Valuation in Integrated Financial and Insurance Markets

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Abstract: A market is presented in which actuarial risk is traded through both insurance and financial contracts. The coexistence of these contracts leads to a new price selection criterion. Financial prices have to be *actuarially consistent* with insurance premiums to exclude arbitrage opportunities in the market. Even though this additional restriction on price dynamics does not imply unique price determination, a representation of actuarially consistent prices is deduced. In this representation, the underlying stochastic structure is separated from the contract's specifications and a link is established between financial prices and insurance premiums. This connection is examined in more detail for commonly used premium calculation principles.

Keywords: risk securitization, no-arbitrage pricing, incomplete markets, actuarial consistency, Fourier transform

1 Introduction

The importance of the interface of capital markets and insurance markets has been increasingly emphasized by both the private and public sector. This economic and political debate has its roots in the growing concerns amongst individuals of the long-term risks over the lifecycle as the nature and magnitude of some of these risks have become apparent only recently. In the past 30 years, financial costs from natural catastrophes have risen, risk to social capital and risk of inflation have become more severe. These developments suggest that innovations in risk management would make a valuable contribution in reducing risk over individuals' lifecycle. In response, one major focus in recent years has been the idea of making risks tradeable in financial markets, that were traditionally spread through insurance and reinsurance contracts. This attempt at risk securitization results in the emergence of financial products that capture insurance related risks, e.g. catastrophe insurance derivatives, index-linked life insurance contracts, index-linked debt, or funded pension schemes.

This overlap of insurance and financial markets raises several questions on risk valuation and suggests examining the similarities and differences between methods that have been developed in both insurance mathematics and mathematical finance. It is possible to classify these issues and the related literature using two factors, the specification of the contracts that are available on the market and the source of uncertainty. To be more precise, our classification is based on whether the economy contains

- financial and/or insurance contracts

that are based on

- financial and/or insurance related risk.

The type of contract is related to the concept upon which valuation is based, whereas the type of underlying risk is connected to the appropriate class of stochastic processes that is used to model the evolution of market uncertainty.

Prior to the convergence of capital and insurance markets, exclusively either financial contracts based on financial risk or insurance contracts based on insurance related risk had been introduced to the market. Stochastic models for the underlying risk processes and methods for the valuation of the corresponding contracts have been developed separately in mathematical finance and insurance mathematics. We refer to Bjørk [4], Duffie [14], and Musiela and Rutkowski [30] and the references therein for the former field of research, and to Bühlmann [7], Gerber [20], and Grandell [22] for the latter.

In a sequence of papers by Brennan and Schwartz [6], Bacinello and Ortu [2], and Nielsen and Sandmann [31], the pricing of equity-linked life insurance contracts is investigated. The benefits of these insurance policies depend on the performance of a reference portfolio that is traded on the capital market. According to our classification, these contracts belong to a market containing insurance contracts that are based on both financial and insurance related risk in form of policyholders' mortality risk.

Aase [1], Cummins and Geman [9], and Embrechts and Meister [16] investigate the valuation of financial contracts that are based on insurance related risk such as catastrophe insurance derivatives. According to our classification, this economy is one that consists of financial contracts based on insurance related risk.

In two articles by Delbaen and Haezendonck [12] and Sondermann [33] the authors show how premium calculation principles for reinsurance contracts can be embedded in a no-arbitrage framework. The important contribution of these papers lies in the construction of an analytical bridge between actuarial and financial valuation. Referring back to our classification, the authors investigate a market that consists of insurance contracts based on insurance related risk.

Recent papers by Schweizer [32] and Møller [28], [29] consider a capital market in which a risk measure is *a priori* given that can be interpreted as an actuarial premium. The authors use an indifference argument based on the possibility of trading in financial instruments to transfer this *a priori* given risk measure into an *a posteriori* risk measure. The resulting measure can be interpreted as a financial premium.

We conclude that the literature, initiated by the convergence of capital and insurance markets, has separately focused on markets consisting of insurance contracts linked to financial market risk, on markets consisting of financial contracts based on insurance related risk, and on embedding actuarial valuation into a no-arbitrage framework.

In a global economy, in which capital and insurance markets merge, financial investors and insurance companies additionally trade in contracts of the other market with the aim of exploiting new investment opportunities and hedging instruments. It is therefore relevant to consider an economy in which both financial and insurance contracts coexist and to investigate price determination in view of this coexistence. This idea of an integrated market and the valuation therein captures exactly the aim of this paper and our contribution to the existing literature.

To be more precise, we assume that an investor facing insurance related risk is able to sell off the risk. This possibility reflects the existence of an insurance contract in which the premium to be paid is specified. In addition, we assume the existence of a traded financial contract that securitizes the underlying risk in the form of a European derivative. To come back to our previous classification, we investigate a market consisting of financial and insurance contracts that are both based on insurance related risk.

One major difficulty in valuation of these contracts is the unpredictable nature of insurance related risk. This feature makes it impossible to synthetically provide a completely secure hedge by continuous trading in existing contracts. Our integrated market is thus incomplete and there exists an infinite collection of financial and insurance price processes that exclude arbitrage strategies.

With the aim of tackling the multiplicity of no-arbitrage prices, we require financial prices to be consistent with the actuarial valuation of the same underlying risk. We therefore introduce a new price process selection criterion in incomplete markets that originates in the coexistence of financial and insurance contracts. In addition to the exclusion of arbitrage opportunities we thus demand financial prices to be robust with respect to this new selection criterion.

It is shown that in general there exists still an infinite collection of financial price processes that are consistent with actuarial valuation. However, the additional selection criterion restricts the set of no-arbitrage price processes and we explicitly characterize this remaining set. Building on a representation of price processes that we deduce in this paper using Fourier analysis, we show that the connection between financial and actuarial prices, emerging from actuarial consistency, is wholly incorporated in the characteristic function of the underlying risk. These results are valid for a very general class of premium calculation principles. We then pick out some principles that are commonly used by the insurance industry and investigate in more detail the set of financial price processes that correspond to the chosen premium principle.

The remainder of the paper is organized as follows: in Section 2 and 3 we introduce the fundamentals of the market, the underlying risk process and the contracts that are available in the market. Section 4 investigates actuarial and financial valuation and introduces the concept of consistency with insurance premiums. In Section 5 we examine certain premium calculation principles in more detail before we conclude in Section 6.

2 The Fundamentals

In this section we introduce the stochastic structure and the underlying process that represents insurance related risk in the market. In addition, we briefly examine the change between equivalent probability measures and the effect that it induces on the dynamics of the risk process.

2.1 Uncertainty

Uncertainty enters through different possible realizations ω of the world. All realizations are collected in a sample space Ω. An event is defined as a subset of Ω and $\mathcal{F}$ denotes the set of all possible events. We assume that $\mathcal{F}$ forms a σ-algebra. The likelihood of events is represented by a probability measure P that assigns probabilities to every event in $\mathcal{F}$. The triple $(\Omega, \mathcal{F}, P)$ thus describes the stochastic foundation of the market on which all following random variables will be defined.

As we consider the stochastic evolution of prices we need to introduce time and the amount of information that is available to market participants at every point in time. We assume that the economy is of finite horizon $T < \infty$ and the flow of information is modeled by a nondecreasing family of σ-algebras

$(\mathcal{F}_t)_{0 \leq t \leq T}$, a filtration. We assume that $\mathcal{F}_T = \mathcal{F}$, each $\mathcal{F}_t$ contains the events in $\mathcal{F}$ that are of P-measure zero, and the filtration is right-continuous, i.e.

$$\mathcal{F}_t = \mathcal{F}_{t+},$$

where $\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$.

In the following section we put more structure on the evolution of uncertainty by taking into account the features of insurance related risk.

2.2 Risk Process

Risk in an insurance context is caused by single events such as accidents, death, or natural catastrophes. One source of uncertainty is therefore the point in time of events. Additionally one has to introduce some variable that measures the impact such an event has on the economy. Let us imagine that this variable measures insured losses and thus claims to be paid by an insurance company. Hence the magnitude of losses represents a second source of uncertainty in the economy.

We model the points in time and magnitudes of events as sequences of random variables $(T_n)_{n \in \mathbb{N}}$ and $(Y_n)_{n \in \mathbb{N}}$ where T_k denotes the point in time of the k-th event causing a corresponding loss of size Y_k. Let us now combine both moment and magnitude risk by introducing a stochastic process $X = (X_t)_{0 \leq t \leq T}$ where for each point in time t the random variable X_t represents the sum of claim amounts incurred in $(0, t]$. Therefore

$$X_t = \sum_{\{k | T_k \leq t\}} Y_k. \tag{1}$$

The stochastic process $X = (X_t)_{0 \leq t \leq T}$ is called accumulated claim process, also referred to as risk process in the literature.

We assume that the past evolution and current state of the risk process X is observable by every agent in the economy, i.e. X is assumed to be adapted to the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$. For simplicity, we shall assume that X generates the flow of information, i.e. $\mathcal{F}_t = \sigma\left(\sigma\left(X_s,\ s \leq t\right) \cup \mathcal{N}\right)$ where $\mathcal{N}$ denotes the events of P-measure zero.

As regards occurrences of events we assume that the counting process $N = (N_t)_{0 \leq t \leq T}$ defined through

$$N_t = \sup\left\{k \geq 1 \,| T_k \leq t\right\} \tag{2}$$

is a homogeneous Poisson process with frequency parameter $\lambda \in \mathbb{R}_+$. The probability of k events occurring in the time interval $(0, t]$ is therefore

$$P\left[N_t = k\right] = e^{-\lambda t} \frac{(\lambda t)^k}{k!},$$

with the expected number of events

$$\mathbf{E}^P\left[N_t\right] = \lambda t,$$

where $\mathbf{E}^P\left[\cdot\right]$ denotes the expectation operator under the measure P.

Furthermore, we assume that loss sizes Y_1, Y_2, Y_3, ... are independent and identically distributed random variables that are independent of the counting process N. Let G denote their common distribution function with support $(0, \infty]$.

In short, we model the risk process X as a compound Poisson process with characteristics $(\lambda, dG(y))$.

2.3 Equivalent Probability Measures

In this section, we briefly review the change between equivalent probability measures and the consequent change in characteristics of the process X.

Let us examine the set of probability measures Q on $(\Omega, \mathcal{F})$ that are equivalent to the "reference" measure P and that preserve the structure of the underlying risk process X, i.e. X is a compound Poisson process under the new probability measure Q. This set has been characterized by Delbaen and Haezendonck [12] and can be parameterized by a pair $(\kappa, v(\cdot))$ consisting of a nonnegative constant κ and a nonnegative, measurable function $v : \mathbb{R}_+ \to \mathbb{R}$ with $\mathbf{E}^P\left[v(Y_1)\right] = 1$. The density process $\xi_t = \mathbf{E}^P\left[\xi_T \mid \mathcal{F}_t\right]$ of the Radon-Nikodym derivative $\xi_T = \frac{dQ}{dP}$ is then given by

$$\xi_t = \exp\left(\sum_{k=1}^{N_t} \ln\left(\kappa v(Y_k)\right) + \lambda(1-\kappa)t\right), \tag{3}$$

for any $0 \leq t \leq T$.

Let us denote the measure Q corresponding to the constant κ and the function $v(\cdot)$ by $P^{\kappa,v}$. Under the new measure $P^{\kappa,v}$ the process X is a compound Poisson process with characteristics $(\lambda\kappa, v(y)\, dG(y))$. κ can therefore be interpreted as market price of frequency risk, and $v(\cdot)$ as market price of claim size risk.

Changing the probability measure plays a central role in the context of valuation of both insurance and financial contracts. In the following section we introduce the contracts that are available on the market before proceeding to the pricing of these contracts.

3 The Market

Suppose an individual or a company is facing the risk process X, e.g. an insurance company that has to pay out claims to their policyholders. The company can make use of three assets that are traded on the market:

- one risk-free bond with price process $B = (B_t)_{0 \le t \le T}$ and associated deterministic short rate of interest r. Without loss of generality, we assume $r \equiv 0$, i.e. $B_t \equiv 1$ for all $0 \le t \le T$;
- one insurance contract that specifies the premium process of the underlying risk process X;
- one European-style financial contract, i.e. at maturity T the contract's payoff depends on the realization of the risk process X_T only.

Let us define the specifications of the latter two risky assets in more detail.

3.1 The Insurance Contract

We consider the setup of Delbaen and Haezendonck [12] in which the insurance (reinsurance) contract allows the individual (insurance company) to sell off the risk of the remaining period. Let p_t denote the premium the individual (insurance company) has to pay at time t to sell the risk $X_T - X_t$ over the remaining period $(t, T]$.

The premium process $p = (p_t)_{0 \le t \le T}$ is a stochastic process that is assumed to be predictable, i.e. it is adapted to $(\mathcal{F}_{t-})_{0 \le t \le T}$, where $\mathcal{F}_{t-} = \bigvee_{s<t} \mathcal{F}_s$.

Remark 1 *Sondermann [33] considers dynamic reinsurance policies, i.e. the insurance company can decide to sell off a certain fraction of their risk and adjust their decision continuously. If the insurance company is allowed to only adjust at finitely many times this approach can be embedded in the framework of Delbaen and Haezendonck [12] by defining the maturities of several contracts accordingly.*

3.2 The Financial Derivative

We assume that the financial derivative securitizes insurance related risk reflected in the underlying risk process X. The buyer of this contract receives a certain payment at expiry T of the contract that depends solely on the realization of X_T. In exchange the seller of the contract receives a certain price that reflects the value of the payoff. The financial contract is therefore of European-style, i.e. early exercise is not allowed and the contract is path-independent.

Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be a measurable function that specifies the buyer's payoff at maturity, i.e. at T the buyer receives $\phi(X_T)$. Let the random variable π_t denote the price at time t that one has to pay in order to enter into the financial contract. Hence the stochastic process $\pi = (\pi_t)_{0 \le t \le T}$ is the financial price process that reflects the value of the payoff $\phi(X_T)$ at maturity T of the contract.

We shall assume the following integrability condition:

$$\phi(\cdot) - k \in \mathbf{L}^2(\mathbb{R}_+), \tag{4}$$

from some $k \in \mathbb{R}$ where $\mathbf{L}^2(\mathbb{R}_+)$ is the set of measurable, square-integrable functions with respect to the Lebesgue measure. Financial contracts with a structure that is similar to existing insurance or reinsurance contracts are spread options that cover a certain layer of losses. These contracts with limited liability fulfill the integrability condition specified in equation (4).

In the following section we investigate the price process p and π of the insurance and financial contract in more detail.

4 No-Arbitrage Valuation

In this section, we examine the valuation of traded assets in the absence of arbitrage strategies. First, we review the equivalence between the existence of equivalent martingale measures and the absence of arbitrage opportunities in the market. We then investigate the valuation of both insurance and financial contracts under the assumption that the corresponding price processes exclude arbitrage opportunities. Thereafter, we introduce the additional restriction that financial prices should be consistent with actuarial pricing principles.

4.1 The Fundamental Theorem of Asset Pricing

The equivalence between the existence of equivalent martingale measures and the absence of arbitrage opportunities in the market plays a central role in mathematical finance. An equivalent martingale measure is a probability measure that is equivalent to the "reference" measure P and under which discounted price processes are martingales. It is important to be aware of the specifications of the model in which this equivalence is used since arbitrage has to be differently defined to guarantee the existence of equivalent martingale measures.

Harrison and Kreps [23], and Harrison and Pliska [24] were the first to establish an equivalence result in a model based on a finite state space Ω. In a discrete infinite or continuous world, the absence of arbitrage is not a sufficient condition for the existence of an equivalent martingale measure. Other definitions of arbitrage opportunity or restricting conditions on the dynamics of price processes have been derived to guarantee the existence of martingale measures. Fritelli and Lakner [17] give a definition of arbitrage, called "free lunch", under which the equivalence result is derived with high level of generality. The only mathematical condition that is imposed on asset prices is that they are adapted to the filtration $(\mathcal{F}_t)_{0 \le t \le T}$ which is a natural requirement.

As asset price processes are not *a priori* assumed to be semimartingales stochastic integrals that reflect achievable gains from continuous trading strategies are not well-defined. To circumvent this problem, the set of trading strategies is restricted to permit trading at either deterministic times or stopping times. The "no free lunch" condition then postulates that the set of achievable gains contains no positive random variables. In a continuous time setting closure of the set of gains has to be considered which essentially depends on the topology on this set. Under a topology that makes use of certain dualities, Fritelli and Lakner [17] prove that there is "no free lunch" with trading strategies at deterministic times if and only if there exists an equivalent martingale measure. Furthermore, if every underlying process is right-continuous, then this result holds additionally for trading strategies at stopping times.

Henceforth, we assume "no free lunch" in the market as outlined above, so that the existence of an equivalent martingale measure is guaranteed.

In the following section, we examine the determination of insurance premiums in a no-arbitrage framework as introduced by Delbaen and Haezendonck [12].

4.2 No-Arbitrage Insurance Premiums

One ad-hoc approach of calculating insurance premiums would be to take the mathematical expected value of the underlying risk. However, an insurance company charging such a "pure premium" would not be able to survive. Therefore, a sensible insurance premium should be greater than the "pure premium" and the additional increase should reflect the insurer's administrative costs, capital costs, and the nature of the underlying risk more specifically. In practice, many different principles are used for calculating insurance premiums. The loading factor could be proportional to the expected value of the underlying risk or it could incorporate higher moments. Another loading factor could depend on agents' preferences that are reflected by some utility function. We refer to Goovaerts et al. [21] for a comprehensive outline of premium calculation principles.

Delbaen and Haezendonck [12] introduced the condition of "no-arbitrage" in an insurance market. Under the additional assumptions of liquidity and divisibility of insurance products, a premium calculation principle is deduced that includes commonly used principles as special cases. In fact, premiums are calculated as expected values under a different, equivalent probability measure. A certain loading factor can then be obtained by choosing the equivalent probability measure accordingly.

Therefore, insurance premiums can be understood as emerging from a hypothetical "no-arbitrage" framework. This standpoint has the advantage of providing a methodological link between financial and actuarial valuation. In this paper, we deduce results for financial prices that are consistent with specific loading factors. Hence our results do not rely on the "no-arbitrage" framework in the insurance market and can be derived independently for different premium calculation principles. In Section 5, we examine some commonly used principles. Nevertheless, let us briefly review the setup given by Delbaen and Haezendonck [12]:

Assume that the company's liabilities are of the form

$$X_t + p_t, \tag{5}$$

for all $0 \leq t \leq T$. The first component X_t denotes accumulated claims up to time t and the second component p_t describes the premium for which the insurance company can sell off the risk of the remaining period $(t, T]$.

A trading strategy in this setup means the possibility of 'take-over' and the company's liabilities thus represent the underlying price process. According to the fundamental theorem of asset pricing, the absence of arbitrage strategies implies the existence of a probability measure Q that is equivalent to the "reference" measure P and under which price processes are martingales.

If one further assumes that the predictable process $p = (p_t)_{0 \leq t \leq T}$ under Q is linear, i.e. of the form

$$p_t = p\,(Q)\,(T - t)\,, \tag{6}$$

then Delbaen and Haezendonck [12] conclude that the existence of sufficiently many reinsurance markets implies that the risk process X under Q is still a compound Poisson process.

As our risk process X has stationary and independent increments the martingale property implies that the premium density takes the form

$$\begin{aligned} p\,(Q) &= \mathbf{E}^Q\,[X_1] \\ &= \mathbf{E}^Q\,[N_1] \cdot \mathbf{E}^Q\,[Y_1]\,. \end{aligned} \tag{7}$$

In Section 2.3, the set of equivalent probability measures that preserve the structure of the underlying risk process X has been characterized by the market price of frequency risk κ and the market price of claim size risk $v\,(\cdot)$. Using the notation $P^{\kappa,v}$ for an equivalent probability measure the premium density corresponding to the pair $(\kappa, v\,(\cdot))$ is given by

$$\begin{aligned} p\,(P^{\kappa,v}) &= \mathbf{E}^{P^{\kappa,v}}\,[X_1] \\ &= \mathbf{E}^{P^{\kappa,v}}\,[N_1]\,\mathbf{E}^{P^{\kappa,v}}\,[Y_1] \\ &= \lambda\kappa \cdot \mathbf{E}^P\,[Y_1 \cdot v\,(Y_1)]\,. \end{aligned} \tag{8}$$

As pointed out and shown in an explicit example by Barfod and Lando [3], the premium density is not in one-to-one correspondence to the set of equivalent measures. This is a crucial difference to the one-to-one correspondence between financial prices of insurance derivatives and the set of equivalent measures.

In fact from the representation of the premium density (8) we deduce that there are infinitely many market prices of risk and therefore equivalent probability measures that lead to the same premium process. It is indeed this indeterminacy that does not pin down a unique financial price process under our additional requirement that financial prices should be consistent with actuarial valuation of the same underlying risk.

Before introducing this additional requirement let us focus on financial valuation of insurance-related risk.

4.3 No-Arbitrage Financial Prices

We denote by π_t the value at time t of a financial contract that pays out $\phi(X_T)$ at maturity T. In the absence of arbitrage strategies the fundamental theorem of asset pricing (see Harrison and Kreps [23], Harrison and Pliska [24], Fritelli and Lakner [17]) implies that the price process $\pi = (\pi_t)_{0\le t\le T}$ is a martingale under an equivalent probability measure $P^{\kappa,v}$. It can therefore be expressed as

$$\pi_t^{\kappa,v} = \mathbf{E}^{P^{\kappa,v}}\left[\phi(X_T)\,|\mathcal{F}_t\right], \tag{9}$$

for all $0 \le t \le T$ where the superscript κ, v states the dependence on the market prices of risk.

As the underlying risk process X is a Markov process and generates the filtration $(\mathcal{F}_t)_{0\le t\le T}$ π_t is of the form

$$\pi_t^{\kappa,v} = f^{\kappa,v}(X_t, t) = \mathbf{E}^{P^{\kappa,v}}\left[\phi(X_T)\,|X_t\right], \tag{10}$$

for some measurable function $f^{\kappa,v}$ with $f^{\kappa,v}(X_T, T) = \phi(X_T)$.

Let us assume that $\phi : \mathbb{R} \to \mathbb{R}$ is a measurable function such that $\phi(\cdot) - k \in \mathbf{L}^2(\mathbb{R}) = \left\{g : \mathbb{R} \to \mathbb{C} \text{ measurable} \mid \int_{-\infty}^{\infty} |g(x)|^2\,dx < \infty\right\}$ for some $k \in \mathbb{R}$. This assumption is satisfied by financial contracts with limited liability. We will now make use of Fourier analysis to calculate the expected payoff in (10).

The Fourier transformation is a one-to-one mapping of $\mathbf{L}^2(\mathbb{R})$ onto itself. In other words, for every $g \in \mathbf{L}^2(\mathbb{R})$ there corresponds one and only one $f \in \mathbf{L}^2(\mathbb{R})$ such that the Fourier transform of f is the function g, that is

$$f(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux} g(x)\,dx \tag{11}$$

is the inverse Fourier transform of g.

Applying the Fourier transform, and thereafter the inverse Fourier transform, to the function $\phi(\cdot) - k \in \mathbf{L}^2(\mathbb{R})$ we deduce

$$\phi(x) - k = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{iux}e^{-iuz}\left(\phi(z) - k\right)dz\,du. \tag{12}$$

With regard to (10) we get

$$\begin{aligned}
\pi_t^{\kappa,v} &= f^{\kappa,v}(X_t,t) = \mathbf{E}^{P^{\kappa,v}}\left[\phi(X_T)\,|\mathcal{F}_t\right] \\
&= \mathbf{E}^{P^{\kappa,v}}\left[\phi(X_T) - k\,|\mathcal{F}_t\right] + k \\
&= \frac{1}{2\pi}\mathbf{E}^{P^{\kappa,v}}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{iuX_T}e^{-iuz}\left(\phi(z)-k\right)dz du\,|\mathcal{F}_t\right] + k \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{E}^{P^{\kappa,v}}\left[e^{iuX_T}\,|\mathcal{F}_t\right]e^{-iuz}\left(\phi(z)-k\right)dz du + k \\
&= \int_{-\infty}^{\infty}\mathbf{E}^{P^{\kappa,v}}\left[e^{iuX_T}\,|\mathcal{F}_t\right]\check{\varphi}(u)\,du + k,
\end{aligned}$$

where we applied Fubini's theorem and $\check{\varphi}(\cdot)$ denotes the inverse Fourier transform of $\phi(\cdot) - k$, i.e.

$$\check{\varphi}(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuz}\left(\phi(z)-k\right)dz. \tag{13}$$

The inverse Fourier transform $\check{\varphi}(\cdot)$ can be derived explicitly by specifying the derivative's payoff structure.

Since a compound Poisson process is a Markov process with stationary and independent increments, we have

$$\begin{aligned}
\mathbf{E}^{P^{\kappa,v}}\left[e^{iuX_T}\,|\mathcal{F}_t\right] &= e^{iuX_t}\mathbf{E}^{P^{\kappa,v}}\left[e^{iu(X_T-X_t)}\,|X_t\right] \\
&= e^{iuX_t}\mathbf{E}^{P^{\kappa,v}}\left[e^{iuX_{T-t}}\right].
\end{aligned}$$

$\mathbf{E}^{P^{\kappa,v}}\left[e^{iuX_{T-t}}\right]$ is the characteristic function of the random variable X_{T-t} under the probability measure Q and given by

$$\chi_{T-t}^{\kappa,v}(u) = \exp\left(\lambda\kappa\left(\int_0^{\infty} e^{iuy}v(y)\,dG(y) - 1\right)(T-t)\right) \tag{14}$$

(see for example Karlin and Taylor [25] p.428).

Hence, the price at time t of the financial contract is given by

$$f^{\kappa,v}(X_t,t) = \int_{-\infty}^{\infty} e^{iuX_t}\chi_{T-t}^{\kappa,v}(u)\,\check{\varphi}(u)\,du + k. \tag{15}$$

This representation of no-arbitrage price processes enables us to derive the inverse Fourier transform of the price process in closed form. For a given value of the loss index $X_t = x$, we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\left(f^{\kappa,v}(x,t) - k\right)dx = \chi_{T-t}^{\kappa,v}(u)\cdot\check{\varphi}(u). \tag{16}$$

The inverse Fourier transform is thus the product of two factors where the first, the characteristic function, contains the whole stochastic structure and the second solely depends on the contract's payoff. Therefore the characteristic function is the important component in linking financial prices with insurance premiums under our concept of consistency that we introduce in the following section.

4.4 Actuarially Consistent No-Arbitrage Prices

This section presents an internal consistency requirement that we impose on financial prices in addition to the exclusion of arbitrage strategies. Although the consistency requirement reflects a further restriction on the possible dynamics of financial prices it is not strong enough to pin down a unique price process. Nevertheless, we characterize the remaining set of price processes and derive a connection between financial and actuarial prices.

As outlined above the market consists of an insurance contract and a financial contract that are both written on the same underlying risk process X. The insurance specifies a premium process $(p_t)_{0\leq t\leq T}$ of the linear form

$$p_t = p \cdot (T-t) \tag{17}$$

with premium density p for selling off the remaining risk $X_T - X_t$. The financial contract specifies a price process $(\pi_t)_{0\leq t\leq T}$ for the payoff $\phi(X_T)$ at maturity.

Internal consistency should require that the financial valuation is consistent with actuarial valuation in the sense that market prices for frequency and claim size risk that lead to the specified premium density are inherent in financial prices.

The following proposition is the main result of this paper linking financial with actuarial prices on the basis of internal consistency as described above.

Proposition 2 *Let $X = (X_t)_{0\leq t\leq T}$ be a compound Poisson process with characteristics $(\lambda, dG(y))$ and let $(p_t)_{0\leq t\leq T}$ be a linear premium process specified in the insurance contract. Suppose that the function $\phi : \mathbb{R}_+ \to \mathbb{R}$ specifies the payoff of the financial contract at time T and satisfies the integrability condition $\phi(\cdot) - k \in \mathbf{L}^2(\mathbb{R}_+)$ for some $k \in \mathbb{R}$. Then for a given market price of severity risk $v : \mathbb{R}_+ \to \mathbb{R}$ with $\mathbf{E}^P[v(Y_1)] = 1$ the function $f^v : \mathbb{R}_+ \times [0,T] \to \mathbb{R}$ defining the financial price process $(f^v(X_t,t))_{0\leq t\leq T}$ that excludes arbitrage strategies and is consistent with the premium process can be represented as*

$$f^v(x,t) = \int_{-\infty}^{\infty} e^{iux} \exp\left(p_t \cdot \frac{\mathbf{E}^P\left[e^{iuY_1} \cdot v(Y_1) - 1\right]}{\mathbf{E}^P\left[Y_1 \cdot v(Y_1)\right]}\right) \check{\varphi}(u)\, du + k, \tag{18}$$

where $\check{\varphi}(\cdot)$ is the inverse Fourier transform of $\phi(\cdot) - k$.

Proof. Internal consistency requires that the market prices of risk characterizing financial no-arbitrage prices lead to the same premium process. This set of market prices of frequency risk κ and claim size risk $v(\cdot)$ can be described by equation (8), that is

$$p = \lambda\kappa \cdot \mathbf{E}^P \left[Y_1 \cdot v(Y_1) \right],$$

and the corresponding premium process $(p_t)_{0 \leq t \leq T}$ is thus given by

$$p_t = \lambda\kappa \cdot \mathbf{E}^P \left[Y_1 \cdot v(Y_1) \right] \cdot (T - t).$$

Substituting this expression into the representation (15) of no-arbitrage financial prices describes financial prices that are consistent with the specified premium process and completes the proof. ■

If we subtract k and apply the inverse Fourier transform on both sides of equation (18) we deduce for every given value $X_t = x$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \left(f^v(x,t) - k \right) dx = \exp\left(p_t \cdot \frac{\mathbf{E}^P \left[e^{iuY_1} \cdot v(Y_1) - 1 \right]}{\mathbf{E}^P \left[Y_1 \cdot v(Y_1) \right]} \right) \cdot \check{\varphi}(u),$$

or alternatively

$$p_t = \ln\left(\frac{\int_{-\infty}^{\infty} e^{-iux} \left(f^v(x,t) - k \right) dx}{\int_{-\infty}^{\infty} e^{-iux} \left(\phi(x) - k \right) dx} \right) \cdot \frac{\mathbf{E}^P \left[Y_1 \cdot v(Y_1) \right]}{\mathbf{E}^P \left[e^{iuY_1} \cdot v(Y_1) - 1 \right]}.$$

We observe that financial prices under the additional requirement of actuarial consistency can still not be determined uniquely. Nevertheless, the set of prices can be parameterized solely by the market price of claim size risk. The indeterminacy is an implication of the fact that there are many market prices of risk that lead to the same actuarial price.

This is an important difference to financial prices where it is possible to back out market prices of risk from financial prices in a unique way. We therefore conclude that a premium process is uniquely determined by requiring it to be consistent with a given financial price process as it uniquely determines the market prices of risk. The consistent premium density is then determined by equation (8).

In the following section, we investigate some premium calculation principles that are commonly used by the insurance industry and derive financial price processes that are actuarially consistent.

5 Premium Calculation Principles

As mentioned in the beginning of Section 4.2, reasonable insurance premiums contain a factor in addition to the "pure" mathematical expectation of the underlying risk. The explicit form of this loading factor differs depending on the risk's nature. In the no-arbitrage framework introduced by Delbaen and Haezendonck [12] this is reflected by the fact that the expected value of the underlying risk is taken under different probability measures. The additional factor is thus inherently related to the choice of the equivalent probability measure, i.e. to the market prices of frequency and claim size risk.

We examine three different premium calculation principles and derive a representation of the corresponding market prices of risk. This allows us to represent financial prices that are in line with the respective insurance premiums.

5.1 Expected Value Principle

Under the expected value principle the premium density is given by

$$p = (1+\delta)\,\mathbf{E}^P\left[X_1\right] = (1+\delta)\,\lambda\mathbf{E}^P\left[Y_1\right],$$

for some $\delta > 0$. This premium calculation principle is mainly used in life insurance because of the homogeneity of the collectives.

If we choose

$$\kappa = (1+\delta)\,\frac{\mathbf{E}^P\left[Y_1\right]}{\mathbf{E}^P\left[Y_1 \cdot v\left(Y_1\right)\right]}$$

as a function of $v(\cdot)$ with $\mathbf{E}^P\left[v\left(Y_1\right)\right] = 1$ we have thus characterized the set of parameters κ and $v(\cdot)$ that correspond to this premium calculation principle.

Furthermore, for any market price of claim size risk $v(\cdot)$ with $\mathbf{E}^P\left[v\left(Y_1\right)\right] = 1$ the function f^v defining the financial price process that is consistent with the expected value principle can be represented as

$$f^v\left(X_t, t\right) = \int_{-\infty}^{\infty} e^{iuX_t}\chi^v_{T-t}\left(u\right)\check{\varphi}\left(u\right)du + k,$$

where the characteristic function is given by

$$\chi^v_{T-t}\left(u\right) = \exp\left(\lambda\left(1+\delta\right)\cdot\frac{\mathbf{E}^P\left[Y_1\right]\cdot\mathbf{E}^P\left[e^{iuY_1}v\left(Y_1\right)-1\right]}{\mathbf{E}^P\left[Y_1 v\left(Y_1\right)\right]}\cdot\left(T-t\right)\right).$$

5.2 Variance Principle

The variance principle is mostly used in property and casualty insurance. It additionally includes fluctuations of X and the premium density is calculated according to

$$p=\lambda\left(\mathbf{E}^P\left[Y_1\right]+\beta\cdot\mathbf{Var}^P\left[Y_1\right]\right),$$

for some $\beta>0$.

To be consistent with this premium density the market price of frequency risk for a given market price of claim size risk $v(\cdot)$ with $\mathbf{E}^P\left[v\left(Y_1\right)\right]=1$ has to be determined through

$$\kappa=\frac{\mathbf{E}^P\left[Y_1\right]+\beta\cdot\mathbf{Var}^P\left[Y_1\right]}{\mathbf{E}^P\left[Y_1\cdot v\left(Y_1\right)\right]}.$$

The function f^v defining financial price processes that are consistent with this premium calculation principle can be represented as

$$f^v\left(X_t,t\right)=\int_{-\infty}^{\infty}e^{iuX_t}\chi_{T-t}^v\left(u\right)\check{\varphi}\left(u\right)du+k,$$

where the characteristic function is given by

$$\chi_{T-t}^v\left(u\right)=\exp\left(\frac{\lambda\cdot\left(\mathbf{E}^P\left[Y_1\right]+\beta\mathbf{Var}^P\left[Y_1\right]\right)\cdot\mathbf{E}^P\left[e^{iuY_1}v\left(Y_1\right)-1\right]}{\mathbf{E}^P\left[Y_1v\left(Y_1\right)\right]}\cdot\left(T-t\right)\right).$$

5.3 Esscher Principle

The last example of premium calculation principles we investigate is the so-called Esscher principle that is gaining more and more attention as it can be derived from equilibrium analysis or from the minimization of a particular loss function. It is defined by a premium density of the form

$$p=\lambda\cdot\frac{\mathbf{E}^P\left[Y_1e^{\gamma Y_1}\right]}{\mathbf{E}^P\left[e^{\gamma Y_1}\right]},$$

for some $\gamma\in\mathbb{R}\backslash\{0\}$.

Here κ depends on the density function $v(\cdot)$ through

$$\kappa=\frac{\mathbf{E}^P\left[Y_1e^{\gamma Y_1}\right]}{\mathbf{E}^P\left[e^{\gamma Y_1}\right]\cdot\mathbf{E}^P\left[Y_1\cdot v\left(Y_1\right)\right]},$$

and the function f^v defining the price process that corresponds to this premium principle for a given market price of claim size risk $v(\cdot)$ can be expressed as

$$f^v\left(X_t,t\right)=\int_{-\infty}^{\infty}e^{iuX_t}\chi_{T-t}^v\left(u\right)\check{\varphi}\left(u\right)du+k,$$

where the characteristic function is given by

$$\chi_{T-t}^v\left(u\right)=\exp\left(\frac{\lambda\cdot\mathbf{E}^P\left[Y_1e^{\gamma Y_1}\right]\cdot\mathbf{E}^P\left[e^{iuY_1}v\left(Y_1\right)-1\right]}{\mathbf{E}^P\left[e^{\gamma Y_1}\right]\cdot\mathbf{E}^P\left[Y_1v\left(Y_1\right)\right]}\cdot\left(T-t\right)\right).$$

6 Conclusion

In this paper we investigated valuation in a market that contains both insurance and financial contracts written on the same underlying compound Poisson process. We examined both corresponding valuation principles - actuarial and financial - on the basis of excluding arbitrage opportunities and deduced a representation of prices for given market prices of frequency and claim size risk.

We introduced a new concept arising from internal consistency that originates in the coexistence of financial and insurance contracts. Financial prices should be consistent with the actuarial valuation of the insurance contract. Although financial prices cannot be uniquely determined, under this additional restriction on their dynamics, we characterized the set of prices that fulfill both absence of arbitrage and actuarial consistency. Through this characterization we established a link between financial price processes and insurance premiums. This connection is wholly incorporated in the characteristic function of the underlying risk process.

We clarified that an important difference between financial and actuarial valuation is contained in the mapping between price processes and market prices of risk. The mapping between financial price processes and market prices of risk is one-to-one whereas there are infinitely many market prices of risk that lead to the same premium process. This implies that premium processes are uniquely determined by assuming them to be consistent with a given financial price process. However, consistency with a given premium process is not strong enough for financial prices to be uniquely determined.

Finally, we examined three premium calculation principles that are widely used by the insurance industry. A representation of financial price processes were derived that are consistent with the respective premium calculation principle.

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