# Optimal Design of the Guarantee for Defined Contribution Funds


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**Abstract:** The question we solve is the optimal design of the minimum guarantee in a Defined Contribution Pension Fund Scheme. We study the investment in the financial market by assuming that the pension fund optimizes its retribution which is a part of the surplus, that is the difference between the pension fund value and the guarantee. Then we define the optimal guarantee as the solution of the contributor's optimization program and find the solution explicitly. Finally, we analyze the impact of the main parameters, and particularly the sharing rule between the contributor and the pension fund. We find that favorable sharing rules for the pension fund lead to conservative guarantees for the contributor: the sharing rule is a way to create a continuum between two extreme pension funding methods that are Defined Benefit and Defined Contribution Pension Schemes, and the sharing rule allows partial risk transfer between the contributor and the pension fund manager.

**Key words:** guarantee, pension funds, portfolio choice, stochastic optimization, variational calculus.

Classification according to **JEL** codes: **G11**

## 1 Introduction

There exist two types of radically different pension funds methods: the "defined benefit" method where the contributions are the random variables but the final benefit is fixed, and the "defined contribution" method where randomness comes from the benefit. Historically, pension funds used mainly the first method which is preferred by the client (see e.g. Davis 1995). However, due to the demographic evolution and the development of the equity markets, new systems have been introduced. Nowadays, the pension funds propose mainly defined contribution schemes which transfer the equity market risk to the clients.

A simple way to moderate this inconvenience for the clients, is to introduce a minimum guarantee on the future benefit that will be paid out to the clients. However, this guarantee can be very complex and the question is to find the optimal form that it should take in order to maximize the utility of the client.

In a general complete financial market framework, we assume that the pension fund's retribution is equal to a fixed part of the surplus (that is the difference between the final value of the portfolio managed by the pension fund and the guarantee). Moreover, the manager of the pension fund will invest the wealth in order to optimize the expected value of the utility of its share of the surplus. Having proved that an auxiliary process is self-financed, we determine the analytic form of the guarantee as the solution of the client's optimization program. This expression is analyzed with respect to the main parameters and expecially with respect to the sharing rule that fixes the repartition of the final surplus between the pension fund and the client. We find that highest guarantees are linked to sharing rules giving all the surplus to the pension fund. Then the choice of the sharing rule is a trade-off between protection (the level of the guarantee) and return (impacted by the part left to the pension fund).

In related literature, Boulier, Huang and Taillard (2001) and Deelstra, Grasselli and Koehl (2001) study the optimal management of a defined contribution plan where the guarantee depends on the level of interest rates at the fixed retirement date. Jensen and Sørensen (2001) measure the effect of a minimum interest rate guarantee constraint through the wealth equivalent in case of no constraints and show numerically that guarantees may induce a significant utility loss for relatively risk tolerant investors. Both the papers by Boulier et al. (2001) and Jensen and Sørensen (2001) choose the Vasiček (1977) specification of the term structure in the spirit of Bajeux-Besnainou, Jordan and Portait (1998), while Deelstra et al. (2001) choose the affine term structure by using the methodology of Deelstra, Grasselli and Koehl (2000).

The paper is organized as follows: in Section 2, we define the market structure and introduce the optimization problems. In Section 3, we obtain the main property of the auxiliary process and deduce a market efficiency test.

In Section 4, we derive the optimal form of the guarantee and comparative statics with respect to the expected value of the benefit. Section 5 is devoted to the analysis of the influence of the sharing rule, and Section 6 provides an example when the client has a power utility function. Section 7 concludes the paper.

# 2 The model

In this section, we describe the financial market and the optimization programs.

## 2.1 The financial market

Randomness is described by $W(t) = \{(W_1(t), ..., W_n(t))' \; ; \; t \in [0, T]\}$, an $n$-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is the real world probability and $T$ is supposed to be finite. The filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$, represents the information structure generated by the Brownian motion and is assumed to satisfy the usual conditions.

Hereafter $\mathbb{E}_t$ stands for $\mathbb{E}(. \mid \mathcal{F}_t)$, the conditional expected value under the real world probability.

The market is composed of $n+1$ financial assets, that the agent can buy or sell continuously without incurring any restriction as short sales constraints or any trading cost.

The first asset is the riskless asset (i.e. the bank account) whose price, denoted by $B(t)$, evolves according to:

$$\frac{dB(t)}{B(t)} = r(t)dt, \qquad B(0) = 1,$$

where $r(t)$ represents the short interest rate.

The remaining $n$ assets are the risky assets, whose prices are denoted by $P_i(t)$, $i = 1, ..., n$. The dynamics of $P_i(t)$ are given by:

$$dP_i(t) = P_i(t) \left[ b_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t) dW_j(t) \right], \qquad P_i(0) = p_i \in (0, +\infty). \quad (1)$$

We assume that the interest rate process $\{r(t), 0 \leq t \leq T\}$, the drift process $\{b(t) = (b_1(t), ..., b_n(t))', 0 \leq t \leq T\}$, and the volatility matrix process

$\left\{\sigma(t) = (\sigma_{ij}(t))_{1\leq i,j\leq n}, 0 \leq t \leq T\right\}$ are progressively measurable w.r.t. $\mathcal{F}$ and satisfy the condition

$$\int_0^T \left( |r(t)| + \|b(t)\| + \sum_{i=1}^n \|\sigma_i(t)\|^2 \right) dt < \infty \text{ a.s.}$$

where $\|.\|$ denotes the Euclidean norm in $\mathbb{R}^n$ and where $\sigma_i(t)$ denotes the $i$-th row of $\sigma(t)$.

We assume also that the financial market is arbitrage-free and complete, i.e. there is only one process $\theta(.)$ satisfying

$$\theta(t) = \sigma^{-1}(t)\left[b(t) - r(t)\mathbf{1}_n\right], \qquad 0 \leq t \leq T,$$

with $\mathbf{1}_n = (1, ..., 1)' \in \mathbb{R}^n$, where $\sigma(t,\omega)$ is non-singular, for $(\lambda \otimes \mathbb{P})$ −a.e. $(t,\omega) \in [0,T] \times \Omega$.

The exponential process

$$Z(t) = \exp\left[-\int_0^t \theta'(s)dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\right], \qquad 0 \leq t \leq T,$$

is assumed to be a martingale, and the risk-neutral equivalent martingale measure, denoted by $Q$, is defined by

$$Q(A) = \mathbb{E}\left[Z(T)1_A\right], \quad A \in \mathcal{F}(T).$$

We further define the state-price density process by

$$H(t) = \frac{Z(t)}{B(t)} = \exp\left[-\int_0^t r(s)ds - \int_0^t \theta'(s)dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\right]. \quad (2)$$

## 2.2 The optimization program of the contributor

The contributor pays a flow to the pension fund. This flow consists in a lump sum at date 0, denoted by $X_0$, and a continuously paid premium, at a rate denoted by $c(t)$, $t \in [0,T]$; the flow of contributions is assumed to be a non-negative, progressive measurable process such that:

$$\int_0^T c^2(t)dt < \infty, \text{ a.s.}$$

The value at date 0 of the cash given by the contributor to the pension fund is equal to:

$$X_0' = X_0 + \mathbb{E}\left[\int_0^T H(s)c(s)ds\right].$$

In exchange, the fund manager will provide at date $T$ a benefit which consists of two parts: The first part $G_T$ is guaranteed, which means that the benefit will be greater than $G_T$ almost surely. We do not constrain the guarantee to be constant, it is only required to be a positive random variable $\mathcal{F}_T$ measurable which is $L^p$ integrable with $p > 2$. In particular, this assumption allows for the case of a stochastic guarantee (for example salary-indexed) whose value will be known at time $T$. The second part of the benefit is a fixed fraction of the surplus $Y_T(G_T)$ (the difference between the terminal wealth $X_T$ of the managed portfolio and the guarantee $G_T$). Indeed, we suppose that the fund manager receives a fixed fraction of the surplus, as a way to incite him. For example Holstrom and Milgrom (1987) have studied the problem of providing incentives over time for an agent with constant absolute risk aversion. They found in their model that the optimal compensation scheme turns out to be a linear function.

Let us denote by $\beta$ the fixed fraction of the surplus that will be kept by the fund manager. Then, the total benefit of the contributor at date $T$ equals:

$$B_T = G_T + (1-\beta)(X_T - G_T). \tag{3}$$

The problem of the contributor is to choose the best contract between those offered by the pension funds, everything else being fixed - that is the value of the cash given by the contributor $X_0'$, the fraction $\beta$ of the surplus kept by the fund manager, and its risk aversion that we introduce more in detail in the next section. The guarantee is then the only remaining variable and the problem is to find its optimal form. This problem is a static one from the contributor point of view since he has a decision to make at date 0 only for a benefit that will be delivered at date $T$.

For $\beta = 0$, the fund manager does not keep any profit from the surplus, so the presence of the guarantee is more an obstacle than a convenience for the client, since guarantees may induce a significant utility loss for relatively risk tolerant investors (see Jensen and Sørensen 2001). On the other hand, $\beta = 1$ means that the contributor will receive the guarantee only, no matter the final surplus. In order to avoid these trivial cases, we will assume $\beta \in (0,1)$ from now on.

We formalize the optimality of the guarantee by assuming that the contributor has an increasing concave utility function $u$; then, the guarantee must lie in the set $\mathcal{G}$ defined as follows:

$$\mathcal{G} = \{G_T\colon \exists k \in [0, X_0'[ \text{ such that } G_T \text{ is solution of (4) defined for } k\},$$

where (4) is the following optimization program:

$$\max_{G_T} \mathbb{E}\left[u\left((1-\beta)Y_T(G_T) + G_T\right)\right] \tag{4}$$

under the constraints:

$$\begin{cases} \mathbb{E}\left[H(T)\left((1-\beta)Y_T(G_T) + G_T\right)\right] = k \\ G_T \geq 0 \text{ a.s.} \end{cases}$$

In order to solve this problem, we need to look more deeply at the way used by the pension fund to manage the portfolio, in order to get the principal features of $Y_T(G_T)$.

## 2.3 The optimization program of the pension fund manager

In this subsection, we describe the portfolio problem faced by the pension fund manager. More precisely, we assume:

(i) that the risk aversion of the fund manager is described by a power utility function

$$U(y) = \frac{y^\gamma}{\gamma},\ \gamma \in (-\infty, 1)\backslash\{0\}, \tag{5}$$

(ii) that he maximizes the expected utility of his terminal wealth (that is, his part of the surplus).

The choice of the power utility is motivated by two reasons.

First, pension funds are in general large companies who define their strategies with respect to the amount of money they are managing, more or less in a scaling way. This feature is well captured by the use of the power utility function.

Secondly, pension funds are regulated in such a way that they can not reach negative values. This is true also in the power utility case, thanks to the infinite marginal utility at zero.

Denoting by $X(t)$ the wealth of the fund at date $t \in [0,T]$, and by $\pi(t)$ the proportion of wealth invested into the $n$ risky assets (in such a way that $1 - \sum_{i=1}^{n} \pi_i(t)$ is the proportion of wealth invested into the riskless asset $B(t)$), the optimization program of the pension fund will be:

$$\max_{(\pi(t))_{t\in[0,T]} \in \mathcal{A}^X} \frac{1}{\gamma}\mathbb{E}\left(X(T) - G_T\right)^{\gamma} \tag{6}$$

under the constraints:

$$dX(t) = \left(X(t)r(t) + X(t)\pi'(t)\left(b(t) - r(t)\right) + c(t)\right)dt + X(t)\pi'(t)\sigma(t)dW(t) \tag{7}$$

with $X(0) = X_0 > 0$ and

$$\begin{aligned} \mathcal{A}^X = \; & \left\{\pi(t) = (\pi_1(t), ..., \pi_n(t))', \, t \in [0,T], \, \mathcal{F} - \text{adapted process such that:} \right. \\ & \int_0^T \left\| X(t)\pi'(t)\sigma(t) \right\|^2 dt < \infty \text{ a.s.} \\ & \left. X(T) - G_T \geq 0 \text{ a.s.} \right\}. \end{aligned}$$

From now on, we assume that:

$$\mathbb{E}\left[H(T)G_T\right] < X_0 + \mathbb{E}\left[\int_0^T H(t)c_t dt\right] = X_0', \tag{8}$$

which is equivalent to say that the set of admissible strategies $\mathcal{A}^X$ is non empty. This assumption is a reasonable assumption as it means that the market value of the contributions is supposed to be larger than the market value of the guarantee.

# 3 Main features of the surplus process

In this section, we define the surplus process. Inspired by the actuarial prospective approach, this surplus process takes into account the future rights of the pension fund, i.e. the contributions that will enter, and the future obligations of the pension fund, i.e. the guarantee that has to be payed at the final date $T$. We prove that the surplus process is self-financing, and deduce a market efficiency test for the pension fund.

**Definition 1** *The surplus process $Y(t)$, $t \geq 0$ is defined by:*

$$Y(t) = X(t) + D(t) - G(t),$$

*where*

$$D(t) = \mathbb{E}_t \int_t^T \frac{H(s)}{H(t)} c(s) ds, \; G(t) = \mathbb{E}_t \left[ \frac{H(T)}{H(t)} G_T \right].$$

This process can be interpreted as a surplus process, in the sense that, at date $t$, it is equal to:

- the value of the portfolio $X(t)$
- plus the discounted value of the future engagements coming from the contributor $D(t)$,
- minus the discounted value of the pension fund future engagement (that is the guarantee) $G(t)$.

Note also that the value of the process at date $T$ is equal to the surplus $X(T) - G_T$.

**Proposition 2** *The surplus process is self-financing, that is there exists a progressive measurable random process* $y(t) = (y_1(t), ..., y_n(t))'$, $t \in [0, T]$ *such that:*

$$dY(t) = Y(t) \left( r(t)dt + y'(t) \left( b(t) - r(t) \right) dt + y'(t)\sigma(t) dW(t) \right) \tag{9}$$

**Proof.** Following Deelstra et al. (2001), for a given process $K(t)$ let denote $\widetilde{K}(t) := H(t)K(t)$. Then

$$d\widetilde{Y}(t) = d\widetilde{X}(t) + d\widetilde{D}(t) - d\widetilde{G}(t).$$

From (1), (2), and (7), easy computations lead to:

$$d\widetilde{X}(t) = \widetilde{X}(t) \left( \pi'(t)\sigma(t) - \theta'(t) \right) dW(t) + \widetilde{c}(t)dt.$$

Using the martingale representation theorem for the Brownian motion, (see e.g. Karatzas and Shreve 1990), it turns out that there exists a unique square integrable process $\zeta(t)_{t\in[0,T]}$, satisfying

$$\int_0^T \left\| \zeta(t) \right\|^2 dt < +\infty \text{ a.s.} \tag{10}$$

such that

$$d\widetilde{D}(t) = -\widetilde{c}(t)dt + \zeta'(t)dW(t). \tag{11}$$

Analogously, there exists a unique square integrable process $\rho(t)_{t\in[0,T]}$, satisfying

$$\int_0^T \|\rho(t)\|^2 \, dt < +\infty \text{ a.s.} \tag{12}$$

such that

$$d\widetilde{G}(t) := d\left(\mathbb{E}_t\left[\widetilde{G}_T\right]\right) = \rho'(t)dW(t).$$

Finally, we get:

$$d\widetilde{Y}(t) = \left(\widetilde{X}(t)\left(\pi'(t)\sigma(t) - \theta'(t)\right) + \zeta'(t) - \rho'(t)\right) dW(t)$$

and therefore the process $Y(t)$ is self-financing. Indeed, in order to prove (9), it suffices to define $y(t)$ as follows:

$$Y(t)y(t) = X(t)\pi(t) + (D(t) - G(t))\,[\sigma'(t)]^{-1}\theta(t) + H^{-1}(t)[\sigma'(t)]^{-1}\left(\zeta(t) - \rho(t)\right) \tag{13}$$

which ends the proof. ■

The following corollary provides an exponential expression for the final surplus $Y_T$, that will be used extensively in the rest of the paper.

**Corollary 3** *The final surplus $Y_T$ satisfies the following equation:*

$$\begin{aligned} Y_T &= Y_0 \exp\left(\int_0^T \left(r(t) + y'(t)\left(b(t) - r(t)\right)\right) dt + \int_0^T y'(t)\sigma(t)dW(t)\right. \\ &\quad \left. -\frac{1}{2}\int_0^T \|y'(t)\sigma(t)\|^2 \, dt\right) \end{aligned} \tag{14}$$

*with*

$$Y_0 = X_0 + \mathbb{E}\left[\int_0^T H(s)c(s)ds\right] - \mathbb{E}\left[H(T)G_T\right] \geq 0.$$

Note that $Y_T$ depends on $G_T$ through $Y_0$ only. From now on we will stress this dependence by denoting $Y_T$ as a function of $G_T$.

Defining the random variable $\varphi_T(\omega)$ as follows

$$\begin{aligned} \varphi_T &= \exp\left(\int_0^T \left(r(t) + y'(t)\left(b(t) - r(t)\right)\right) dt + \int_0^T y'(t)\sigma(t)dW(t)\right. \\ &\quad \left. -\frac{1}{2}\int_0^T \|y'(t)\sigma(t)\|^2 \, dt\right), \end{aligned} \tag{15}$$

(14) can be rewritten as

$$Y_T(G_T) = (X_0' - \mathbb{E}\left[H(T)G_T\right])\, \varphi_T. \tag{16}$$

For two different minimum guarantees $G_T^1$ and $G_T^2$, we can write

$$Y_T(G_T^2) = Y_T(G_T^1)\frac{X_0' - \mathbb{E}\left[H(T)G_T^2\right]}{X_0' - \mathbb{E}\left[H(T)G_T^1\right]}. \tag{17}$$

If we compare the surplus in case with a guarantee $G_T$ with the no-guarantee-case, we have a strong relationship between the variance of the surplus and the expectation. This relationship can be used as a market efficiency test: the variance of the surplus turns out to be proportional to the square of the expectation. This comes from the decreasing quadratic relation between the variance of the surplus and the market value of the minimum guarantee (see (18)).

**Proposition 4** *(Market efficiency test) There exists a (positive) constant k such that*

$$Var\left[Y_T(G_T)\right] = k\left(E\left[Y_T(G_T)\right]\right)^2.$$

**Proof.** From (17) it follows that

$$Var\left[Y_T(G_T)\right] = Var\left[Y_T(0)\right]\left(1 - \frac{\mathbb{E}\left[H(T)G_T\right]}{X_0'}\right)^2 \tag{18}$$

and

$$E\left[Y_T(G_T)\right] = E\left[Y_T(0)\right]\left(1 - \frac{\mathbb{E}\left[H(T)G_T\right]}{X_0'}\right),$$

and therefore

$$\frac{Var\left[Y_T(G_T)\right]}{Var\left[Y_T(0)\right]} = \left(\frac{E\left[Y_T(G_T)\right]}{E\left[Y_T(0)\right]}\right)^2.$$

■

This result is strongly linked to the power utility of the pension fund which, as well known in literature since Merton (1992), provides this kind of mean-variance relationships. However, the last proposition delivers a strong test to check with data from some pension funds whether their investment strategies follow the model proposed in this paper, which is an interesting application of Merton's results.

# 4 The optimal guarantee for the contributor

By the analysis of the pension fund manager problem, we have obtained the principal features of the final surplus. Now, we come back to the initial problem of the contributor (4), that is:

$$\max_{G_T} \mathbb{E}\left[u\left((1-\beta)Y_T(G_T)+G_T\right)\right]$$

under the constraints:

$$\begin{cases} \mathbb{E}\left[H(T)\left((1-\beta)Y_T(G_T)+G_T\right)\right]=k \\ G_T \geq 0 \text{ a.s.} \end{cases}$$

## 4.1 The main result

Substituting the expression (16) of $Y_T(G_T)$ into the constraint of the optimization program, we find

$$\mathbb{E}\left[H(T)\left((1-\beta)\varphi_T\left(X_0'-\mathbb{E}\left[H(T)G_T\right]\right)+G_T\right)\right]=k,$$

which is equivalent to the constraint

$$\mathbb{E}\left[H(T)G_T\right]=\frac{k-\mathbb{E}\left[H(T)(1-\beta)\varphi_T X_0'\right]}{1-\mathbb{E}\left[H(T)(1-\beta)\varphi_T\right]},$$

or using the fact that the surplus process is self-financing and thus $\mathbb{E}\left[H(T)Y_T\right]=Y_0$ and $\mathbb{E}\left[H(T)\varphi_T\right]=1$:

$$\mathbb{E}\left[H(T)G_T\right]=\frac{k-(1-\beta)X_0'}{\beta}.$$

Noticing that under this constraint,

$$Y_T(G_T)=\varphi_T\left(X_0'-\mathbb{E}\left[H(T)G_T\right]\right)=\frac{X_0'-k}{\beta}\varphi_T,$$

one sees that the $Y_T(G_T)$ is a constant in this minimization problem with respect to $G_T$. Therefore, it will be denoted in this section $Y_T$ only.

The problem is then

$$\max_{G_T} \mathbb{E}\left[u\left(\frac{1-\beta}{\beta}(X_0'-k)\varphi_T+G_T\right)\right]$$

under the constraints:

$$\begin{cases} \mathbb{E}\left[H(T)\left(\frac{1-\beta}{\beta}(X_0' - k)\varphi_T + G_T\right)\right] = k \\ G_T \geq 0 \text{ a.s.} \end{cases}$$

or equivalently

$$\max_{B_T} \mathbb{E}\left[u\left(B_T\right)\right] \tag{19}$$

s.to:

$$\begin{cases} \mathbb{E}\left[H(T)B_T\right] = k \\ B_T \geq \frac{1-\beta}{\beta}(X_0' - k)\varphi_T \text{ a.s.} \end{cases}$$

where $B_T$ is the final benefit defined by (3). Now the problem (19) can be solved by using variational calculus.

**Proposition 5** *The solution of the contributor problem takes the following form:*

$$G_T^* = \left(I\left(\lambda H(T)\right) - \frac{1-\beta}{\beta}\left(X_0' - k\right)\varphi_T\right) 1_{I(\lambda H(T)) - \frac{1-\beta}{\beta}\left(X_0' - k\right)\varphi_T > 0} \tag{20}$$

*where* $I(.) = (u')^{-1}$, *and the real number* $\lambda$ *is defined implicitly by:*

$$\mathbb{E}\left[H_T\left(G_T^* + \frac{1-\beta}{\beta}\left(X_0' - k\right)\varphi_T\right)\right] = k \tag{21}$$

**Proof.** See Appendix.

Notice that the expression of the solution is close to the one obtained without considering the positivity constraint on the guarantee, which is simply equal to $I\left(\lambda H(T)\right) - \frac{1-\beta}{\beta}\left(X_0' - k\right)\varphi_T$.

## 4.2 Comparative statics with respect to the expected value of the benefit

Having the explicit expression of the optimal guarantee, we analyze now the sensibility with respect to the key parameters, namely the expected value of the benefit $k$, and the sharing rule entirely defined by the parameter $\beta$. The former is done in this subsection. Basicaly the result is quite intuitive: the guarantee is increasing $\omega$ by $\omega$ with the expected value of the benefit.

We begin with two technical Lemmas and then state formally the result in the Proposition.

Let us denote by $B_T^{\lambda,k}$ the solution of the problem (19), where $\lambda$ is the corresponding Lagrangian multiplier implicitly defined by (21). Notice that the benefit depends on $k$ only ($\beta$ is kept constant), while $\lambda$ is an auxiliary parameter implicitly defined through the equality $\mathbb{E}\left[H(T)B_T^{\lambda,k}\right] = k$. However, a shift on $k$ changes the value of the corresponding $\lambda$, so we need to investigate first the relationship between $k$ and $\lambda$, everything else being fixed.

**Lemma 6** *Let us consider the problem (19).*
*a) If $\lambda_1 < \lambda_2$, then $B_T^{\lambda_1,k} \geq B_T^{\lambda_2,k}$ a.s. for $k \in [0, X_0'[$.*
*b) If $k_1 < k_2$, then $B_T^{\lambda,k_1} \geq B_T^{\lambda,k_2}$ a.s. for a fixed $\lambda$.*

**Proof.** See Appendix.

**Corollary 7** *Let $B_T^{\lambda_1,k_1}$, $B_T^{\lambda_2,k_2}$ satisfy*

$$\begin{aligned} \mathbb{E}\left[H(T)B_T^{\lambda_1,k_1}\right] &= k_1, \\ \mathbb{E}\left[H(T)B_T^{\lambda_2,k_2}\right] &= k_2. \end{aligned}$$

*If $k_1 < k_2$, then $\lambda_2 < \lambda_1$.*

**Proof.** From the previous lemma, $\mathbb{E}\left[H(T)B_T^{\lambda_2,k_1}\right] \geqslant k_2 > k_1 = \mathbb{E}\left[H(T)B_T^{\lambda_1,k_1}\right]$, then $\lambda_2 < \lambda_1$. ■

We can now state the comparative static result of the guarantee with respect to the expected value of the benefit.

**Proposition 8** *Let $\beta$ be fixed, $k_1 < k_2$ and $G_T^1, G_T^2$ the correponding optimal guarantees given by (20). Then, $G_T^1 \leq G_T^2$ almost surely.*

**Proof.** See Appendix.

# 5 Influence of the sharing rule

In this section, we study the impact of the parameter $\beta$ which describes the sharing rule of the surplus.

We follow the same procedure than the one adopted in the previous section; the main difference is that we now write the benefit $B_T^{\lambda,\beta}$ as a function of $\lambda$ and $\beta$ (and no more of $\lambda$ and $k$).

**Lemma 9** *Let us denote by* $B_T^{\lambda,\beta}$ *the solution of the problem (19) where* $k$ *is fixed. If* $\beta_1 < \beta_2$*, then* $B_T^{\lambda,\beta_1} \geq B_T^{\lambda,\beta_2}$ *a.s.*

**Proof.** See Appendix.

**Corollary 10** *Let* $B_T^{\lambda_1,\beta_1}$, $B_T^{\lambda_2,\beta_2}$ *satisfy*

$$\mathbb{E}\left[H(T)B_T^{\lambda_1,\beta_1}\right] = k = \mathbb{E}\left[H(T)B_T^{\lambda_2,\beta_2}\right].$$

*If* $\beta_1 < \beta_2$*, then* $\lambda_1 > \lambda_2$*.*

**Proof.** From the previous lemma $\mathbb{E}\left[H(T)B_T^{\lambda_1,\beta_1}\right] = \mathbb{E}\left[H(T)B_T^{\lambda_2,\beta_2}\right] < \mathbb{E}\left[H(T)B_T^{\lambda_2,\beta_1}\right]$, then $\lambda_1 > \lambda_2$ from Lemma 6. ■

**Proposition 11** *Let* $k$ *be fixed,* $\beta_1 < \beta_2$ *and* $G_T^{\beta_1}, G_T^{\beta_2}$ *the corresponding optimal guarantees given by (20). Then* $G_T^{\beta_1} \leq G_T^{\beta_2}$ *almost surely.*

**Proof.** See Appendix.

This result can be viewed as follows : the sharing rule is a way to transfer the risk from the client to the pension fund. With a low $\beta$, the guarantee is low too, which means that the risk is high and mainly taken by the client. To the opposite, when $\beta$ is high, the protection is higher and the pension fund plays an intermediary role between the financial market and the client.

# 6 The solution with power utility function for the contributor

In this section, we compute explicitely the guarantee when the utility function of the contributor has the same power form as the one of the fund, in order to illustrate more concretely our results:

$$u(y) = \frac{y^{\overline{\gamma}}}{\overline{\gamma}}, \ \overline{\gamma} \in (-\infty, 1)\backslash\{0\}. \tag{22}$$

We want to study the implications of (22) on the optimal guarantee and in particular on its positiveness.

**Proposition 12** *Consider the problem (4), where the utility $u(.)$ is given by (22) with $\gamma = \overline{\gamma}$. Then the optimal guarantee has the following form:*

$$G_T^* = \frac{\max\{k - (1-\beta)X_0', 0\}}{\beta} \frac{H(T)^{\frac{1}{\gamma-1}}}{\mathbb{E}\left[H(T)^{\frac{\gamma}{\gamma-1}}\right]} \tag{23}$$

*Moreover, if market parameters are constant, i.e. $r(t) \equiv r, b(t) \equiv b, \sigma(t) \equiv \sigma$, then the guarantee can be expressed in terms of the basic securities as follows:*

$$G_T^* = \frac{\max\{k - (1-\beta)X_0', 0\}}{\beta} e^{\alpha_0 T} \prod_{i=1}^{n} \left(\frac{P_i(T)}{P_i(0)}\right)^{\alpha_i}, \tag{24}$$

*with*

$$\begin{aligned} \alpha_0 &= \left(r + \frac{1}{2}\|\theta\|^2 - \frac{1}{2}\left(\frac{\gamma\|\theta\|}{\gamma-1}\right)^2 - \frac{1}{\gamma-1}\theta'\sigma^{-1} vec\left(\frac{1}{2}\sum_{j=1}^{n}\sigma_{ij}^2 - b_i\right)\right) \in \mathbb{R} \\ \alpha &= -\frac{1}{\gamma-1}\left(\sigma^{-1}\right)'\theta \in \mathbb{R}^n, \end{aligned}$$

*where $vec[a_i]$ denotes the vector of $\mathbb{R}^n$ whose i-th component is $a_i$.*

**Proof.** See Appendix.

We stress that the parameter $k$ cannot be arbitrary, but it must satisfy (29), that is

$$(1-\beta)X_0' \leq k < X_0',$$

otherwise the optimal guarantee is zero a.s.: the idea is that if $k < (1-\beta)X_0'$, the preference for the surplus is so high that the contributor could accept (to pay) a negative guarantee, i.e. the positivity constraint is binding.

We also notice that the form (23) is the optimal form of a guarantee in very general market frameworks as introduced in Section 2.1 and in particular, in the interest rate framework models of Boulier et al. (2001) and Deelstra et al. (2001). In Deelstra et al. (2001), the different moments of the state-price density process can be found for the affine term structure case, so that it is possible to study more in detail this optimal guarantee.

However, it is difficult to explain to the concerned parties of a pension fund (client, manager, government,...) what the state-price density process stands for. In this respect, the optimal form (24) in case of constant parameters is very explicit and is expressed in terms of the prices of the risky assets.

In order to obtain an explicit formula in case of $\gamma \neq \overline{\gamma}$, we now look immediately at the case of constant market parameters.

**Proposition 13** *Consider the problem (4), where the utility $u(.)$ is given by (22) with $\gamma \neq \overline{\gamma}$, and the market parameters are constant.*

*Then the optimal guarantee has the following form:*

$$\begin{aligned} G_T^* &= \lambda^{\frac{1}{\overline{\gamma}-1}} e^{-\frac{1}{\overline{\gamma}-1}\left(r+\frac{1}{2}\|\theta\|^2\right)T-\frac{1}{\overline{\gamma}-1}\theta' W(T)} \\ &\quad -\frac{1-\beta}{\beta}\left(X_0'-k\right) e^{\left(r+\frac{1-2\gamma}{2(1-\gamma)^2}\|\theta\|^2\right)T-\frac{1}{\gamma-1}\theta' W(T)}, \end{aligned}$$

*where the parameter $\lambda$ is implicitly given by*

$$\begin{aligned} \tfrac{k-(1-\beta)X_0'}{\beta} &= \lambda^{\frac{1}{\overline{\gamma}-1}} e^{-\frac{\overline{\gamma}}{\overline{\gamma}-1}\left(r+\frac{1}{2}\|\theta\|^2\right)T+\frac{1}{2}\left(\frac{\overline{\gamma}}{\overline{\gamma}-1}\|\theta\|^2 T\right)^2} \Phi\left(\pm f(\lambda) \pm \tfrac{\overline{\gamma}}{\overline{\gamma}-1}\|\theta\|^2 T\right) \\ &\quad -\frac{1-\beta}{\beta \exp\left\{\frac{1}{2}\left(\frac{\gamma\|\theta\|}{\gamma-1}\right)^2 T\right\}}\left(X_0'-k\right) e^{\frac{1}{2}\left(\frac{\gamma}{\gamma-1}\|\theta\|^2 T\right)^2} \Phi\left(\pm f(\lambda) \pm \tfrac{\gamma}{\gamma-1}\|\theta\|^2 T\right), \end{aligned}$$

*where the sign $+$ (resp. $-$) is according to the case $\gamma > \overline{\gamma}$ (resp. $\gamma < \overline{\gamma}$), the function $f(\lambda)$ is defined by*

$$\begin{aligned} f(\lambda) &= \left[\ln\left(\frac{1-\beta}{\beta \mathbb{E}\left[H(T)^{\frac{\gamma}{\gamma-1}}\right]}\left(X_0'-k\right)\right)-\frac{1}{\overline{\gamma}-1}\ln\lambda\right]\frac{1}{\left(\frac{\gamma}{\gamma-1}-\frac{\overline{\gamma}}{\overline{\gamma}-1}\right)\|\theta\|^2 T} \\ &\quad -\left(r+\frac{1}{2}\|\theta\|^2\right)\frac{1}{\|\theta\|^2}, \end{aligned}$$

*and $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}t^2}dt$.*

**Proof.** See Appendix.

The expression of the optimal guarantee is semi-explicit in the case of $\gamma \neq \overline{\gamma}$ since $\lambda$ should first be determined numerically, which however turns out to be an easy task by using a numerical package as Matlab or Maple. As in the previous Proposition, the guarantee can be easily expressed in terms of the prices of the risky assets.

# 7 Conclusion

In a Defined Contribution framework, we obtained the optimal guarantee that maximizes the expected utility function of the benefit.

Moreover, we analyze the influence of the key parameters (the expected value of the client's terminal wealth and the sharing rule between the client and the pension fund).

We obtain that the sharing rule can be used not only as a way to incitate the pension fund, but also as a parameter amortizing the risks of the financial market for the client. In this sense, we think that the kind of pension funds we have studied makes a bridge between the two classical polar cases, the defined contribution and defined benefit pension funds schemes.

It would be very interesting to extend our analysis to the case of more sophisticated sharing rules. Nevertheless, we leave that point for further research, since our methodology cannot be applied to that extended framework, the main features of the surplus process we have used in this paper being not true in general.

Using numerical methods should permit to extend our analysis and to include it in a more general reflexion on the definition (by the state or a regulator) of optimal sharing rules.

**Acknowledgements**

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# 8 Appendix

**Proof of Proposition 5.**

Let us introduce the Hamiltonian $H(B_T, \lambda, \mu)$ associated to the problem (19):

$$\begin{aligned} H(B_T, \lambda, \mu, \omega) &= \mathbb{E}\left[u\left(B_T\right)\right] - \lambda\left(\mathbb{E}\left[H(T)B_T\right] - k\right) \\ &\quad + \int_\Omega \left(B_T(\omega) - \frac{1-\beta}{\beta}(X_0' - k)\varphi_T(\omega)\right)\mu(d\omega), \end{aligned}$$

where the Lagrangian multiplier $\lambda$ is a real number, while the Hamiltonian multiplier $\mu(d\omega)$ is a (positive) real measure.

Suppose that the minimum is attained at $B_T^*$. Then at $B_T^\varepsilon = B_T^* + \varepsilon\delta$ with $\delta \in L_+^p$ with $p > 2$ and $\varepsilon > 0$, one has

$$\frac{\partial}{\partial\varepsilon}\left[\mathbb{E}\left[u\left(B_T^\varepsilon\right)\right] - \lambda\left(\mathbb{E}\left[H(T)B_T^\varepsilon\right] - k\right) + \int_\Omega \left(B_T^\varepsilon(\omega) - \frac{1-\beta}{\beta}(X_0' - k)\varphi_T(\omega)\right)\mu(d\omega)\right]_{|\varepsilon=0} = 0.$$

Using the uniform integrability which is a consequence of the $L^p$ integrability with $p > 2$, one finds

$$\mathbb{E}\left[u'\left(B_T^\varepsilon\right)\delta - \lambda H(T)\delta\right]_{|\varepsilon=0} + \int_\Omega \delta(\omega)\mu(d\omega) = 0,$$

or equivalently, for all $\delta \in L_+^p$ with $p > 2$

$$\mathbb{E}\left[\delta\left(u'\left(B_T^*\right) - \lambda H(T) + \frac{\mu(d\omega)}{dP(\omega)}\right)\right] = 0,$$

therefore almost surely

$$B_T^*(\omega) = I\left(\lambda H(T,\omega) - \frac{\mu(d\omega)}{dP(\omega)}\right). \tag{25}$$

From (25) and the complementarity conditions, we can divide the set $\Omega$ into two subsets:

i) For $\omega \in \Omega$ such that the guarantee is strictly positive, i.e. $B_T(\omega) > \frac{1-\beta}{\beta}(X_0' - k)\varphi_T(\omega)$, then $\mu(d\omega) = 0$ and

$$B_T(\omega) = I\left(\lambda H(T,\omega)\right).$$

ii) For $\omega \in \Omega$ such that the guarantee is zero, i.e. $B_T(\omega) = \frac{1-\beta}{\beta}(X_0' - k)\varphi_T(\omega)$, then

$$B_T(\omega) = I\left(\lambda H(T,\omega) - \frac{\mu(d\omega)}{dP(\omega)}\right) > I\left(\lambda H(T,\omega)\right),$$

since $\mu(.)$ is a positive measure and $I$ is decreasing.

These two properties are equivalent with the following:

i) for $\omega \in \Omega$ such that $I\left(\lambda H(T,\omega)\right) > \frac{1-\beta}{\beta}(X_0' - k)\varphi_T(\omega)$, i.e.

$$\lambda < \frac{1}{H(T,\omega)}u'\left(\frac{1-\beta}{\beta}(X_0' - k)\varphi_T(\omega)\right),$$

then $B_T(\omega) = I(\lambda H(T,\omega))$;

ii) for $\omega \in \Omega$ such that $I(\lambda H(T,\omega)) < \frac{1-\beta}{\beta}(X_0' - k)\varphi_T(\omega)$, i.e.

$$\lambda > \frac{1}{H(T,\omega)} u'\left(\frac{1-\beta}{\beta}(X_0' - k)\varphi_T(\omega)\right),$$

then

$$\begin{aligned} B_T(\omega) &= I\left(\lambda H(T,\omega) - \frac{\mu(d\omega)}{dP(\omega)}\right) \\ &= \frac{1-\beta}{\beta}(X_0' - k)\varphi_T(\omega), \end{aligned}$$

which permits to compute all values of $\mu(.)$.

In conclusion, there exists a threshold for $\lambda$ such that

$$B_T^* = I(\lambda H(T))\, 1_{\lambda < \frac{1}{H(T)} u'\left(\frac{1-\beta}{\beta}(X_0' - k)\varphi_T\right)} + \frac{1-\beta}{\beta}(X_0' - k)\varphi_T 1_{\lambda > \frac{1}{H(T)} u'\left(\frac{1-\beta}{\beta}(X_0' - k)\varphi_T\right)},$$

which gives (20). ■

**Proof of Lemma 6.**

a) In order to prove the first assertion, we consider different possible cases and we check that the property is fulfilled in all cases.

a.i) For $\omega \in \Omega$ such that $B_T^{\lambda_2}(\omega) = I(\lambda_2 H(T,\omega))$, we have

$$\frac{1}{H(T,\omega)} u'\left(\frac{1-\beta}{\beta}(X_0' - k)\varphi_T(\omega)\right) > \lambda_2 > \lambda_1,$$

then $B_T^{\lambda_1}(\omega) = I(\lambda_1 H(T,\omega)) > I(\lambda_2 H(T,\omega)) = B_T^{\lambda_2}(\omega)$, since the function $I(.)$ is decreasing.

a.ii) For $\omega \in \Omega$ such that $B_T^{\lambda_2}(\omega) = \frac{1-\beta}{\beta}(X_0' - k)\varphi_T(\omega) = B_T^{\lambda_1}(\omega)$ (independent of $\lambda$) the statement is obvious.

a.iii) For $\omega \in \Omega$ such that $B_T^{\lambda_2}(\omega) = \frac{1-\beta}{\beta}(X_0' - k)\varphi_T(\omega)$ and $B_T^{\lambda_1}(\omega) = I(\lambda_1 H(T,\omega))$ we have

$$\lambda_2 > \frac{1}{H(T,\omega)} u'\left(\frac{1-\beta}{\beta}(X_0' - k)\varphi_T(\omega)\right) > \lambda_1,$$

that is

$$I(\lambda_1 H(T,\omega)) > \frac{1-\beta}{\beta}(X'_0 - k)\varphi_T(\omega) > I(\lambda_2 H(T,\omega)),$$

which gives the result.

b) We now prove the second assertion by considering the different possible cases:

b.i) For $\omega \in \Omega$ such that $B_T^{k_1}(\omega) = I(\lambda H(T,\omega)) = B_T^{k_2}(\omega)$, the statement is obvious.

b.ii) For $\omega \in \Omega$ such that

$$\begin{aligned} B_T^{k_1}(\omega) &= \frac{1-\beta}{\beta}(X'_0 - k_1)\varphi_T(\omega) \\ B_T^{k_2}(\omega) &= \frac{1-\beta}{\beta}(X'_0 - k_2)\varphi_T(\omega), \end{aligned}$$

the statement is also true.

b.iii) For $\omega \in \Omega$ such that $B_T^{k_1}(\omega) = \frac{1-\beta}{\beta}(X'_0 - k_1)\varphi_T(\omega)$ and $B_T^{k_2}(\omega) = I(\lambda H(T,\omega))$, then

$$\lambda \geq \frac{1}{H(T,\omega)} u'\left(\frac{1-\beta}{\beta}(X'_0 - k_1)\varphi_T(\omega)\right),$$

that is $\frac{1-\beta}{\beta}(X'_0 - k_1)\varphi_T(\omega) \geq I(\lambda H(T,\omega))$.

b.iv) The last possibility, i.e.

$$\begin{aligned} B_T^{k_1}(\omega) &= I(\lambda H(T,\omega)) \\ B_T^{k_2}(\omega) &= \frac{1-\beta}{\beta}(X'_0 - k_2)\varphi_T(\omega), \end{aligned}$$

implies

$$\frac{1-\beta}{\beta}(X'_0 - k_2)\varphi_T(\omega) \geq I(\lambda H(T,\omega)) \geq \frac{1-\beta}{\beta}(X'_0 - k_1)\varphi_T(\omega)$$

that is $k_1 \geq k_2$, which is a contradiction. ■

**Proof of Proposition 8.**

i) For $\omega \in \Omega$ such that $G_T^1(\omega) = 0$ and $G_T^2(\omega) \geq 0$, the statement is obvious.

ii) For $\omega \in \Omega$ such that

$$\begin{aligned} G_T^1(\omega) &= I(\lambda_1 H(T,\omega)) - \frac{1-\beta}{\beta}(X_0' - k_1)\varphi_T(\omega) \\ G_T^2(\omega) &= I(\lambda_2 H(T,\omega)) - \frac{1-\beta}{\beta}(X_0' - k_2)\varphi_T(\omega), \end{aligned}$$

since $\lambda_1 > \lambda_2$ by Corollary 7, then $I(\lambda_1 H(T,\omega)) < I(\lambda_2 H(T,\omega))$, and the statement follows by comparing term by term.

iii) The last possibility, i.e. $G_T^1(\omega) > 0$ and $G_T^2(\omega) = 0$ cannot happen with positive probability. In fact, it should be $\lambda_1 < \frac{1}{H(T)} u'\left(\frac{1-\beta}{\beta}(X_0' - k_1)\varphi_T\right)$ and $\lambda_2 > \frac{1}{H(T)} u'\left(\frac{1-\beta}{\beta}(X_0' - k_2)\varphi_T\right)$, but since $\lambda_1 > \lambda_2$, this implies $k_2 < k_1$, a contradiction. ■

**Proof of Lemma 9.**

Let us first introduce the parameter $\overline{\beta}$ given by

$$\overline{\beta} = \frac{1-\beta}{\beta} \tag{26}$$

and $B_T^{\overline{\beta}_1} = B_T^{\beta_1}$.

i) For $\omega \in \Omega$ such that $B_T^{\overline{\beta}_1}(\omega) = B_T^{\overline{\beta}_2}(\omega) = I(\lambda H(T,\omega))$, the statement is obvious.

ii) For $\omega \in \Omega$ such that

$$\begin{aligned} B_T^{\overline{\beta}_1}(\omega) &= \overline{\beta}_1 (X_0' - k)\varphi_T(\omega) \\ B_T^{\overline{\beta}_2}(\omega) &= \overline{\beta}_2 (X_0' - k)\varphi_T(\omega), \end{aligned}$$

we obtain $B_T^{\overline{\beta}_1}(\omega) > B_T^{\overline{\beta}_2}(\omega)$ since $\overline{\beta}(\beta)$ is decreasing, and thus also $B_T^{\beta_1}(\omega_i) > B_T^{\beta_2}(\omega_i)$.

iii) For $\omega \in \Omega$ such that $B_T^{\overline{\beta}_1}(\omega) = I(\lambda H(T,\omega))$ and $B_T^{\overline{\beta}_2}(\omega) = \overline{\beta}_2 (X_0' - k)\varphi_T(\omega)$, from (20) it follows

$$\begin{aligned} B_T^{\overline{\beta}_1}(\omega) &= I(\lambda H(T,\omega)) > \overline{\beta}_1 (X_0' - k)\varphi_T(\omega), \\ I(\lambda H(T,\omega)) &< \overline{\beta}_2 (X_0' - k)\varphi_T(\omega) = B_T^{\overline{\beta}_2}(\omega), \end{aligned}$$

which leads to $\beta_1 > \beta_2$, a contradiction.

iv) For $\omega \in \Omega$ such that $B_T^{\overline{\beta}_1}(\omega) = \overline{\beta}_1 (X_0' - k) \varphi_T(\omega)$ and $B_T^{\overline{\beta}_2}(\omega) = I(\lambda H(T,\omega))$, we have $I(\lambda H(T,\omega)) < \overline{\beta}_1 (X_0' - k) \varphi_T(\omega)$, and the proof is complete. ■

**Proof of Proposition 11.**

We suppose thus that $\overline{\beta}_1 > \overline{\beta}_2$, where $\overline{\beta}$ is given by (26), and where we denote also $G_T^{\overline{\beta}_1} = G_T^{\beta_1}$.

i) For $\omega \in \Omega$ such that $G_T^{\overline{\beta}_1}(\omega) = 0$ and $G_T^{\overline{\beta}_2}(\omega) \geq 0$, the statement is obvious.

ii) For $\omega \in \Omega$ such that

$$\begin{aligned} G_T^{\overline{\beta}_1}(\omega) &= I(\lambda_1 H(T,\omega)) - \overline{\beta}_1 (X_0' - k) \varphi_T(\omega) \\ G_T^{\overline{\beta}_2}(\omega) &= I(\lambda_2 H(T,\omega)) - \overline{\beta}_2 (X_0' - k) \varphi_T(\omega), \end{aligned}$$

since $\lambda_1 > \lambda_2$, then $I(\lambda_1 H(T,\omega)) < I(\lambda_2 H(T,\omega))$, and the statement follows by comparing term by term.

iii) The last possibility, i.e. $G_T^{\overline{\beta}_1}(\omega) > 0$ and $G_T^{\overline{\beta}_2}(\omega) = 0$ cannot happen with positive probability. In fact, it should be

$$\begin{aligned} \overline{\beta}_1 (X_0' - k) \varphi_T(\omega) &< I(\lambda_1 H(T,\omega)) \\ \overline{\beta}_2 (X_0' - k) \varphi_T(\omega) &\geqslant I(\lambda_2 H(T,\omega)) \end{aligned}$$

and $\lambda_1 > \lambda_2$, then $\overline{\beta}_2 > \overline{\beta}_1$, a contradiction. ■

**Proof of Proposition 12.**

We have to find $\lambda$ in (21), i.e. we have to solve

$$\mathbb{E}\left[H(T)\left(\max\left\{\left(I(\lambda H(T)) - \frac{1-\beta}{\beta}(X_0' - k)\varphi_T\right), 0\right\} + \frac{1-\beta}{\beta}(X_0' - k)\varphi_T\right)\right] = k$$

where $I(x) = x^{\frac{1}{\gamma-1}}$.

It is easy to check that the function (15) assumes the following form

$$\varphi_T = \frac{H(T)^{\frac{1}{\gamma-1}}}{\mathbb{E}\left[H(T)^{\frac{\gamma}{\gamma-1}}\right]},$$

so that the previous equation can be written as follows:

$$\frac{k-(1-\beta)X_0'}{\beta} = \mathbb{E}\left[\max\left\{\left(\lambda^{\frac{1}{\overline{\gamma}-1}}H(T)^{\frac{\overline{\gamma}}{\overline{\gamma}-1}} - \frac{1-\beta}{\beta}\left(X_0'-k\right)\frac{H(T)^{\frac{\gamma}{\gamma-1}}}{\mathbb{E}\left[H(T)^{\frac{\gamma}{\gamma-1}}\right]}\right),0\right\}\right]. \tag{27}$$

If $\gamma = \overline{\gamma}$, we obtain

$$\frac{k-(1-\beta)X_0'}{\beta} = \max\left\{\left(\lambda^{\frac{1}{\gamma-1}}\mathbb{E}\left[H(T)^{\frac{\gamma}{\gamma-1}}\right] - \frac{1-\beta}{\beta}\left(X_0'-k\right)\right),0\right\},$$

which admits the solution

$$\lambda = \left(\frac{k}{\mathbb{E}\left[H(T)^{\frac{\gamma}{\gamma-1}}\right]}\right)^{\gamma-1} \tag{28}$$

provided that

$$\lambda^{\frac{1}{\gamma-1}}\mathbb{E}\left[H(T)^{\frac{\gamma}{\gamma-1}}\right] - \frac{1-\beta}{\beta}\left(X_0'-k\right) \geq 0,$$

i.e.

$$(1-\beta)X_0' \leq k < X_0' \tag{29}$$

Now, if we plug (28) into (20) we obtain (23).

Now, let us consider the case with constant market parameters. From (1), it results

$$W(T) = \sigma^{-1}vec\left[\ln\left(\frac{P_i(T)}{P_i(0)}\right) + \left(\frac{1}{2}\sum_{j=1}^{n}\sigma_{ij}^2 - b_i\right)T\right],$$

so that

$$\begin{aligned} H(T) &= \exp\left\{-\left(r+\frac{1}{2}\|\theta\|^2\right)T - \theta' W(T)\right\} \\ &= \exp\left\{-\left(r+\frac{1}{2}\|\theta\|^2 + \theta'\sigma^{-1}vec\left(\frac{1}{2}\sum_{j=1}^{n}\sigma_{ij}^2 - b_i\right)\right)T - \theta'\sigma^{-1}vec\left[\ln\left(\frac{P_i(T)}{P_i(0)}\right)\right]\right\} \end{aligned}$$

and

$$\mathbb{E}\left[H(T)^{\frac{\gamma}{\gamma-1}}\right] = e^{-\frac{\gamma}{\gamma-1}\left(r+\frac{1}{2}\|\theta\|^2\right)T+\frac{1}{2}\left(\frac{\gamma\|\theta\|}{\gamma-1}\right)^2 T}.$$

We have

$$\begin{aligned}
\frac{H(T)^{\frac{1}{\gamma-1}}}{\mathbb{E}\left[H(T)^{\frac{\gamma}{\gamma-1}}\right]} &= e^{\left(r+\frac{1}{2}\|\theta\|^2-\frac{1}{2}\left(\frac{\gamma\|\theta\|}{\gamma-1}\right)^2\right)T-\frac{1}{\gamma-1}\theta' W(T)} \\
&= e^{\left(r+\frac{1}{2}\|\theta\|^2-\frac{1}{2}\left(\frac{\gamma\|\theta\|}{\gamma-1}\right)^2-\frac{1}{\gamma-1}\theta'\sigma^{-1}vec\left(\frac{1}{2}\sum_{j=1}^n\sigma_{ij}^2-b_i\right)\right)T-\frac{1}{\gamma-1}\theta'\sigma^{-1}vec\left[\ln\left(\frac{P_i(T)}{P_i(0)}\right)\right]},
\end{aligned}$$

and since

$$\theta'\sigma^{-1}vec\left[\ln\left(\frac{P_i(T)}{P_i(0)}\right)\right] = vec\left[\ln\left(\frac{P_i(T)}{P_i(0)}\right)\right]'\left(\sigma^{-1}\right)'\theta,$$

we obtain

$$\begin{aligned}
G_T^* &= \frac{\max\left\{k-(1-\beta)X_0',0\right\}}{\beta}\exp\left\{\left(r+\frac{1}{2}\|\theta\|^2-\frac{1}{2}\left(\frac{\gamma\|\theta\|}{\gamma-1}\right)^2\right.\right. \\
&\quad \left.\left.-\frac{1}{\gamma-1}\theta'\sigma^{-1}vec\left(\frac{1}{2}\sum_{j=1}^n\sigma_{ij}^2-b_i\right)\right)T-\frac{1}{\gamma-1}vec\left[\ln\left(\frac{P_i(T)}{P_i(0)}\right)\right]'\left(\sigma^{-1}\right)'\theta\right\},
\end{aligned}$$

which gives the statement (24), once the parameters $\alpha_0, \alpha$ are replaced. ■

**Proof of Proposition 13.**

If market parameters are constant, $H(T)$ is log-normally distributed:

$$H(T) = \exp\left\{-\left(r+\frac{1}{2}\|\theta\|^2\right)T-\theta' W(T)\right\},$$

and

$$\mathbb{E}\left[H(T)^{\frac{\gamma}{\gamma-1}}\right] = e^{-\frac{\gamma}{\gamma-1}\left(r+\frac{1}{2}\|\theta\|^2\right)T+\frac{1}{2}\left(\frac{\gamma\|\theta\|}{\gamma-1}\right)^2 T}.$$

From (27), we look for $\lambda$ such that

$$\begin{aligned}
\frac{k-(1-\beta)X_0'}{\beta} &= \mathbb{E}\left[\max\left\{\left(\lambda^{\frac{1}{\overline{\gamma}-1}}e^{-\frac{\overline{\gamma}}{\overline{\gamma}-1}\left(r+\frac{1}{2}\|\theta\|^2\right)T-\frac{\overline{\gamma}\|\theta\|^2 T}{\overline{\gamma}-1}N(0,1)}\right.\right.\right. \\
&\quad \left.\left.\left.-\frac{1-\beta}{\beta\mathbb{E}\left[H(T)^{\frac{\gamma}{\gamma-1}}\right]}\left(X_0'-k\right)e^{-\frac{\gamma}{\gamma-1}\left(r+\frac{1}{2}\|\theta\|^2\right)T-\frac{\gamma\|\theta\|^2 T}{\gamma-1}N(0,1)}\right),0\right\}\right],
\end{aligned}$$

where $N(0,1)$ denotes the Normal standard variable.

Suppose that $\gamma < \overline{\gamma}$ (the case $\gamma > \overline{\gamma}$ is perfectly analogous), so that the max is strictly positive iff

$$\begin{aligned} N(0,1) &> \left[\ln\left(\frac{1-\beta}{\beta\mathbb{E}\left[H(T)^{\frac{\gamma}{\gamma-1}}\right]}\left(X_0'-k\right)\right)-\frac{1}{\overline{\gamma}-1}\ln\lambda\right]\frac{1}{\left(\frac{\gamma}{\gamma-1}-\frac{\overline{\gamma}}{\overline{\gamma}-1}\right)\|\theta\|^2 T} \\ &\quad -\left(r+\frac{1}{2}\|\theta\|^2\right)\frac{1}{\|\theta\|^2} \\ &= :f(\lambda), \end{aligned}$$

then (27) becomes

$$\begin{aligned} \frac{k-(1-\beta)X_0'}{\beta} &= \frac{1}{\sqrt{2\pi}}\int_{f(\lambda)}^{+\infty}\lambda^{\frac{1}{\overline{\gamma}-1}}e^{-\frac{\overline{\gamma}}{\overline{\gamma}-1}\left(r+\frac{1}{2}\|\theta\|^2\right)T-\frac{\overline{\gamma}\|\theta\|^2 T}{\overline{\gamma}-1}x-\frac{1}{2}x^2}dx \\ &\quad -\frac{1}{\sqrt{2\pi}}\int_{f(\lambda)}^{+\infty}\frac{1-\beta}{\beta\mathbb{E}\left[H(T)^{\frac{\gamma}{\gamma-1}}\right]}\left(X_0'-k\right)e^{-\frac{\gamma}{\gamma-1}\left(r+\frac{1}{2}\|\theta\|^2\right)T-\frac{\gamma\|\theta\|^2 T}{\gamma-1}x-\frac{1}{2}x^2}dx. \end{aligned}$$

We complete the square and we arrive to

$$\begin{aligned} &\tfrac{k-(1-\beta)X_0'}{\beta} = \lambda^{\frac{1}{\overline{\gamma}-1}}e^{-\frac{\overline{\gamma}}{\overline{\gamma}-1}\left(r+\frac{1}{2}\|\theta\|^2\right)T+\frac{1}{2}\left(\frac{\overline{\gamma}}{\overline{\gamma}-1}\|\theta\|^2 T\right)^2}\Phi\left(-f(\lambda)-\tfrac{\overline{\gamma}}{\overline{\gamma}-1}\|\theta\|^2 T\right) \\ &-\tfrac{1-\beta}{\beta\mathbb{E}\left[H(T)^{\frac{\gamma}{\gamma-1}}\right]}\left(X_0'-k\right)e^{-\frac{\gamma}{\gamma-1}\left(r+\frac{1}{2}\|\theta\|^2\right)T+\frac{1}{2}\left(\frac{\gamma}{\gamma-1}\|\theta\|^2 T\right)^2}\Phi\left(-f(\lambda)-\tfrac{\gamma}{\gamma-1}\|\theta\|^2 T\right), \end{aligned}$$

which gives the result once the term $\mathbb{E}\left[H(T)^{\frac{\gamma}{\gamma-1}}\right]$ is replaced. ■

# References


[1] Bajeux-Besnainou, I., Jordan, J. V., Portait, R. 1998. Dynamic Asset Allocation for Stocks, Bonds and Cash over Long Horizons. Presented at the 1998 Southern Finance Association conference and at the October 1998 Bachelier Seminar in Paris.

[2] Boulier, J. F., Huang, S. J., Taillard, G., 2001. Optimal Management Under Stochastic Interest Rates: the Case of a Protected Pension Fund. Insurance: Mathematics and Economics 28, 173-189.

[3] Davis, E. P., 1995. Pension funds: retirement-income security and capital markets: an international perspective. (Oxford University Press).

[4] Deelstra, G., Grasselli, M., Koehl, P. F., 2000. Optimal Investment Strategies in a CIR Framework. Journal of Applied Probability 37, 1-12.

[5] Deelstra, G., Grasselli, M., Koehl, P. F., 2001. Optimal Investment Strategies for Defined Contribution Funds. Submitted.

[6] Holstrom, B., Milgrom, P., 1987. Aggregation and Linearity in the Provision of Intertemporal Incentives. Econometrica 55, 303-328.

[7] Jensen, B.A., Sørensen, C., 2001. Paying for minimum interest rate guarantees: Who should compensate who? European Financial Management 7 no. 2, 183-211.

[8] Karatzas, I., Shreve, S., 1990. Brownian Motion and Stochastic Calculus. (Springer Verlag).

[9] Merton, R., 1992. Continuous Time Finance. (Blackwell).

[10] Vasiček, O. A., 1977. An Equilibrium Characterization of the Term Structure. Journal of Financial Economics 5, 177-188.