Equilibrium Open Interest

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Abstract

Open interest in a financial contract describes the total number that are held long at the close of the exchange and is quoted at the end of each trading day in addition to daily closing prices and volume. Our paper investigates the risk-sharing rationale for option demand and the resulting shape of the open interest curve in calls across strikes in an equilibrium setup. We argue that agent's preferences over skewness drives equilibrium demand in options and that the observed shape of the open interest curve is the result of trade-offs between co-skewness and variance. We explain that open interest curves are sensitive to the distributional assumptions made for the underlying security; an analysis of open interest in addition to price and volume could therefore enrich current empirical studies.

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Open interest in a financial contract describes the total number that are held long at the close of the exchange and is quoted at the end of each trading day in addition to daily closing prices and volume. Our paper investigates the risk-sharing rationale for option demand and the resulting shape of the open interest curve in calls across strikes in an equilibrium setup. We argue that agent's preferences over skewness drives equilibrium demand in options and that the observed shape of the open interest curve is the result of trade-offs between co-skewness and variance. We explain that open interest curves are sensitive to the distributional assumptions made for the underlying security; an analysis of open interest in addition to price and volume could therefore enrich current empirical studies.

Keywords

Heterogeneity, Equilibrium, Demand, Open Interes, Co-Skewness

1 Introduction

Options have been a tremendous success story over the last quarter century: their volume grew at (compounded) annual rates of 127% from 1973 to 2000[1] and option like features are used to insure increasingly complex underlyings, e.g. credit risk, earthquakes, wheather. However, options are under scrutiny by regulators and the public as they have been involved in some of the largest corporate losses (Metallgesellschaft, Barings, LTCM, Enron). Despite their omnipresence, little is known in the finance literature about the driving forces behind derivatives demand; the literature also provides little practical advice as to how much options should be added to a stock portfolio and with which specifications.

In this paper we develop an equilibrium theory of why options are traded, describe agent's holdings of a portfolio of call options with differing strikes and relate this to the shape of the open interest curve across strikes[2]. To describe individual demand and the equilibrium allocation through explicit equations we make use of an asymptotic expansion.

Our asymptotic expansion decomposes the trade-offs between risk and return into two components: a component based on agent's risk-tolerance and means/covariances of the securities distribution as well as a component based on "skew-tolerance" and covariances/coskewness. In the market-clearing equilibrium the two components contribute separetely to demand and prices: The mean-variance component leads to a linear sharing of the stock with relative weights given by agent's risk-tolerances and does not lead to option demand. This result corresponds to previous analyses where agent's preferences are mean-variance, see, e.g. Huang and Litzenberger (1988). Agents have preference over skewness for a variety of reasons: For example, they care about liquidity effects that lead to higher premia/lower prices due

[1]According to the Office of the Comptroller of the Currency (OCC) total volume was 1,119,245 in 1973 and 726,727,939 in 2000; year end open interest was 245,825 in 1973 and 71,249,929 in 2000.

[2]Open interest in a financial assets denotes the total number of that contract that are held long at the daily close of the exchange and is quoted at the end of each day for all financial contracts traded.

to rare adverse economic shocks. Agents' preferences over skewness leads to non-linear sharing rules. Options are contracts with non-linear payoffs; agents can achieve such payoff profiles by trading options. This justifies that agents hold options in equilibrium.

We find that co-skewness of options with the stock determines the actual size of demand and thereby the shape of the open interest curve across strikes. In particular we argue that the presence of skewness risk is driving the demand in options. Other authors have documented that skewness risk is priced in the market. In addition to these theoretical contributions we also provide an algorithm for computing portfolio allocations of portfolio managers that want to hedge co-skewness risk.

Our paper extends the previous literature by providing an equilibrium based explanation for option demand and presents explicit quantitative results for the entire portfolio allocation in a stock and options. Previous studies have only looked at the underlying structural assumptions of the economy that lead to non-linear sharing rules in markets and derived quantitative predictions as to which side of the market an agent takes (Leland (1980), Brennan and Solanki (1981), Carr and Madan (2001), Franke, Stapelton, and Subrahmanyam (1998)).

In this paper we also link demand and prices in options whereas so far the literature takes one of three approaches: No-arbitrage pricing works only when the option has no social value. The technique of Cox and Huang (1989) works only in complete markets, but it is unclear if markets are effectively complete. Our technique does not require complete markets. The third technique is the pricing of a representative agent; however such an agent can not have demand in an asset that is in zero net-supply.

While our paper is focussed on option demand and the open interest curve across strikes, it also has pricing implications for options and the stock. In particular our paper finds theoretically that co-skewness should be priced. Harvey and Siddique (2000), Dittmar (2002) and Chung, Johnson and Schill (2002) provided empirical support for this.

The remainder of the paper is organized as follows: Section 2 presents our setup. Section 3 describes our expansion; in particular it discusses individual demand schedules, the equilibrium allocation and explains why option demand is driven by co-skewness and preferences over it. Section 4 discusses the shape of the open interest curve across strikes and relates it to co-skewness. Section 5 looks at a snapshot of open interest across strikes that is taken from the market and calibrates our setup to that data. Section 6 concludes the paper.

2 The Setup

We consider a two–period financial economy populated by two agents; they trade only today (date 0) and at date T. Agents can invest into a riskless bond with constant price over time[3] and $N+1$ risky securities: a stock (security 0) and N call options with maturity T written on that stock with strike prices $K_1, \ldots, K_N$.

We introduce a parameter $\epsilon \geq 0$ that parametrizes a series of economies and corresponding portfolio problems. In the ϵ-economy the payoff from the stock is random and given as

$$\Pi_0(\epsilon) = 1 + \xi_0 \epsilon;$$

here ξ_0 is a pure random component, i.e. $E[\xi_0] = 0$, such that $E[\Pi_0(\epsilon)] = 1$ for all ϵ. The distribution ξ_0 is known to both agents. We denote its date 0 price by

$$P_0(\epsilon) = 1 - \pi_0(\epsilon)\epsilon^2$$

and interpret $\pi_0(\epsilon)$ as the risk–premium in the ϵ-economy. The variance of the stock is $var(\Pi(\epsilon)) = var(\xi_0)\epsilon^2$ and we expand throughout the premium $\pi_0(\epsilon)$ by the ϵ^2 term. When $\pi_0(\epsilon)$ would be constant in ϵ this would model risk premia that are linear in variance; here we allow the premium $\pi_i(\epsilon)$ to depend on ϵ to capture nonlinear dependence on the variance. A characteristic feature of the expansion is also that $\epsilon = 0$ corresponds to an

[3] We do not analyze the consumption-savings decision here and therefore set the riskless rate equal to 0 for simplicity.

economy without risk such that the return form the bond and the stock then coincide, i.e., $\Pi_0(0) = P_0(0)$.

The distribution of payoffs at maturity from option i is determined through the stock as

$$\begin{aligned} \Pi_i(\epsilon) &= \epsilon(1+\xi_0-K_i)^+ \\ \text{and } P_i(\epsilon) &= E[\epsilon(1+\xi_0-K_i)^+]-\pi_i(\epsilon)\epsilon^2. \end{aligned}$$

Note that the variance of options is $var(\Pi_i(\epsilon)) = var((1+\xi_0-K_i)^+)\epsilon^2$; our expansion of the option payoff is set up such that the relation between stock and option variance

$$\frac{var(\Pi_i(\epsilon))}{var(\Pi_0(\epsilon))} = \frac{var((1+\xi_0-K_i)^+)}{var(\xi_0)} \tag{1}$$

is constant in ϵ. Also the cumulative probability of exercise is unaffected by ϵ. Basically, as we expand the distribution of stock prices we also expand strikes. Our expansion may initially seem odd, but we get a standard option for all $\epsilon > 1$.

Setting the $\epsilon = 0$ expansion term for the options ensures that in the no-risk economy ($\epsilon = 0$) the return on bond, stock and the call options coincide, i.e., $\Pi_i(0) = P_i(0)$ for all $i = 0, \ldots, N$. In particular here we force the payoff and the price of the derivative to be zero when $\epsilon = 0$. Throughout we expand the premium $\pi_i(\epsilon)$ by the ϵ^2 term; again we allow the premium $\pi_i(\epsilon)$ to depend on ϵ to capture nonlinear dependence of the premium $\pi_i(0)$ on the variance.

Each agent is endowed with 1/2 units of the stock, i.e. the total supply consists of one unit of stock, which is infinitely divisible. All call options are in zero net–supply. In each ϵ-economy agents pursue trading strategies over time: between 0 and T agent i invests \$$b_i(\epsilon)$ of his initial wealth $W_{0i}(\epsilon) = \frac{P_0(\epsilon)}{2}$ into the riskless security and holds $d_{ij}(\epsilon)$ units of risky security j ($0 \leq j \leq N$), i.e. $W_{0i}(\epsilon) = b_i(\epsilon) + \sum_{j=0}^{N} d_{ij}(\epsilon)P_j(\epsilon)$. We re-express this as $b_i(\epsilon) = W_{0i}(\epsilon) - \sum_{j=0}^{N} d_j(\epsilon)P_j(\epsilon)$ and derive his total payoff (wealth) at date T as $W_{Ti}(\epsilon) = W_{0i}(\epsilon) + \sum_{j=0}^{N} d_{ij}(\epsilon) \cdot (\Pi_j(\epsilon) - P_j(\epsilon))$. We assume he does not consume today and his date T preferences over wealth can be represented by von Neumann-Morgenstern utility functions $E[u_i(W_{Ti}(\epsilon))]$, where u_i is an increasing and strictly concave function. The agent then chooses the strategy that maximizes his expected utility

$$E\left[u_i\left(W_{Ti}(\epsilon)\right)\right] = E\left[u_i\left(W_{0i}(\epsilon) + \sum_{j=0}^{N} d_{ij}(\epsilon)\left(\Pi_j(\epsilon) - P_j(\epsilon)\right)\right)\right],$$

over all holdings $d_{ij}(\epsilon)$ in the risky securities. Both agents trade competitively; we derive both agents' demand and prices in each security by the following equilibrium concept:

Definition 1 *A financial equilibrium in the ϵ-economy consists of prices $P_j(\epsilon)$ for the available assets, and portfolio demand vectors $(b_i(\epsilon), d_i(\epsilon))$ for both agents, such that $(b_i(\epsilon), d_i(\epsilon))$ maximizes agent i's utility, and stock and option markets clear, i.e. $d_{10}(\epsilon) + d_{20}(\epsilon) = 1$ and $d_{1i}(\epsilon) + d_{1i}(\epsilon) = 0$ $(i = 1, \dots, N)$.*

In each ϵ-economy open interest in every option corresponds to the absolute value of either agent's demand in that option. The setup will therefore provide us with a risk–sharing prediction of the open interest in options.

Our goal is to expand demand, d_i, and premia, π_i, into series with respect to ϵ. The expansion will be around $\epsilon = 0$, i.e. in vector notation we aim for agent i at an expansion

$$d_i(\epsilon) = d_i(0) + d_i'(0)\epsilon + \dots \text{ and } \pi_i(\epsilon) = \pi_i(0) + \pi_i'(0)\epsilon + \dots, \qquad (2)$$

where $\pi_i'(\epsilon) = \frac{\partial \pi}{\partial \epsilon}(\epsilon)$, and $d_i'(\epsilon) = \frac{\partial d_i}{\partial \epsilon}(\epsilon)$.

For each agent $i = 1, 2$ we define the $(N+1)$-dimensional function $H(\epsilon)$ by

$$H_{ij}(\epsilon) = \frac{1}{\epsilon} \cdot \frac{\partial E[u_i(W_{Ti})]}{\partial d_{ij}} = E\left[\frac{\partial u_i}{\partial W}(W_{Ti}(\epsilon)) \cdot (\xi_j + \pi_j(\epsilon)\epsilon)\right]. \qquad (3)$$

The *first order conditions*[4] of agent i in the ϵ economy are then $H_i(\epsilon) = 0$. Our idea is then to derive agent's holdings through an application of the Implicit Function Theorem; taking derivatives with respect to ϵ we get

[4]To simplify the exposition we divided by ϵ when defining H_i in equation 3. A repeated application of our procedure would yield the same result.

$$0 = \frac{\partial H_i}{\partial d_i} \cdot \frac{\partial d_i}{\partial \epsilon} + \frac{\partial H_i}{\partial \epsilon}.$$

The way our expansion is set up we have $\frac{\partial H_i}{\partial d_i}(0) = 0$ since in the $\epsilon = 0$ economy all assets coincide and therefore their demand is indeterminate. (A proof is provided in the appendix.) In the spirit of l'Hopital's rule we then "choose" $\frac{\partial H_i}{\partial \epsilon}(0)$ such that it is also equal to zero.

This small–noise expansion is a perturbation technique that was first introduced by Samuelson (1970) (see also Merton and Samuelson (1974)). While Samuelson's analysis is asymptotically valid only for the zero–order term, Judd and Guu (1999), using bifurcation methods, develop this analysis to a technique that is asymptotically valid for all terms in the polynomial expansion. We follow here their procedure; for completeness we quote their generalized Implicit Function Theorem in the appendix.

The first three derivatives of agent i's utility function are denoted $u_i' = \frac{\partial u_i}{\partial W}$, $u_i'' = \frac{\partial^2 u_i}{\partial W^2}$, and $u_i''' = \frac{\partial^3 u_i}{\partial W^3}$, which are all evaluated at his "safe" wealth $W_{0i}(0) = \frac{P_0(0)}{2} = \frac{1}{2}$ in the ϵ-economy. Moreover, for agent i we define the indices

$$\tau_i = -\frac{u_i'}{u_i''}, \;\; \rho_i = \frac{\tau_i^2}{2} \frac{u_i'''}{u_i'} = \frac{1}{2} \frac{u_i'}{u_i''} \frac{u_i'''}{u_i''}. \tag{4}$$

where τ_i is his risk-tolerance and ρ_i is his skew-tolerance.

We also denote by V the $(N+1) \times (N+1)$ dimensional *variance/covariance matrix* of pure random components (of the stock and all options), by χ the $N+1$ dimensional *co-skewness vector*, and by ζ_i the $N+1$ dimensional *(third order) co-moment vector* [5], i.e.

$$V_{jk} = E[\xi_j \xi_k], \chi_j = E[\xi_0^2 \xi_j], \zeta_{ij} = \sum_{k,l=0}^{N} d_{ik}(0) d_{il}(0) E[\xi_j \xi_k \xi_l]. \tag{5}$$

for agent $i = 1, 2$, securities $j, k = 1, ..., N$. These terms will describe demand and will be explained in due course. For technical reasons we assume throughout that the variance-covariance matrix V of the $N + 1$ securities

[5] Harvey and Siddique (2000) and Dittmar (2002) refer to co-skewness in extensions of the CAPM framework. We will not pursue this interpretation here.

is not singular, i.e. that V is invertible. (This assumption is weaker than assuming that a security could not be redundant. Redundant securities need to be excluded, since its demand would be indeterminate.)

3 The Small-Noise Expansion

3.1 Individual Demand

In our asymptotic analysis we focus on the first term that is not equal to zero; in our equilibrium analysis in the next subsection we will find $d_i(0) = 0$ for options and therefore the next higher order term $d_i'(0)$ will need to be analyzed. Applying theorem 3 in Judd and Guu (1999) we prove (see appendix 7):

Theorem 1 *Agent i's demand vector is asymptotically described as*

$$d_i\left(0\right) = \tau_i \cdot \left(V^{-1}\pi\left(0\right)\right) \text{ and } d_i'\left(0\right) = \tau_i \cdot \left(V^{-1} \cdot \pi'\left(0\right)\right) + \frac{\rho_i}{\tau_i}\left(V^{-1} \cdot \zeta_i\right). \quad (6)$$

To interpret these demand terms let us first look at the case where $N = 0$, i.e. only the stock and the riskless investment, but no options, can be traded. Then $\zeta_0 = d_{i0}^2(0)\chi_0$ (in theorem 1) and the demand equation (6) for agent i reduces to

$$d_{i0}(0) = \tau_i \frac{\pi_0}{V_{00}} \text{ and } d_{i0}'(0) = \tau_i \frac{\pi_0'(0)}{V_{00}} + \frac{\rho_i}{\tau_i} d_{i0}^2(0) \frac{\chi_0}{V_{00}}.$$

We interpret the asymptotic expansion $d_{i0}(\epsilon) = d_{i0}(0) + d_{i0}'(0)\epsilon + \mathcal{O}(\epsilon^2)$ as follows: The $d_{i0}(0)$ term consists of the premium, $\pi_0(0)$, standardized by the variance, V_{00}, of the stock risk; the agent's position is the product of his risk-tolerance τ with this premium per unit risk. The agent here cares how his wealth risk $\eta_i = d_{i0}(0)\xi_0$ covaries with stock risk. This result is common to economies in which an agent's preferences can be summarized in terms of means and variances, see, Huang and Litzenberger (1988).

The $d_{i0}'(0)$ term takes the third moment χ_0 and the premium correction term $\pi_0'(0)$ into account. The premium component $\pi_0'(0)$ induces demand by a similar rationale as the $d_{i0}(0)$ term. We interpret the χ based term as a correction term to the "mean-variance" demand term $d_{i0}(0)$: there the agent cared how his wealth risk $\eta_i = d_{i0}(0)\xi_0$ covaries with stock risk. For the correction term the agent is interested in how his squared wealth risk, $\eta_i^2 = d_{i0}^2(0)\xi_0^2$, covaries with the stock risk: he cares about $Cov(\eta^2, \xi_0) = E[d_{i0}^2(0)\xi_0^3] = \zeta_0 = d_{i0}^2(0)\chi_0$. The agent standardizes with respect to the risk V_{00} of a unit stock position and then adjusts his position relative to the product of this covariance with $\frac{\rho_i}{\tau_i}$. Kimball (1990) refers to $\frac{\rho_i}{\tau_i} = -\frac{u_i'''}{2u''}$ as prudence. We therefore interpret the $\frac{\rho_i}{\tau_i} d_{i0}^2(0)\frac{\chi_0}{V_{00}}$ term as a precautionary adjustment.

In the case where multiple options can be traded agents' zero-order risk η_i becomes $\eta_i = \sum_{j=0}^{N} d_{ij}(0)\xi_j$, the third moment becomes a co-moment vector, the standardization w.r.t. variance becomes a matrix V^{-1} and so agent i's demand (an $N+1$ dimensional vector) becomes

$$d_i(\epsilon) = \tau_i V^{-1}\pi(0) + \left(\tau_i V^{-1}\pi'(0) + \frac{\rho_i}{\tau_i}V^{-1}\zeta_i\right)\epsilon + \mathcal{O}\left(\epsilon^2\right).$$

Again, the $d_i(0) = \tau_i V^{-1}\pi(0)$ term corresponds to a premium induced holding position in the N assets. We continue to interpret this part of demand as the result of linear regression of a security's risk ξ_i on agent i's (zero-order) wealth risk $\eta_i = \sum_{j=0}^{N} d_{ij}(0)\xi_j$. Since $\zeta_i = E[\eta_i^2\xi_i] = Cov(\eta^2, \xi_i)$ we interpret the $d_i'(0)$ term as a correction to that zero–order wealth risk in which the agent cares how his squared wealth risk covaries with security i. Also via the $\tau_i V^{-1}\pi'(0)$ term the agent aims in the $d_i'(0)$ term to gain from the premium.

3.2 The Equilibrium Allocation

In equilibrium, agents have to agree on a market clearing price, i.e. $d_{10}(\epsilon) + d_{20}(\epsilon) = 1$ for the stock and $d_{1i}(\epsilon) + d_{2i}(\epsilon) = 0$ for the options ($i = 1, \ldots, N$). For the zero and first order expansion terms of $d_i(\epsilon)$ in our series expansion this translates into

$$d_{10}(0) + d_{20}(0) = 1, d_{1i}(0) + d_{2i}(0) = 0, \text{ and } d_1'(0) + d_2'(0) = 0.$$

for options $i = 1, \ldots, N$. Appendix 7.3 proves:

Theorem 2 *The premium of asset $i = 0, \ldots, N$ is:*

$$\pi_i(\epsilon) = \frac{1}{\tau_1 + \tau_2} V_{0i} - \frac{\rho_1 \tau_1 + \rho_2 \tau_2}{(\tau_1 + \tau_2)^3} \chi_i \epsilon + \mathcal{O}\left(\epsilon^2\right) \tag{7}$$

and the first agent's demand is:

$$\begin{aligned} d_{10}(\epsilon) &= \frac{\tau_1}{\tau_1 + \tau_2} - \tau_1 \tau_2 \frac{\rho_1 - \rho_2}{(\tau_1 + \tau_2)^3} \cdot \left(V^{-1}\chi\right)_0 \epsilon + \mathcal{O}\left(\epsilon^2\right) \text{ for the stock, and} \\ d_{1i}(\epsilon) &= -\tau_1 \tau_2 \frac{\rho_1 - \rho_2}{(\tau_1 + \tau_2)^3} \cdot \left(V^{-1}\chi\right)_i \epsilon + \mathcal{O}\left(\epsilon^2\right) \text{ for options } (i = 1, \ldots, N). \end{aligned}$$

Note that agent's zero order wealth risk therefore simplifies to $\eta = d_{i0}(0)\xi_0$ and that this implies that in the correction term $d_i'(0)$ in which he cares about how his wealth squared zero-order wealth risk covaries with that of security i becomes $\zeta_i = Cov(\eta^2, \xi_i) = d_{i0}^2(0)\chi_i$. In particular this implies that cross-risk terms cancel out in pricing; however they matter for demand to achieve the optimal risk position. In particular the third moment here has simplified, in equilibrium, to $\zeta = \chi$, since $d_i(0) = 0$ for options.

Note also that we have separation of tastes and moments of the distribution in $d_i(0)$ and $d_i'(0)$. This is a convenient feature for empirical studies: in a regression of the premium on covariance V_{0i} and co-skewness χ_i our approach predicts that the coefficients would be the inverse of the weighted risk-tolerance of the agents $\frac{1}{\tau_1+\tau_2}$ and the weighted skew-tolerances $\frac{\rho_1\tau_1+\rho_2\tau_2}{(\tau_1+\tau_2)^3}$.

Theorem 2 implies that for options the zero order equilibrium demand term is zero since this term corresponds to a mean-variance framework; this part of demand is in line with common knowledge about mean-variance frameworks: it is well known that in such frameworks two-fund separation holds, i.e. agents hold the bond and the market portfolio. Options are not contained in the "market" portfolio, since they are in zero net-supply. Therefore, in *our* economy the "market" portfolio consists of one unit of the stock only.

Based on theorem 2 we find that prices are (up to second order terms) equal to

$$\begin{aligned} P_i(\epsilon) &= E[\Pi_i(\epsilon)] - \pi_i(0)\epsilon^2 - \pi_i'(0)\epsilon^3 + \mathcal{O}\left(\epsilon^4\right) \\ &= E[\Pi_i(\epsilon)] - \frac{1}{\tau_1 + \tau_2} V_{0i}\epsilon^2 + \frac{\rho_1\tau_1 + \rho_2\tau_2}{(\tau_1 + \tau_2)^3} \chi_i \epsilon^3 + \mathcal{O}\left(\epsilon^4\right). \end{aligned}$$

Here the $\frac{1}{\tau_1+\tau_2}V_{01}\epsilon^2$ term is exactly the pricing term that would result in a CAPM world. However, in addition we have the $\pi'(0) = \frac{\rho_1\tau_1+\rho_2\tau_2}{(\tau_1+\tau_2)^3}\chi_i\epsilon^3$ term which depends on preferences $\tau_1, \tau_2, \rho_1, \rho_2$ and co-skewness χ_i. Therefore, in our models co-skewness risk is priced. This confirms from a theoretical perspective the empirical evidence in Harvey and Siddique (2000), Dittmar (2002) and Chung, Johnson and Schill (2002)[6]. We will not pursue this relation further, since our focus is on option demand and open interest; we elaborate in the next subsection that co-skewness induces option demand.

3.3 Equilibrium Option Demand and Co-skewness

Theorem 3 *With one option, the first agent's option demand is given by*

$$\begin{aligned} d_{11}(\epsilon) &= -\tau_1\tau_2\frac{\rho_1-\rho_2}{(\tau_1+\tau_2)^3}\frac{V_{01}\chi_0 - V_{00}\chi_1}{\det(V)}\epsilon + \mathcal{O}\left(\epsilon^2\right) \\ &= -\tau_1\tau_2\frac{\rho_1-\rho_2}{(\tau_1+\tau_2)^3}\frac{E\left[\left(\frac{V_{01}}{V_{00}}\xi_0 - \xi_1\right)\xi_0^2\right]}{V_{11}-\frac{V_{01}}{V_{00}}} \end{aligned}$$

Note that $\frac{V_{01}}{V_{00}}\xi_0 - \xi_1$ describes the component of the option risk ξ_1 that is orthogonal to (unhedgeable by) the stock and that $var(\frac{V_{01}}{V_{00}}\xi_0 - \xi_1) = V_{11} - \frac{V_{01}^2}{V_{00}} = \frac{1}{V_{00}}det(V)$. Therefore demand is given of the covariance of the unhedgeable component standardized by the risk of that part.

The demand equation with multiple options is a generalization of this: since it is a correction to the zero-order stock holdings we replace the skewness by co-skewness of the option with the stock and the variance in the denominator becomes the inverse of the variance-covariance matrix. The equation then states that demand is driven by coskewness divided by the variance of the part of the option's risk that is not spanned by option i.

Note that option demand is driven by $V^{-1}\chi$ and skew-tolerances, whereas their prices are given by risk-tolerances, skew-tolerances and χ. An analysis of option demand has therefore the potential to single out these terms better.

We will now assume that the stock distribution is trinomial, i.e. ξ_0 takes randomly one of exactly three values $(-\Delta, 0, -\Delta)$; the probability of these

[6]Since our market portfolio consists of the stock, only, our definition of co-skewness and their coincide

is supposed to be $(p, 1-2p, p)$. (For simplicity we take the distribution to be symmetric.) We then calculate $E[\xi_0] = 0, E[\xi_0^2] = 2\Delta^2 p, E[\xi_0^3] = 0$. The market with only the stock and the bond is incomplete; we introduce exactly one option and thereby make the market complete. For simplicity of exposition we take its strike price $K = 1$. The option then only pays in the highest state Δ and calculate $E[(1+\xi_0-K)^+] = \Delta p$, $E[((1+\xi_0-K)^+)^2] = \Delta^2 p$, $E[(1+\xi_0)\cdot(1+\xi_0-K)^+)] = \Delta(\Delta+1)p$, and $E[(1+\xi_0)^2(1+\xi_0-K)^+] = (1+\Delta)^2\Delta p$. Then we can calculate the variance-covariance matrix V and the inverse V^{-1} of V as

$$V^{-1} = \frac{1}{\det(V)}\begin{pmatrix} V_{11} & -V_{01} \\ -V_{01} & V_{00} \end{pmatrix} = \frac{1}{\det(V)}\begin{pmatrix} \Delta^2 p(1-p) & -\Delta^2 p \\ -\Delta^2 p & 2\Delta^2 p \end{pmatrix}. \tag{8}$$

Proposition 4 *For two random variables X, Y we have $E[(X-E[X])^3] = E\left[(X)^3]]\right] - E\left[X]]\right]^3 - 3E\left[X]]\right]\cdot Var\left(X\right)$ and $E\left[(X-E[X])^2\cdot(Y-E[Y])\right] = E[X^2Y] - E\left[X^2\right]\cdot E\left[Y\right] - 2E[X]\cdot Cov(X,Y)$.*

Also we calculate $\chi_0 = 0$ (since the distribution is symmetric) and $\chi_1 = (1+\Delta)^2 p - 2\Delta^3 p^2 - 2\Delta^2 p = \Delta(1+\Delta^2 p) - 2\Delta^3 p^2$ Using this inverse of the variance/covariance matrix we find:

$$V^{-1}\chi \;=\; \frac{\Delta^2 p}{det(V)}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{\chi_1}{\Delta^2 p(1-2p)}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

since $det(V) = 2\Delta^4 p^2(1-p) - \Delta^4 p^2 = \Delta^4 p^2(1-2p)$.

4 Equilibrium Open Interest Across Strikes and Co-skewness

Equilibrium demand in options arises from skew-tolerance and co-skewness as long as[7] $\rho_1 \neq \rho_2$. In our two agent setup open interest is the absolute value

[7]In general we expect $\rho_1 \neq \rho_2$ and demand in options to arise. Note that when both agents are identical, both in their initial endowment and their risk-preferences we expect that no option demand arises. Our demand equation for $d'(0)$ confirms our intuition, since for identical agents $\rho_1 = \rho_2$, so that $d'(0) = 0$.

of either agent's demand, i.e. according to theorem 2 we have separattion of tastes and distributional characteristics: we have that the open interest is

$$\tau_1\tau_2\frac{|\rho_1-\rho_2|}{(\tau_1+\tau_2)^3}\cdot\left|V^{-1}\chi\right|\epsilon+\mathcal{O}(\epsilon^2)=\text{tastes}\cdot|\phi|_i$$

Risk-preferences then enter only as a multiplicative factor for the demand in all contracts ($\tau_1\tau_2\frac{|\rho_1-\rho_2|}{(\tau_1+\tau_2)^3}$); the relative size (shape of the open interest curve) is determined through (co)variances and skewness terms by $\phi=V^{-1}\chi$.

This implies for all economies with $\epsilon>0$ that

$$\frac{d_{1i}(\epsilon)}{d_{1j}(\epsilon)}=\frac{\phi_i}{\phi_j}+\mathcal{O}(\epsilon),\frac{|d_{1i}(\epsilon)|}{|d_{1j}(\epsilon)|}=\frac{|\phi_i|}{|\phi_j|}+\mathcal{O}(\epsilon),$$

as long as $d_j(\epsilon)\neq 0$. Therefore the relative size of demand and open interest across strikes neither depends on the particular ϵ nor on agent's tastes; it can be described entirely through ϕ. The shape of the open interest curve in our model will therefore be characterized by distributional characteristics (truncated moments of the underlying stock distribution.)

To gain further insights we will now, entirely for illustrative purposes, assume that ξ_0 is uniformly distributed over the interval $(1-\epsilon,1+\epsilon)$, i.e. $1+\xi_0\epsilon$ is uniformly distributed over the interval $(0,2)$. Throughout we will restrict ourselves to strikes K_i between 0 and 2. Straightforward calculations reveal that $E[1+\xi_0]=1, Var(1+\xi_0)=\frac{1}{3}$, $E[(\Pi_0-K_i)^+]=\frac{(K_i-2)^2}{4}$, $E[\Pi_0(\Pi_0-K_i)]=\frac{(K_i-2)^2(K_i-4)}{12}$, $E[((\Pi_0-K_i)^+)^2]=-\frac{K_i-2)^3}{6}$ and for $K_i<K_j$ that $E[(\Pi_0-K_i)^+(\Pi_0-K_j)^+]=-\frac{(K_j-2)^2(3K_i-K_j-4)}{12}$. Also we calculate $E[\Pi_0^2(\Pi_0-K_i)^+]=\frac{(K_i^2+2)(-2+K_i)^2}{24}$.

Note that $\chi_0=0$, since the distribution is symmetric. For all other terms we use proposition 4 to get:

Theorem 5 *We have* $V_{0i}=\frac{(K_i+1)(-2+K_i)^2}{12}$, $V_{ii}=-\frac{(3K_i+2)(-2+K_i)^3}{48}$, $V_{ij}=-\frac{(-2+K_j)^2(-4K_j-4+3K_i^2)}{48}$ *and* $V_{00}=\frac{1}{3}$. *Skewness is* $\chi_0=0$, $\chi_i=\frac{K_i^2(-2+K_i)^2}{24}$.

Whenever K tends to 0 the option becomes more like the stock. The agent try to leverage his position in these "close" assets by holding a long position in one of them and a short in the other. However we expect the utility gain from such leveraging to be small compared to transaction costs

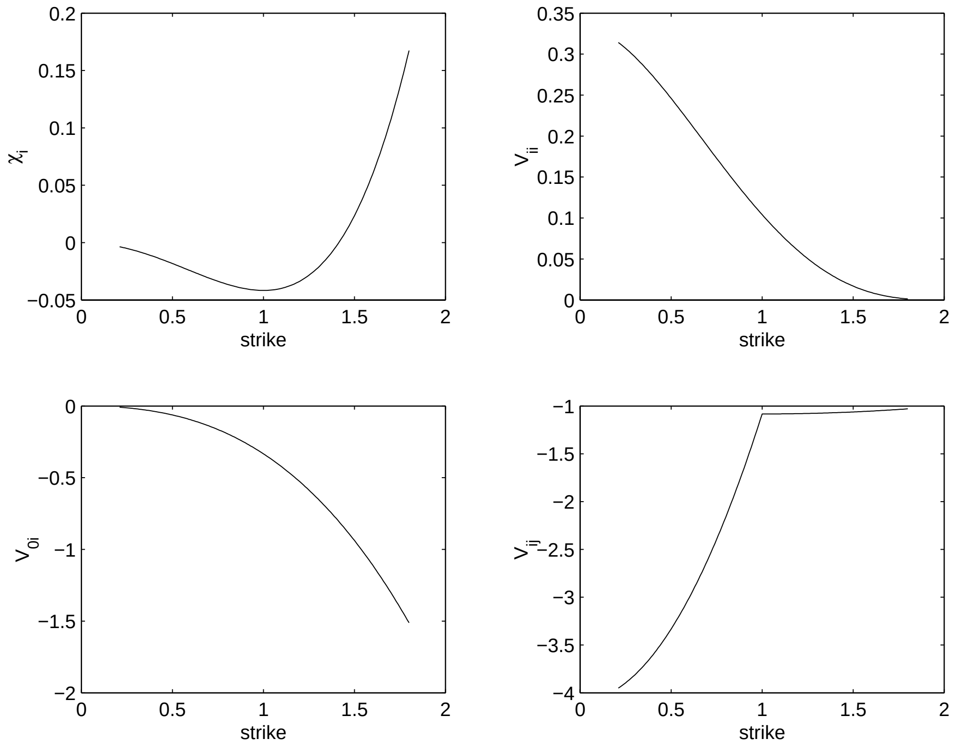

Figure 1: Moments for the uniform distribution.

agents would have to pay for such behavior. A similar problem arise on the right side: when K tends to 2 the option becomes similar to an asset paying nothing and therefore demand can exhibit "odd" for those options, too.We therefore focus on the center behavior.

To further analyze open interest we will look at an odd number of strikes with the center strike at 1 and such that the distance $|K_i - K_{i-1}| = \delta$ is constant (uniform strike grid). We will now look at the setup in which three options can be traded. To simplify our presentation we look only at the symmetric case where the middle option has a strike of one and the other two are symmetric around that one, i.e. we set $K_1 = 1-\delta, K_2 = 1$, $K_3 = 1+\delta$ and vary δ between 0 and 1. We then get

$$\begin{pmatrix} d_1'(0) \\ d_2'(0) \\ d_3'(0) \end{pmatrix} = \text{tastes} \cdot \begin{pmatrix} -\frac{1}{4\delta(-1+\delta)} \\ \frac{2\delta^2+3\delta-1}{2\delta} \\ -\frac{1}{4\delta(-1+\delta)} \end{pmatrix}$$

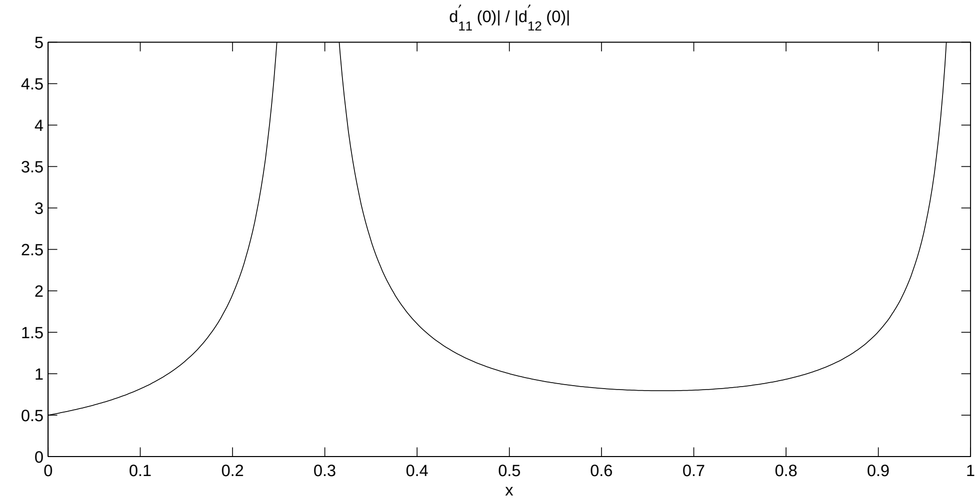

Figure 2: Ratio of demand when 3 options are traded.

Proposition 6 *Up to higher order terms in ϵ the open interest curve is symmetric, i.e. $|d_1'(0)| = |d_3'(0)|$ and open interest peaks at the center strike, i.e. $|d_1'(0) < d_2'(0)|$, if and only if either* $0.5 < \delta < 0.8229$ *or* $0 < \delta < 0.1304$. *Also:* $\lim_{\delta\to 0}\left|\frac{d_1'(0)}{d_2'(0)}\right| = 0.5$ *and* $\lim_{\delta\to 1}\left|\frac{d_1'(0)}{d_2'(0)}\right| = \infty$.

With four options we get

$$\begin{pmatrix} d_1'(0) \\ d_2'(0) \\ d_3'(0) \\ d_4'(0) \end{pmatrix} = \text{tastes} \cdot \begin{pmatrix} -\frac{(3\delta^2-4)(\delta-2)^2}{4(-10+3\delta)\delta(3\delta-2)} \\ \frac{1}{4\delta(9\delta^3-34\delta^2-36\delta+8)(-10+3\delta)} \\ \frac{1}{4\delta(9\delta^3-34\delta^2-36\delta+8)(-10+3\delta)} \\ -\frac{(3\delta^2-4)(\delta-2)^2}{4(-10+3\delta)\delta(3\delta-2)} \end{pmatrix}$$

(Here, again we took strikes symmetrically around 1, i.e. $K_1 = 1 - \frac{3}{2}\delta$, $K_2 = 1 - \frac{1}{2}\delta$, $K_3 = 1 + \frac{1}{2}\delta$, $K_4 = 1 + \frac{3}{2}\delta$.)

Proposition 7 *Up to higher order terms in ϵ the open interest curve is symmetric, i.e. $|d_1'(0)| = |d_4'(0)|$, $|d_2'(0)| = |d_3'(0)|$ and open interest peaks at the center strike, i.e. $|d_1'(0)| < |d_2'(0)|$ if and only if either* $0.4 < \delta < 0.5752$ *or* $0.7104 < \delta < 1$. *Also:* $\lim_{\delta\to 0}\left|\frac{d_1'(0)}{d_2'(0)}\right| = 1$ *and* $\lim_{\delta\to 1}\left|\frac{d_1'(0)}{d_2'(0)}\right| = 1$.

We would like to point out that for a picture where one strike would be approaching 0 or 2 demand and open interest for that option would tend to infinity so that ultimately the picture would be destroyed.

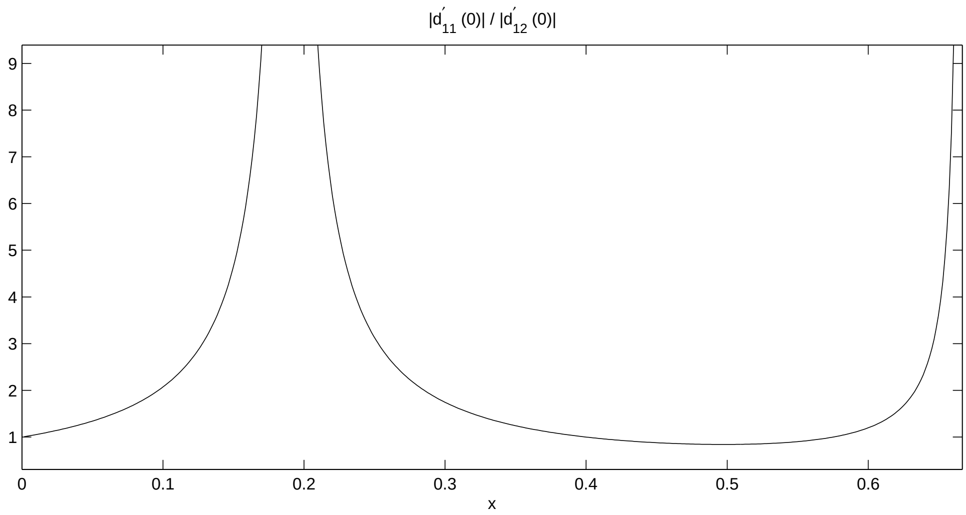

Figure 3: Ratio of demand when 4 options are traded.

With five options we get

$$\begin{pmatrix} d_1'(0) \\ d_2'(0) \\ d_3'(0) \\ d_4'(0) \\ d_5'(0) \end{pmatrix} = \text{tastes} \cdot \begin{pmatrix} \frac{(-4\delta+3)(\delta-1)^2}{2\delta(-1+2\delta)(6\delta-7)} \\ \frac{2}{\delta(4\delta^2-6\delta+1)(6\delta-7)} \\ \frac{(18\delta^3-25\delta^2+6\delta-1)}{\delta(6\delta-7)} \\ \frac{2}{\delta(4\delta^2-6\delta+1)(6\delta-7)} \\ \frac{(-4\delta+3)(\delta-1)^2}{2\delta(-1+2\delta)(6\delta-7)} \end{pmatrix}$$

5 Comparing Our Theory of Open Interest with the Markets

Open interest in a call option denotes the total number of that contract that are held long at the close of the exchange and is quoted at the end of each day for all financial contracts traded.

A series of strikes on a contract has some particular feastures: it has a lifetime of roughly two years, is assigned a unique letter code for reference purposes and the strikes for which exchanges allow trading allow for a very wide range that are most likely covering all eventual stock price movements over the two year life-span. It is common market knowledge that open interest

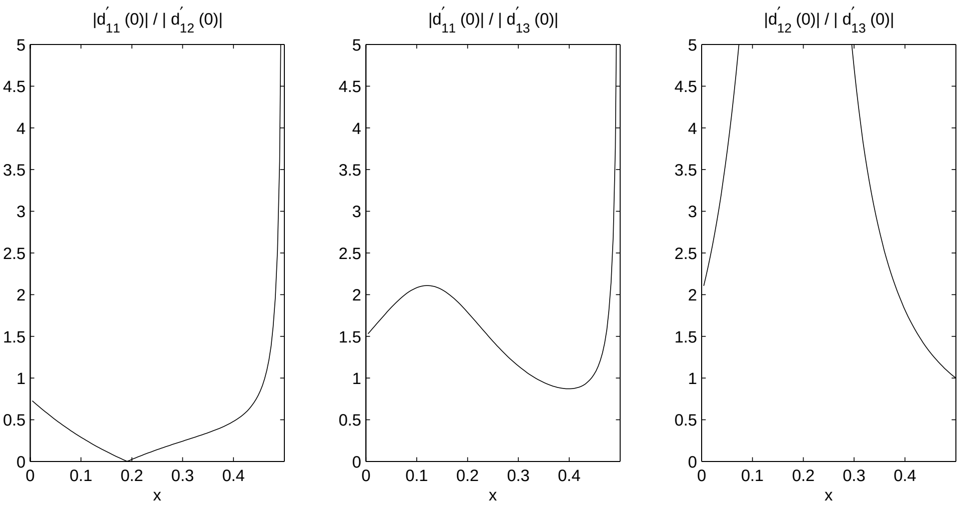

Figure 4: Ratios of demand when 5 options are traded.

in call and put options is largest for contracts at-the-money and decreases (to zero) the more the option contract is in or out of the money contract.

Our focus in this paper is only on highlighting the presence of these facts; we will not run an empirical analysis. Figure 8 plots the open interest on March 18, 2002 on Microsoft[8] depending on the strike-to-asset ratio (strike divided by the asset's price on that day). The first row looks at the open interest in calls across strikes: we find the first of three facts in the data; we also see a particular shape of that curve. We find that shape also in the put curve and in the sume of the open interest in calls and puts with the same strike.

Put-call parity allows replicating a long put position in a strike K through joint long positions in the stock, the call with strike K and an investment of \$ K in the bond. In this paper we focus on the equilibrium analysis of call options. Therefore we translate all positions into corresponding call positions and create a new summary open interest in a call with strike K by adding[9]

[8]Data is from the American Stock Exchange.

[9]Implicitly we assume here that agents hold either only calls, only puts or straddles (positions that are jointly long in calls and puts or jointly short in calls and puts). We exclude that a market participant is holding both a long call and a short put. Such a strategy would replicate a long stock position and seems unrealistic to us.

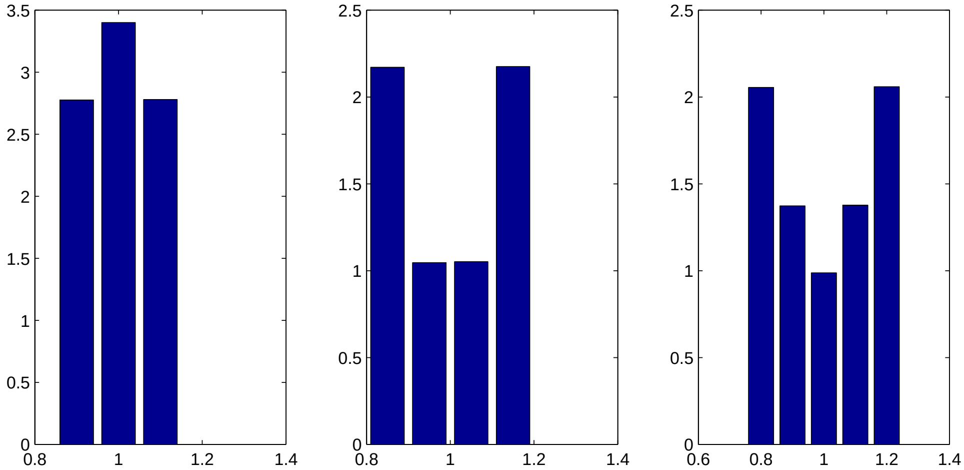

Figure 5: Open interest when $\delta = 0.1$: comparison of the shape when 3 options can be traded with the cases when 4 or 5 can be traded.

to it the open interest in the put with strike K.

This shape of the curve is stable for the contract with the shortest maturity and does not depend on the specific underlying we chose to present. Note that we are looking here only at the contract with the shortest maturity and that contracts with other maturity might look different. E.g. figure 9 plots the open interest for the series with the longest maturity; besides this figure is organized as figure 9. We find that this curve does not present such a clear picture. It seems that puts and calls have modes at distinctly different strikes and the sum of both curves seems to have two modes.

We will now calibrate our model to the Microsoft series: we have set up our expansion such that the ration between option and stock standard deviation is independent of ϵ (see equation 1). Also, multiplying strikes and the random variables by ϵ means that the ratio between these is independent of ϵ. We will consequently analyze the distance between strikes in terms of standard deviations $\sqrt{V_{00}}$.

We compare here two distributions and assume that ξ_0 is either the tent distribution on the interval -1 to 1 or the normal distribution with mean 0 and standard deviation parameter 0.44. Both distributions have a mean of zero and variance 1/6. (Note that we truncate the normal distribution at -1

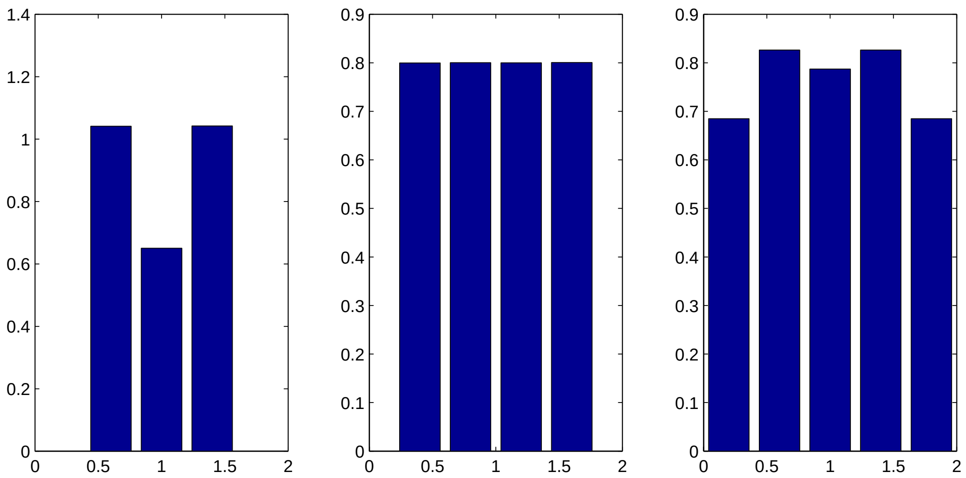

Figure 6: Open interest when $\delta = 0.4$: comparison of the shape when 3 options can be traded with the cases when 4 or 5 can be traded.

and 1; therefore here the variance does not coincide with the square standard deviation parameter 0.44^2.) The density functions for these two distributions are depicted in figure 10.

We now calibrate our model to the Microsoft strike series that is contained in figure 8: volatility of Microsoft is stochastically fluctuating around 62% over time. We will therefore compare the prediction for three cases: a volatility of 55%, 60%, and 65%. Figures 11, 12 and 13 look at the cases with these volatility when 8, 9 and 10 options are traded.

This outcome enforces our view that adding an analysis of open interest to those of stock prices can enrich studies of financial markets.

6 Conclusion

This paper derived the leading terms in an asymptotic expansion that describes agents' individual demand schedules and the equilibrium allocation. We related them to preference over and moments of mean, variance and co-skewness. We explained that mean variance terms do not induce option demand and highlighted that option demand is driven by co-skewness of the option with the underlying security. We discussed that the shape of the open

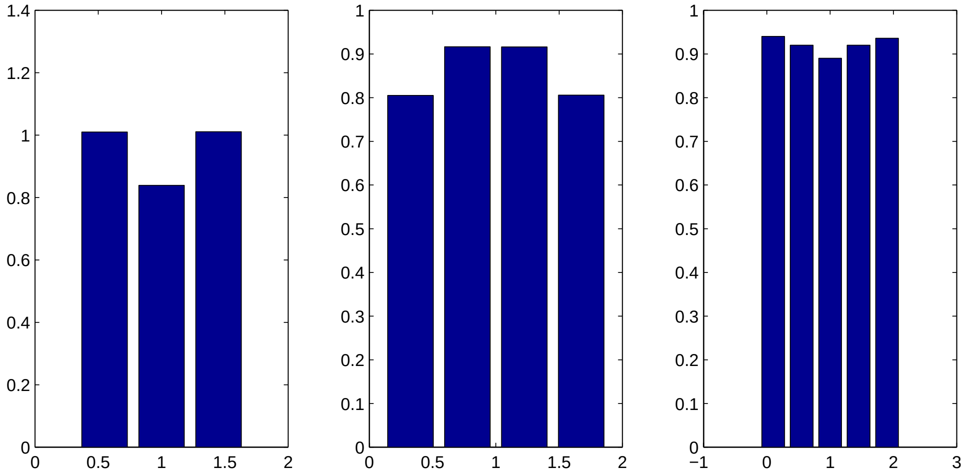

Figure 7: Open interest when $\delta = 0.45$: comparison of the shape when 3 options can be traded with the cases when 4 or 5 can be traded.

interest curve across strikes is in our model independent of tastes and the expansion parameter; it only depends on distributional characteristics (variance and co-skewness). We calibrated a tent and a normal distribution to the strike set for Microsoft (April 2002 maturity) and found that qualitative features can be matched with the curve that we see on the market: a peak around the at-the-money contracts and decreasing open interest the more we get and out-of-the-money.

7 APPENDIX: Perturbation Analysis

We prove in this appendix theorem 1 for any agent $i = 1, 2$. We denote $\eta_i = \sum_{j=0}^{N} d_{ij}(0)\xi_j$, and find that $\eta_i = \frac{\partial W_{Ti}}{\partial \epsilon}(0)$ and $\frac{\partial^2 W_{Ti}}{\partial \epsilon^2}(0) = \pi_0(0) + \sum_{j=0}^{N} d_{ij}(0) 2\pi_j(0)$ since $W_{0i}(\epsilon) = \frac{1}{2}P_0(\epsilon) = \frac{1}{2}(1 - \pi_0(\epsilon)\epsilon^2)$ and therefore the wealth dynamics is

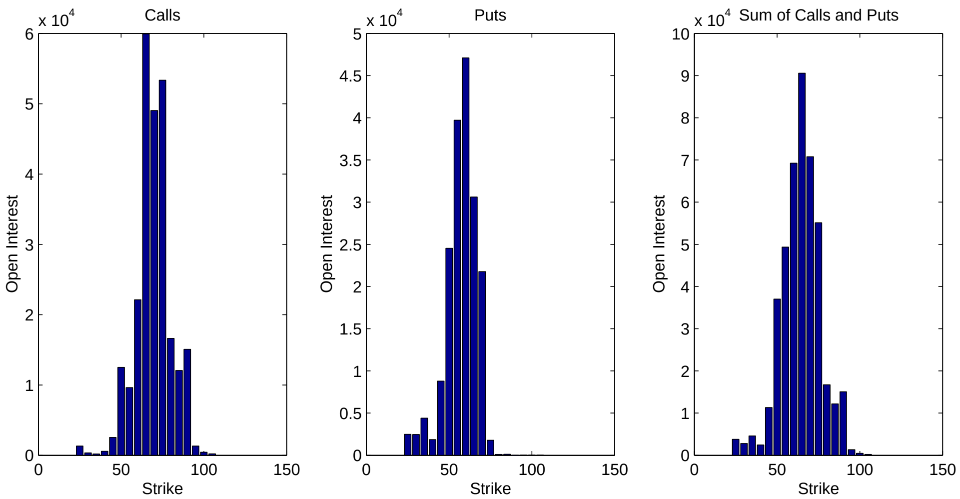

Figure 8: Empirically observed open interest on Monday March 18, 2002 for April 2002 maturity.

$$
\begin{aligned}
W_{Ti}(\epsilon) &= \frac{1}{2}(1-\pi_0(\epsilon)\epsilon^2) + \sum_{j=0}^{N} d_{ij}(\epsilon)\cdot\left(\xi_j\epsilon + \pi_j(\epsilon)\cdot\epsilon^2\right) \\
\frac{\partial W_{Ti}}{\partial\epsilon} &= \frac{1}{2}(\pi_0'(\epsilon)\epsilon^2 + \pi_0(\epsilon)2\epsilon) + \sum_{j=0}^{N} d_{ij}(\epsilon)\cdot\left(\xi_j + \pi_j'(\epsilon)\cdot\epsilon^2 + \pi_j'2\epsilon\right) \\
\frac{\partial^2 W_{Ti}}{\partial\epsilon^2} &= \frac{1}{2}(\pi_0''(\epsilon)\epsilon^2 + \pi_0'(\epsilon)4\epsilon + \pi_0(\epsilon)2) + \sum_{j=0}^{N} d_{ij}(\epsilon)\cdot\left(\pi_j''(\epsilon)\epsilon^2 + \pi_j'(\epsilon)4\epsilon + \pi_j'2\right).
\end{aligned}
$$

Note that $E[\eta_i]=0$ and $\zeta_{ij} = E\left[\left(\sum_{k=0}^{N} d_{ik}(0)\xi_k\right)^2\cdot\xi_j\right] = E[\eta_i^2\xi_j]$.

Theorem 8 *(Theorem 4 in Judd and Guu (1999)) Suppose $H : R^n \times R \to R^n$ is analytic, and $H(x,0) = 0$ for all $x \in R^n$. Furthermore, suppose that for some $(x_0, 0)$*

$$\frac{\partial H}{\partial x}(x_0,0) = 0_{n\times n}, \frac{\partial H}{\partial \epsilon}(x_0,0) = 0_n, \text{ and det } \left(\frac{\partial^2 H}{\partial x \partial \epsilon}(x_0,0)\right) \neq 0$$

Then there is an open neighborhood $\mathcal{N}$ of $(x_0,0)$, and a function $h(\epsilon) : R \to R^n, h(\epsilon) \neq 0$ for $\epsilon \neq 0$, such that

$$H(h(\epsilon),\epsilon) = 0 \text{ for } (h(\epsilon),\epsilon) \in \mathcal{N}$$

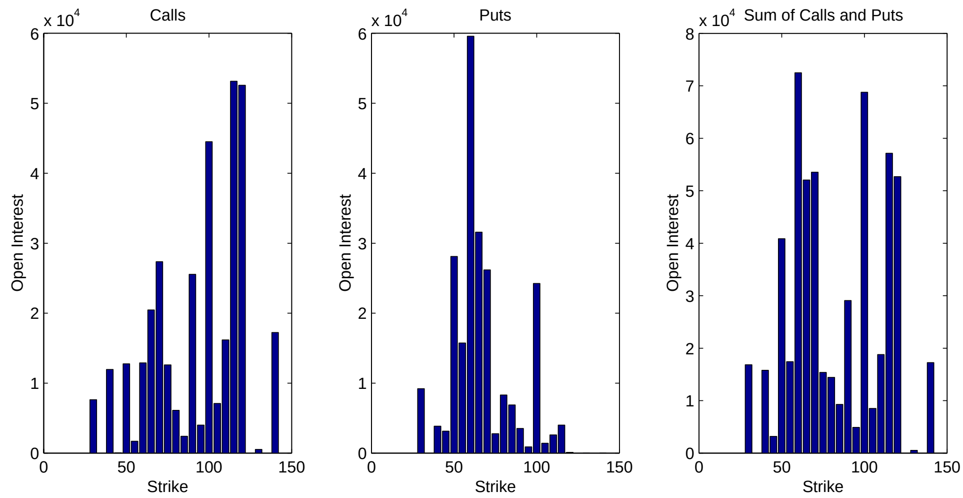

Figure 9: Empirically observed open interest on Monday March 18, 2002 for January 2004 maturity.

Furthermore, h is analytic and can be approximated by a Taylor series. In particular, the first order derivatives equal

$$h'(0) = -\frac{1}{2} \cdot \left(\frac{\partial^2 H}{\partial x \partial \epsilon} \right)^{-1} \cdot \frac{\partial^2 H}{\partial \epsilon^2}.$$

7.1 Calculating Derivatives

To apply theorem 8 we will now calculate the (first order) derivatives of H_i with respect to d and ϵ and then the (second order) derivatives with respect to (d, ϵ) and (ϵ, ϵ) and evaluate them at $\epsilon = 0$:

$$\begin{aligned}
\frac{\partial H_{ij}}{\partial d_{ik}} &= E\left[\frac{\partial^2 u_i}{\partial W^2}(W_{Ti}(\epsilon)) \cdot (\xi_j + \pi_j(\epsilon)\epsilon) \cdot \left(\xi_k \epsilon + \pi_k(\epsilon)\epsilon^2\right) \right] \\
\frac{\partial H_{ij}}{\partial \epsilon} &= E\left[\frac{\partial^2 u}{\partial W^2}(W_{Ti}(\epsilon)) \cdot \frac{\partial W_{Ti}}{\partial \epsilon} \cdot (\xi_j + \pi_j(\epsilon) \cdot \epsilon) + \frac{\partial u_i}{\partial W}(W_{Ti}(\epsilon)) \cdot \left(\pi_j(\epsilon) + \pi_j'(\epsilon)\epsilon\right) \right].
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 H_{ij}}{\partial d_{ik} \partial \epsilon} = \; & E\left[\frac{\partial^3 u_i}{\partial W^3}(W_{Ti}(\epsilon)) \cdot \frac{\partial W_{Ti}}{\partial \epsilon} \cdot (\xi_j + \pi_j(\epsilon)\epsilon) \left(\xi_k \epsilon + \pi_k(\epsilon)\epsilon^2\right) \right. \\
& + \frac{\partial^2 u_i}{\partial W_{Ti}^2}(W_{Ti}(\epsilon)) \left(\pi_j(\epsilon) + \pi_j'(\epsilon)\epsilon\right) \left(\xi_k \epsilon + \pi_k(\epsilon)\epsilon^2\right) \\
& \left. + \frac{\partial^2 u_i}{\partial W^2}(W_{Ti}(\epsilon))(\xi_j + \pi_j(\epsilon)\epsilon^2) \left(\xi_k + \pi_k'(\epsilon)\epsilon^2 + \pi_k 2\epsilon\right). \right]
\end{aligned}$$

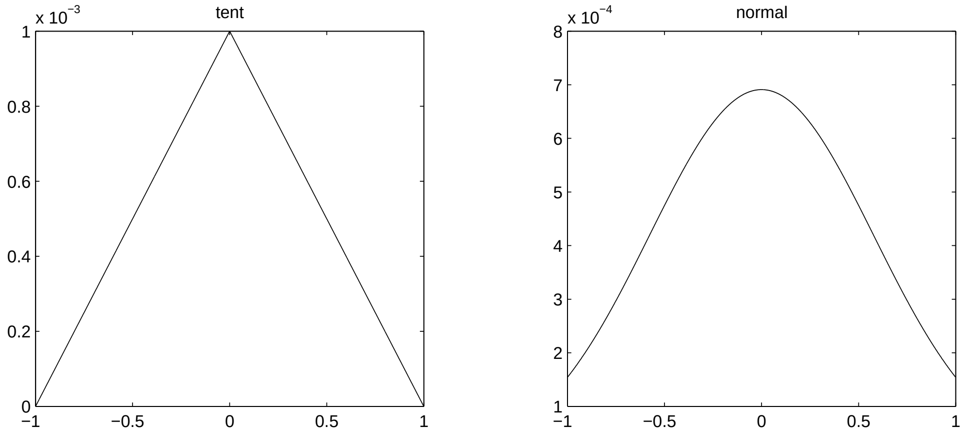

Figure 10: Tent and normal distribution for ξ_0.

We also calculate

$$\begin{aligned}\frac{\partial^2 H_{ij}}{\partial \epsilon^2} &= E\Bigg[\frac{\partial^3 u_i}{\partial W^3}(W_{Ti}(\epsilon)) \cdot \left(\frac{\partial W_{Ti}}{\partial \epsilon}\right)^2 \cdot (\xi_j + \pi_j(\epsilon) \cdot \epsilon) \\ &\quad + \frac{\partial^2 u_i}{\partial W^2}(W_{Ti}(\epsilon)) \cdot \frac{\partial W_{Ti}}{\partial \epsilon} \cdot \left(\pi_j(\epsilon) + \pi_j'(\epsilon)\epsilon\right) \\ &\quad + \frac{\partial^2 u_i}{\partial W^2}(W_{Ti}(\epsilon)) \cdot \frac{\partial W_{Ti}}{\partial \epsilon} \cdot (\xi_j + \pi_j(\epsilon)) + \frac{\partial u_i}{\partial W}(W_{Ti}(\epsilon)) \cdot \left(2\frac{\partial \pi_j}{\partial \epsilon} + \pi_j''(\epsilon)\epsilon\right)\Bigg].\end{aligned}$$

7.2 Deriving The Zero and First Order Terms

We see that $\frac{\partial H_{ij}}{\partial d_{ik}}$ is equal to zero at $\epsilon = 0$ for any j, k so that $\frac{\partial H_i}{\partial d}(\epsilon = 0) = 0$. In the next steps we will check that $det\left(\frac{\partial^2 H_i}{\partial d_i \partial \epsilon}\right)_{|\epsilon=0} \neq 0$ and require $\frac{\partial H_i}{\partial \epsilon}(\epsilon = 0) = 0$ to apply theorem 8: We can deduce from the above equations that $\frac{\partial^2 H_{ij}}{\partial d_{jk} \partial \epsilon}(\epsilon = 0) = u_i'' E[\xi_j \xi_k]$, i.e.

$$\frac{\partial^2 H_i}{\partial d \partial \epsilon}(\epsilon = 0) = u_i'' \cdot V.$$

(All other terms are equal to zero at $\epsilon = 0$.) Therefore $det\left(\frac{\partial^2 H_i}{\partial d_i \partial \epsilon}\right)_{|\epsilon=0} = u_i'' \cdot det(V)$ and $\left(\frac{\partial H_i^2}{\partial d \partial \epsilon}(0)\right)^{-1} = \frac{1}{u_i''} V^{-1}$. Since $u_i'' > 0$ (u_i is strictly concave), and $det(V) \neq 0$ we have that $det\left(\frac{\partial^2 H_i}{\partial d_i \partial \epsilon}\right)_{|\epsilon=0} \neq 0$.

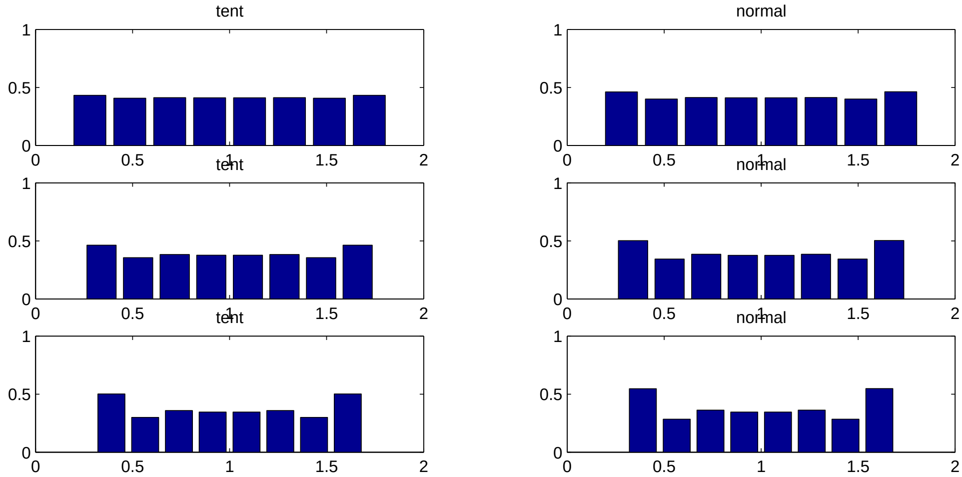

Figure 11: Theoretical open interest curve When our Model is calibrated to Microsoft and 8 options can be traded: the first row assumes that volatility is 55%, the second takes 60% and the third row takes 65% volatility.

The condition $H_\epsilon(\epsilon = 0) = 0$ becomes the following system of equations:

$$\begin{aligned} 0 &= \frac{\partial H_{ij}}{\partial \epsilon}(0) = u_i'' \cdot \sum_{k=0}^{N} d_{ik}(0) Cov(\xi_j, \xi_k) + u_i' \cdot \pi_j(0), \\ \text{i.e. } 0 &= u_i'' \cdot V \cdot d_i + u_i' \cdot \pi(0) \text{ or equivalently } d_i(0) = \tau_i \cdot \left(V^{-1}\pi(0)\right). \end{aligned}$$

This gives us the zero order demand term. Now that the conditions have been checked to apply theorem 8 we calculate

$$\frac{\partial^2 H_{ij}}{\partial \epsilon^2}(0) = u_i''' E[\eta_i^2 \xi_j] + 2u_i'' E[\eta_i]\pi_j(0) + 2u_i'\pi_j'(0) = u_i'''\zeta_j + 2u_i'\pi_j'(0).$$

This follows since $E[\eta_i] = 0$. Theorem 8 tells us

$$\frac{\partial d_i}{\partial \epsilon}(0) = -\frac{1}{2}\left(\left(\frac{\partial^2 H_i}{\partial d_i \partial \epsilon}(\epsilon = 0)\right)^{-1} \frac{\partial^2 H_i}{\partial \epsilon^2}\right) = -\frac{u_i'''}{2u_i''}\left(V^{-1}\zeta_i\right) - \frac{u_i'}{u_i''}\left(V^{-1}\pi'(0)\right).$$

7.3 Calculating The Equilibrium

Using $d_i(0) = \tau_i \cdot (V^{-1}\pi)$ (equation 6) we get that the zero order aggregate demand is $(\tau_1 + \tau_2) \cdot (V^{-1}\pi)$. This translates into (for assets $j = 1, \ldots, N$)

$$\pi_0(0) = \frac{1}{\tau_1 + \tau_2} V_{00}, \pi_j(0) = \frac{1}{\tau_1 + \tau_2} V_{0j}, d_{i0}(0) = \frac{\tau_i}{\tau_1 + \tau_2}, d_{ij}(0) = 0.$$

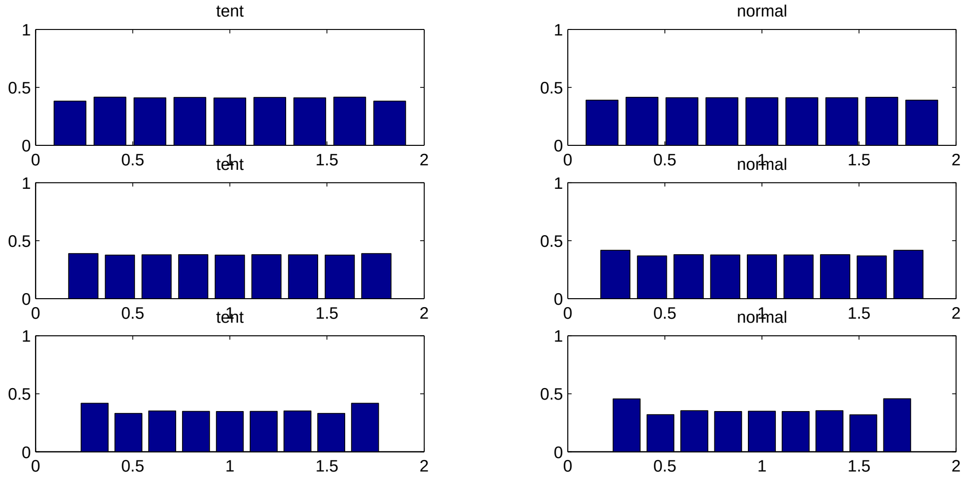

Figure 12: Theoretical open interest curve When our Model is calibrated to Microsoft and 8 options can be traded: the first row assumes that volatility is 55%, the second takes 60% and the third row takes 65% volatility.

This implies that $\zeta_{ij} = E\left[\left(\sum_{k=0}^{N} d_{ik}(0)\xi_k\right)^2 \xi_j\right] = d_{i0}^2(0)E[\xi_0^2\xi_j] = d_{i0}^2(0)\cdot\chi_j$. Therefore, the ζ_i vector reduces to the vector $d_{i0}^2(0)\cdot\chi$. We calculate the first order equilibrium demand and price terms using equation (6) as

$$d_i'(0) = V^{-1}\left(\tau_i\pi'(0) + \frac{\rho_i}{\tau_i}\left(\frac{\tau_i}{\tau_1+\tau_2}\right)^2\chi\right).$$

The market clearing condition for the first order demand is then:

$$0 = d_1'(0) + d_2'(0) = V^{-1}\left((\tau_1+\tau_2)\,\pi'(0) + \left(\frac{\tau_1\rho_1}{(\tau_1+\tau_2)^2} + \frac{\tau_2\rho_2}{(\tau_1+\tau_2)^2}\right)\chi\right)$$

which implies

$$\pi'(0) = -\frac{\rho_1\tau_1+\rho_2\tau_2}{(\tau_1+\tau_2)^3}\chi,$$

and so

$$\begin{aligned} d_1'(0) &= \left(\frac{\rho_1\tau_1}{(\tau_1+\tau_2)^2} - \frac{\tau_1}{(\tau_1+\tau_2)^3}(\rho\tau_1+\rho_2\tau_2)\right)V^{-1}\chi \\ &= -\tau_1\tau_2\frac{\rho_1-\rho_2}{(\tau_1+\tau_2)^3}V^{-1}\chi. \end{aligned}$$

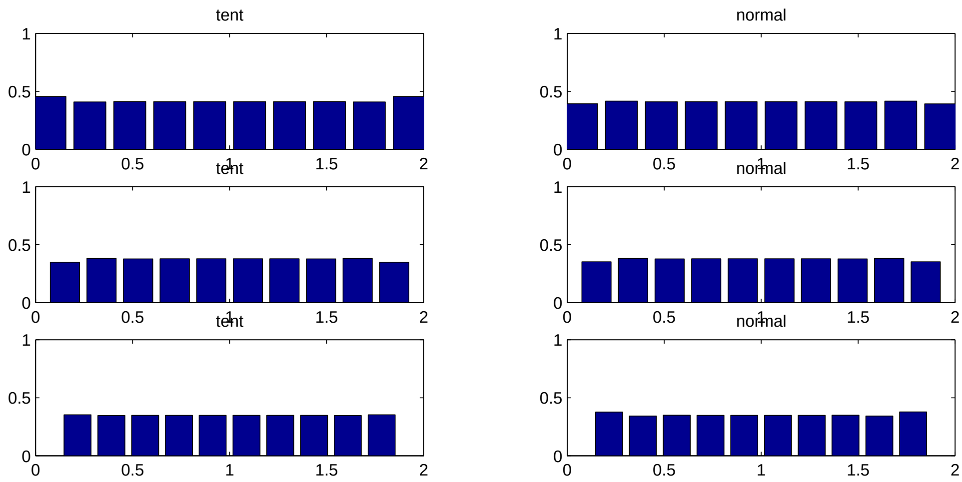

Figure 13: Theoretical open interest curve When our Model is calibrated to Microsoft and 10 options can be traded: the first row assumes that volatility is 55%, the second takes 60% and the third row takes 65% volatility.

8 Reference

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