ_{Ft}, the solution of Equation (F4) is

$$\gamma_{Ft} = \gamma_{Fu}\left[\frac{\gamma_{Fss}}{\gamma_{Fu}} - \frac{\gamma_{F0} - \gamma_{Fss}}{\gamma_{F0} - \gamma_{Fu}}e_t\right]\left[1 - \frac{\gamma_{F0} - \gamma_{Fss}}{\gamma_{F0} - \gamma_{Fu}}e_t\right]^{-1}, \tag{F7}$$

where

$$e_t \triangleq \exp\left[\left[-\frac{A_1^2}{B}\right](\gamma_{Fss} - \gamma_{Fu})t\right],$$

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[8] Alternatively, if in FE $a_0 = a_1 = b_1 = 0$, it is reduced to RE. In hindsight, a crucial difference is the assumption of $a_1 < 0$ in FE.

$$\gamma_{Fss} \triangleq k_2\left\{(k_1 - \rho) + [(k_1 - \rho)^2 + (1 - \rho^2)]^{1/2}\right\},$$

$$\gamma_{Fu} \triangleq k_2\left\{(k_1 - \rho) - [(k_1 - \rho)^2 + (1 - \rho^2)]^{1/2}\right\},$$

and

$$k_1 \triangleq \frac{a_1 / b^{1/2}}{A_1 / B^{1/2}}, \; k_2 \triangleq \frac{(bB)^{1/2}}{A_1}, \; \rho \triangleq \frac{D}{(bB)^{1/2}}.$$

Note that, from Equation (F7), the quality of information, γ_{Ft} converges exponentially from a non-negative initial condition, γ_{F0}, to a non-negative absorbing steady state (root) γ_{Fss}.[9] Technically, $\lim_{t\to\infty} \gamma_{Ft} = \gamma_{Fss}$.

From Equation (F6), the distribution of m_{Ft} is normal. This is the estimate of the growth rate in the original economy and the growth rate in the σ-algebra equivalent economy [see also Liptser and Shiryayev (1978)]. Using the solutions to equations (F6) and (F7), we can calculate the moments of this distribution. Its mean is

$$\mathrm{E}_0(m_{Ft}) = -\frac{a_0}{a_1}\left(1 - e^{a_1 t}\right) + m_{F0}e^{a_1 t} \xrightarrow[t\to\infty]{} -\frac{a_0}{a_1}, \tag{F8}$$

where E_0 is the expectation operator with respect to the information at time 0.[10] For any current date 0, the mean at any future date t is a weighted average of the initial condition, or current mean m_{F0}, and the asymptotic mean $-a_0/a_1$. As the future date t advances, the weight of the initial condition decays exponentially to zero, and the weight of the asymptotic mean increases exponentially to one.

The variance of m_{Ft} is

$$\mathrm{Var}_0(m_{Ft}) = \int_0^t \left(f(\gamma_{Fs})e^{a_1(t-s)}\right)^2 ds, \tag{F9a}$$

[9] See Feldman (1989) Proposition 2, page 797, and its proof, pages 808-809.

[10] Note that because m_{Ft} is observable, the initial condition is known and does not contribute to the variance. The same will be the case (below) with respect to m_{Rt}.

where Var_0 is the variance operator with respect to the information at time 0. While the diffusion component contributes to a diverging variance for further future dates, the damping effect of the reverting mean offsets it and, overall, the variance of m_{Ft} converges to a constant as the future date t advances. For a quality of information γ_{Ft} that is already in steady state, we have

$$\mathrm{Var}_0(m_{Ft}) = -(2a_1 B)^{-1}(D + A_1\gamma_{Fss})^2\left(1 - e^{2a_1 t}\right) \xrightarrow[t\to\infty]{} -(2a_1 B)^{-1}(D + A_1\gamma_{Fss})^2 . \quad \text{(F9)}$$

Note that the assumption that the quality of information is already in steady state simplifies the notation of the center expression in Equation (F9) above, but the limit on the right-hand side of Equation (F9) does not depend on this assumption. Now, consider the possible values of $\mathrm{Var}_0(m_{Ft})$ between the initial time and steady state of the quality of information γ_{Ft}. We can identify an upper bound for the variance $\mathrm{Var}_0(m_{Ft})$. Because of the monotonic nature of the quality of information, γ_{Ft}, and because $\mathrm{Var}_0(m_{Ft})$ is a quadratic function of γ_{Ft}, $\mathrm{Var}_0(m_{Ft})$ is bounded below the value *Vmax,* where $Vmax \triangleq \mathrm{Max}\left[-(2a_1)^{-1} f(\gamma_{F0})^2, -(2a_1)^{-1} f(\gamma_{Fss})^2\right]$ and Max is the maximum operator.[11] That is,

$$0 \le \mathrm{Var}_0(m_{Ft}) \le Vmax, \quad \forall t,\ t \in [0,\infty].$$

The proof is straightforward and is omitted.

Note that at each date, neither the asymptotic mean nor the asymptotic variance are functions of current conditions. That is, at each date, the asymptotic moments depend neither on the date nor on the state of the economy.

[11] If D and A_1 are of the same sign, $\mathrm{Var}_0(m_{Ft})$ is an increasing (decreasing) function of time if $\gamma_{F0}<\gamma_{Fss}$ ($\gamma_{F0}>\gamma_{Fss}$). If D and A_1 are of different signs, depending on the relative sizes of the absolute values of D and $A_1\gamma_{Ft}$, the previous case or the opposite might hold. If D and A_1 are of different signs and the relative sizes of the absolute values of D and $A_1\gamma_{Ft}$, change as γ_{Ft} changes, $\mathrm{Var}_0(m_{Ft})$ is a decreasing-increasing function of time.

The Solution to Riedel's Equilibrium

The solution of the stochastic differential Equation (R3) is

$$m_{Rt} = m_{R0} + \int_0^t \gamma_{Rs} d\overline{W}_{Rs} \ , \tag{R6}$$

and the solution of Equation (R4) is

$$\gamma_{Rt} = \frac{\gamma_{R0}}{1+\gamma_{R0}t} \xrightarrow[t\to\infty]{} 0 \,. \tag{R7}$$

The distribution of m_{Rt}, the estimate of the growth rate in the original economy and the growth rate in the σ-algebra equivalent economy, is normal. Using the previous solutions in Equations (R6) and (R7), we can calculate the moments of this distribution. At each date 0, the mean at any future date is the current date mean or the initial condition. That is,

$$\mathrm{E}_0(m_{Rt}) = m_{R0} \,. \tag{R8}$$

The variance is

$$\mathrm{Var}_0(m_{Rt}) = \int_0^t {\gamma_{Rs}}^2 ds = \gamma_{R0} \frac{\gamma_{R0}t}{1+\gamma_{R0}t} \xrightarrow[t\to\infty]{} \gamma_{R0} \,. \tag{R9}$$

While the diffusion component contributes to a diverging variance in time, the damping effect of the improving quality of information, or decreasing estimation error, offsets it; and, overall, the variance of m_{Rt} converges to a constant. At each date 0, the variance at any future date t is the current date value, or the initial condition γ_{R0}, of the estimation error.

We find the intuition behind the result in Equation (R9) most interesting. As the information flow is basically a random walk [see Equation (R6)], on one hand it should not make us worse off if we appropriately process it, and on the other hand it could not make us better off because it has no systematic information. Thus, asymptotically, we cannot do better and should not do worse than our initial uncertainty. Indeed, at each date, the asymptotic variance of the observable growth rate in the σ-algebra equivalent complete information economy is exactly the current estimation error of the unobservable growth rate in the original economy.

Note that at each date, the stochastic asymptotic mean is a function of current conditions. That is, at each date, the asymptotic mean depends on both the date and the state of the economy. The asymptotic variance, though deterministic, also depends on the current economic conditions, particularly on the investors' information.

4 Constant/Stochastic Asymptotic Moments (CAM/SAM)

We use $\mathrm{M}_{\mathrm{s}}^{\mathrm{n}}(m_\infty) = \lim_{u\to\infty} \mathrm{M}_{\mathrm{s}}^{\mathrm{n}}(m_{s+u})$ to denote the n^{th} probability distribution moment, where n is a positive integer, with respect to the information at time *s,* of the random variable m_{s+u} as u goes to infinity, or the time s n^{th} asymptotic moment. In particular, for the first two moments we use $\mathrm{M}_{\mathrm{s}}^{1}(m_\infty) = \mathrm{E}_{\mathrm{s}}(m_\infty) = \lim_{u\to\infty} \mathrm{E}_{\mathrm{s}}(m_{s+u})$ and $\mathrm{M}_{\mathrm{s}}^{2}(m_\infty) = \mathrm{Var}_{\mathrm{s}}(m_\infty) = \lim_{u\to\infty} \mathrm{Var}_{\mathrm{s}}(m_{s+u})$ to denote the mean and variance, respectively.

Definition. *Constant asymptotic moments (CAM)*. We call a stochastic process m_t CAM if $\mathrm{M}_{t}^{\mathrm{n}}(m_\infty) = \mathrm{M}_{t+h}^{\mathrm{n}}(m_\infty),\ \forall t,h,n,$ and $t,h,n > 0$.

Definition. *Stochastic asymptotic moments (SAM)*. We call a stochastic process m_t SAM if at least one of its asymptotic moments (as defined above) is stochastic.

Intuitively, a stochastic process is CAM if, at each date, the moments of the asymptotic distribution of the process' value have the same values. Similarly, a stochastic process is SAM if, for at least one of its asymptotic moments, the future date values of the asymptotic moments are random variables. Using this definition, we are now ready to state the following proposition.

Proposition. *m_{Ft} is CAM and m_{Rt} is SAM.*

Proof. Because both m_{Ft} and m_{Rt} are Gaussian, it is sufficient to examine the first two moments. From Equations (F8) and (F9), neither $\mathrm{E}_s(m_{F\infty})$ nor $\mathrm{Var}_s(m_{F\infty})$ is a function of s or m_{Fs}. Thus, m_{Ft} is CAM. In addition, from Equations (R9), $\mathrm{Var}_s(m_{R\infty})$ is a function of s. Thus, m_{Ft} is not CAM. Furthermore, from Equation (R8), $\mathrm{E}_s(m_{R\infty})$ is a function of m_{Rs}. Because m_{Rs} is stochastic, m_{Rt} is SAM. □

Examining FE, we see from Equations (F8) and (F9) that at any date s, the asymptotic mean and asymptotic variance of m_{Ft} are constant functions of the model parameters. Examining RE, we see from Equation (R8) that at each date s, the asymptotic mean of m_{Rt}, $\lim_{t\to\infty}\mathrm{E}_s(m_{Rt})$ is that date productivity level m_{Rs}. Similarly, we see from Equation (R9) that at each date s, the asymptotic variance of m_{Rt}, $\lim_{t\to\infty}\mathrm{Var}_s(m_{Rt})$ is that date's value of the estimation error, γ_{Rs}.

In other words, in FE, at each date, and regardless of the current state of the economy, investors' long-term estimate of the unobservable economic productivity, or growth rate, is equal to some constant. This constant is the long-term mean of the mean reverting stochastic process that describes the evolution of the unobservable productivity factor. Similarly, in FE, at each date, and regardless of the current state of the economy, the variance of investors' long-term distribution of the productivity factor, or growth rate, is a constant function of the model parameters and of γ_{Fss}, the steady state value of the quality of information.

In contrast, in RE, at each date, investors' long-term estimate of the growth rate is the current state of the economy, or the current estimate of the unobservable growth rate. As the growth rate of the economy in the σ-algebra equivalent economy evolves stochastically (as a random walk), so does the estimate of its long-term value. Similarly, in RE, at each date, the variance of investors' long-term distribution of the growth rate is the prevailing value of the estimation error of the unobservable growth rate. Thus, as the current estimation error of the unobservable growth rate changes in time, so does the variance of the value of the long-term distribution of the growth rate.

Note that neither FE- nor RE-equivalent economies are stationary. The ongoing change in the values of the estimation errors of the growth rates is sufficient to deem them non-stationary.

5 Resolution

We will now argue that the term structure in the FE is bounded because the growth rate in its σ-algebra equivalent complete information economy is CAM, and that the term structure in the RE is not bounded because the growth rate in its σ-algebra equivalent complete information economy is a random walk SAM. The argument involves the following steps.

1) It is sufficient to analyze the term structure in the σ-algebra equivalent complete information economies of FE and RE because demand and prices in these economies are equivalent to those in the original corresponding incomplete information economies [see Feldman (1992)].

2) In both the FE and the RE, the asymptotic rate of the term structure of interest rates is a deterministic function of the growth rate. This is one of the important results of CIR, and it is manifested particularly in Feldman (1989) and Riedel (2000). Thus, for the sake of brevity, we will not repeat FE and RE term structure functions.

3) Finally, DIR show that the long forward rate must equal the lowest possible value of its possible future values to prevent arbitrage.[12] Because the possible future long rates in the CAM FE are all equal to some constant, the long rate is equal to this constant. In contrast, because in the SAM RE the possible future long rates are stochastic and follow a random walk, the long rate goes to negative infinity.[13]

Thus, the added structure in FE actually simplifies the asymptotic properties of the perceived growth rate and, with that, simplifies the equilibrium. In contrast, the simple RE results in a random asymptotic behavior of the perceived growth rate, leading to a complex equilibrium. In this sense, the SAM RE is not a special case of the CAM FE.

6 Conclusion

We see how a more "complex" ("simple") model induces a simpler (more complex) behavior of the asymptotic growth rate which, in turn, induces an "interior" ("corner") equilibrium. The question, then, becomes one of modeling. RE is obviously more parsimonious, but FE has an advantage in explaining term structure stylized facts: we do not observe term structures that fall to negative infinity. The empirical question, however, is not fully resolved. As DIR point out, the data so far does not seem to support a no-arbitrage condition of the term structure, which says that the long forward rate can never fall. It is this condition that is behind the conflict between FE and RE. This calls for further empirical studies.

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[12] See Reidel (2000), p.62, for an intuitive explanation of the result in Dybvig, Ingersoll, and Ross (1996).

[13] *Note that it is the random walk stochastic asymptotic mean of the growth rate that makes the impact in RE's* σ-algebra equivalent economy. The asymptotic variance is deterministic, bounded, and decreases to zero as a function of the age of the economy. Recall that in FE the asymptotic variance is constant.

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