# **Model Uncertainty and Portfolio Insurance**

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by

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### **Model Uncertainty and Portfolio Insurance**

**Abstract**. Some real-world insurance products contain a minimum wealth or a income stream guarantee, both of which have to be met irrespective of capital market conditions. Therefore, the seller of such products will have to choose that portfolio strategy that performs best in a reasonable worst case capital market scenario, as the literature under model uncertainty (in particular Anderson/Hansen/Sargent (2000)) suggests, if he wants to avoid additional cash payments.

This paper shows that this solution to the portfolio problem crucially hinges on the assumption that model uncertainty is taken into account by adding an explicit preference for models' similarity to the objective function of the decision problem, a so-called preference for robustness. If there are strictly to meet minimum investment goals instead, as in the case of the real-world insurance products cited above, the Anderson/Hansen/Sargent (2000) solution will not exist in general. Then, only one trivial portfolio strategy is able to defend minimum investment goals, namely invest in the riskless asset the amount guaranteed discounted at the riskfree rate.

**Key Words**: model uncertainty, portfolio selection, minimum wealth or income stream guarantee, Portfolio Insurance, robustness

JEL Classification numbers: G11

### Model Uncertainty and Portfolio Insurance

#### **1** Preliminaries

#### 1.1 Introduction to the problem

In real-world financial markets, insurance products are observable that offer their buyers a minimum wealth or a income stream guarantee. Two prominent examples are money back guarantees at maturity (guaranteed minimum wealth), in Germany known under the label "Riester products", and life annuities (guaranteed income stream). By construction, sellers of these products are obliged to meet their guarantees irrespective of capital market conditions. Therefore, they will be well-advised to pursue a portfolio strategy that is able to fulfill this requirement if they want to avoid additional cash payments.

The literature under model uncertainty (see in particular Anderson/Hansen/Sargent (2000)) seems to offer a solution to this portfolio problem by proposing the following non-trivial portfolio strategy, non-trivial in the sense that wealth is not completely invested in the risk-less asset: follow that portfolio strategy that performs best in a reasonable worst case capital market scenario. Reasonable is particularized as in a certain way close to a reference model (see Cagetti/Hansen/Sargent/Williams (2002), p. 374).

It is the objective of this paper to show that Anderson/Hansen/Sargent's (2000) solution does not work in general under model uncertainty. The reason for this outcome is that their solution procedure crucially hinges on the assumption that model uncertainty is taken into account by adding an explicit preference for models' similarity to the objective function of the decision problem, a so-called preference for robustness. This preference for robustness indeed leads to a more conservative investment strategy (see Cagetti/Hansen/Sargent/Williams (2002), p. 373). However, this investment strategy will be still too aggressive if there are strictly to meet minimum investment goals. As a rule, only a full investment of the amount guaranteed in the riskless asset is able to meet these minimum investment goals.

To offer a proof of this statement, portfolio strategies for defending a guaranteed minimum wealth (Option Based Portfolio Insurance) and a guaranteed income stream (Constant Proportion Portfolio Insurance) are calculated for several capital market scenarios.

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Option Based Portfolio Insurance calls for duplication of the option implied by this minimum wealth guarantee. A duplication portfolio, though, is fitted to a concrete price process and can handle this price process only. For that reason, there is just one price process that allows for duplication and this price process is the worst case scenario at the same time which means that the idea of a worst case scenario is reduced to absurdity. Consequently, the only portfolio strategy that is capable of defending guaranteed wealth irrespective of the capital market environment reads: invest in the riskless asset guaranteed wealth discounted at the riskfree rate.

In the case of Constant Proportion Portfolio Insurance, things turn out to be slightly more sophisticated. Whenever there is a non-trivial portfolio strategy, there is no pronounced worst case scenario, in that all portfolio weights have the same structure irrelevant of the capital market scenario chosen. A clear-cut worst case scenario will occur, however, if stock market crashes are added to capital market conditions: the maximum possible negative jump amplitude  $\varphi_{extr}$ . But then the optimum portfolio weight turns out to be trivial; it invests in the riskless asset  $(1 - \varphi_{extr})$  times guaranteed income per period discounted at the riskfree rate.

The results of the paper sketched above extend the literature in two ways. First, they integrate an environment observable in the pragmatic world (minimum investment goals) into portfolio selection under model uncertainty. The model uncertainty literature so far, follows Anderson/Hansen/Sargent's (2000) setup and copes with model uncertainty solely with the help of a theoretical concept, an explicit preference for robustness – although this preference shows different degrees of complexity in Maenhout (2001), Trojani/Vanini (2001), and Uppal/Wang (2002). Second, the paper supplements the Portfolio Insurance literature. It shows how to adapt Option Based Portfolio Insurance even to an environment with several sources of uncertainty by using roll-over Option Based Portfolio Insurance. Thereby, Rubinstein (1985) is generalized and a guess formulated in Geman (1992) and Leland (1992) proven. In addition, Constant Portfolio Insurance strategies are modified to work in a stochastic volatility environment, which means Black/Jones (1987) is extended.

This article is organized as follows. The rest of Section 1 gives some definitions, and outlines the framework of the model used. Section 2 calculates dynamic Portfolio Insurance strategies for various classes of stock price processes. These results are interpreted in the light of model uncertainty in Section 3. Section 4 concludes the paper; an appendix follows.

#### 1.2 Characterization of model uncertainty

#### 1.2.1 Definition and particularization

A comprehensive definition of model uncertainty is missing in the literature. There are just some scattered, not always consistent descriptions. For example, Xia (2001, p. 211) denotes a situation in which the drift parameter of a geometric Brownian motion is not exactly known "parameter risk" (or estimation risk), whereas the same situation is called "model misspecification stemming from parameter risk" by Maenhout (2001, p. 3). Anderson/Hansen/Sargent (2000, p. 6) argue that model misspecification/uncertainty results in an additional risk term g(t) besides the increment d z(t) of a Wiener process. Therefore, total risk reads g(t) + d z(t), i.e., is the sum<sup>1</sup> of model uncertainty g(t) and market risk dz(t), where the decision maker does not know the distribution of g(t) (see Anderson/Hansen/Sargent (2000, p. 7)). Finally, Anderson/Hansen/Sargent's (2000, p. 9) and Cagetti/Hansen/Sargent/Williams (2002, p. 374) point out that model uncertainty must be close to the reference model dz(t) – a reasonable worst case scenario must exist – because otherwise the reference model would be easily rejected empirically.

Structuring the arguments of the above paragraph, it is first necessary to distinguish between parameter risk and model uncertainty. Under parameter risk, the parameters of a stochastic process change stochastically over time like, e.g., in Merton's (1973) stochastic volatility model. However, the presence of two types of risk, estimation and market risk, is not sufficient for model uncertainty to be present; model uncertainty means that the shape of market risk itself must shift unforeseeably as Anderson/Hansen/Sargent's (2000) and Uppal/Wang's (2002) formulation g(t) + dz(t) implies. Therefore, second, the question arises how to particularize an unforeseeable shift of the shape of market risk. "Unforeseeably" means that the probabilities of this shift are unknown<sup>2</sup> to the decision maker. A shift of the shape of market risk signifies that the form of the stochastic process shifts (and not just its parameters). Reviewing the examples contained in Anderson/Hansen/Sargent's (2000), Maenhout (2001), and Trojani/Vanini (2001), one particularization of a shift of the

<sup>&</sup>lt;sup>1</sup> Alternatively, this statement can be formulated as follows: the distribution function of market risk is subject to a multiplicative distortion – a Radon-Nikodym derivative unequal to one – (see Anderson/Hansen/Sargent (2000), p. 22 and Uppal/Wang (2002), p 4).

<sup>&</sup>lt;sup>2</sup> That is the reason why I speak of "model uncertainty" and not "model risk". Since for estimation risk there is a stochastic process describing its distribution, see, e.g., Merton's (1973) stochastic volatility model, the term "parameter risk" is justified.

form of a stochastic process encompasses a shift between diffusion processes, e.g., the transition from a constant opportunity set to a deterministic or stochastic opportunity set, or between jump processes in the same mold. Another way to particularize Anderson/Hansen/Sargent's (2000) and Uppal/Wang's (2002) formulation of model uncertainty is to make shifts between diffusion and jump/diffusion processes admissible. Yet, a shift from "normal" to extraordinary price movements seems to violate Cagetti/Hansen/Sargent/Williams's (2002) empirical detection criterion. There are two arguments, though, that indicate this conclusion might be too hasty. First, the inclusion of jumps does not necessarily change stocks' distribution. For example, if 1 + jump amplitude is logarithmic-normally distributed as in Merton (1976), investor's wealth has the same type of distribution as in the case of a (pure) geometric Brownian motion. Second, Bates (2000, p. 182, 183) points out that there are two models vying to explain negative skewness in stocks' returns after the '87 crash: stochastic volatility and jumps. Since both explanation describe the same phenomenon, they are rather difficult to distinguish empirically.

To be able to differentiate between both particularizations of model uncertainty, I call an intra-model-shift (from diffusion to diffusion or jump/diffusion to jump/diffusion) homogenous model uncertainty, an inter-model-shift (from diffusion to jump/diffusion) heterogeneous model uncertainty. It is important to remember, however, that both homogenous and heterogeneous model uncertainty are nevertheless particularizations of Cagetti/Hansen/Sargent/Williams's (2002, p. 374) phrase "close to the reference model".

Based on the discussion above, I define and particularize model uncertainty as follows: model uncertainty denotes an unforeseeable shift between different classes of stock price processes where the probabilities of this shift are unknown. This shift might involve homogenous model uncertainty (e.g., shift from a diffusion process with one source of risk to one with two sources of risk) or heterogeneous model uncertainty (e.g., shift from a diffusion to a combined jump/diffusion process).

Comparing my definition and particularization of model uncertainty with the literature, there are two differences. First, I particularize Anderson/Hansen/Sargent's (2000) and Up-pal/Wang's (2002) general formalization of model uncertainty by identifying it as an unforeseeable shift from one class of stochastic process to another. Second, I generalize existing particularizations of model uncertainty in continuous time, which all (see Anderson/Hansen/Sargent's (2000), p. 25, 26, Maenhout (2001), p. 13, Trojani/Vanini (2001), p. 7, and Uppal/Wang (2002), p. 10) work within homogenous model uncertainty, by increasing the degree of heterogeneity of model uncertainty.

#### 1.2.2 Integration of model uncertainty into portfolio selection

To integrate model uncertainty into the decision problem "portfolio selection", the literature follows two different directions. First, a Bayesian framework can be used, like in Xia (2001) for pure estimation risk, and in Avramov (2001) for pure model risk.<sup>3</sup> This means, decision makers integrate in the calculation of expected utility not just the probabilities for certain payoffs within market risk, but also their probability estimates with respect to the class of the stock price model itself. Critical for this procedures is the knowledge of the probability that a certain class of price process will occur. Since this information is missing according to the definition of model uncertainty proposed in this article, the Bayesian approach is not pursued any further. Second, Anderson/Hansen/Sargent (2000) take model uncertainty into account by adding an explicit preference for models' similarity, so-called robustness, to the objective function of the decision problem. The economic reason for this is explained best by Cagetti/Hansen/Sargent/Williams's (2002, p. 373): a preference for robustness leads to a more conservative investment strategy. Technically, a preference for robustness is obtained from a two person game (see Anderson/Hansen/Sargent (2000), p. 9). A malevolent player chooses the state of nature while large deviations from a reference model are penalized so that the state chosen remains (rather) close to the reference model. The decision maker (second player) then selects the best portfolio strategy depending on the (worst) state selected by the first player.

Although this approach is undoubtedly able to integrate model uncertainty into the decision problem, its resulting portfolio strategies are not appropriate to handle the obligations from real-world insurance products with minimum investment goal. A preference for robustness results in a portfolio strategy that tries to minimize the utility loss in the worst case state. This utility loss, however, might be too low to persuade a decision maker to defend minimum investment goals. The only way to achieve this, will be to impose a utility of  $-\infty$  on the decision maker if minimum investment goals are not met, i.e., to add a strict minimum wealth or a income stream constraint to the decision problem.

For that reason, I take model uncertainty into account by analyzing the following decision problem: decision makers maximize their expected utility subject to a minimum wealth or a income stream constraint. That is, I fall back on the ideas of the Portfolio Insurance literature (e.g. Black/Perold (1992)) to analyze model uncertainty although this literature

<sup>&</sup>lt;sup>3</sup> Since the Bayesian approach works with probabilities, the term "model risk" instead of "model uncertainty" is justified.

does not refer to model uncertainty in an explicit way. I do not follow the model uncertainty literature in narrower sense, who always works with an explicit preference for robustness: Anderson/Hansen/Sargent (2000), the pioneering work, and their extensions in form of several formalizations of robustness (minimum entropy, homothetic, and constrained robustness) (see Tojani/Vanini (2001)) or "differences in the degree of ambiguity" about the returns of different assets (Uppal/Wang (2002), p. 3).

#### 1.3 Framework of the analysis

To be able to elaborate what consequences the transition from a preference for robustness to minimum investment goal constraints has for portfolio selection under model uncertainty, a framework of the analysis is needed that corresponds with the one of the literature aside from the objective function of the decision problem. Therefore, it is well-advised to use the literature's standard set of assumptions:<sup>4</sup>

- 1. Capital markets are free of arbitrage and perfect, i.e., short selling constraints or transaction costs do not prevent minimum investment goals from being met.
- 2. Trading happens in continuous time, i.e., a potential lack of transaction speed does not cause a violation of minimum investment goals.
- 3. There is a riskless asset in the market with dynamic

$$dP(t) = r P(t) d t$$
<sup>(1)</sup>

where P(t) denotes the price of the riskless asset at time t, r its interest rate, and dt a time period of infinitesimal length

4. There is one risky asset (stock market index), i.e., there is no basis risk that endangers minimum investment goals.

In addition to these four standard assumptions, another one is added that reflects the particularization of model uncertainty developed in this paper:

5. The price process of the stock index is subject to model uncertainty since it can follow a geometric Brownian motion (one source of risk), Merton's (1973) stochastic volatility model (two homogenous sources of risk due to estimation risk), and a combined jump/

<sup>&</sup>lt;sup>4</sup> These assumptions are extracted from the core articles of the model uncertainty and the Portfolio Insurance literature, i.e., And erson/Hansen/Sargent (2000), Maenhout (2001), Trojani/Vanini (2001), Cagetti/ Hansen/Sargent/Williams (2002), and Uppal/Wang (2002) as well as Black/Perold (1992).

diffusion model (infinitesimal and non-infinitesimal price changes and, thus, two heterogeneous sources of risk).

Start with the formalization of the geometric Brownian motion, the reference model (or with g(t) = 0 in the language of Anderson/Hansen/Sargent (2000)):

$$dS(t) = \alpha S(t) dt + \sigma S(t) dz(t)$$
<sup>(2)</sup>

where S(t) denotes the price of the stock index at time t, dS(t) its infinitesimal price change,  $\alpha$  its per unit time mean,  $\sigma$  its per unit time standard deviation, and dz(t) the increment of a Wiener process, the only source of risk in this class of stock price model.

Under a combined jump/diffusion process the index evolves according to (or g(t) = the stochastic differential equation for a Poisson process, see Merton (1976), p. 128)

$$dS(t) = \alpha S(t) dt + \sigma S(t) dz(t)$$
(3)

with probability  $1 - \lambda dt$  (diffusion case)

$$\Delta \mathbf{S}(t) = \mathbf{S}(t^{-})(1+\phi) - \mathbf{S}(t^{-})$$

with probability  $\lambda dt$  (jump case)

with  $\Delta$  signifying a large jump-induced, i.e., non-infinitesimal price change,  $\lambda$  d t denoting the probability that a jump occurs between time t and t + dt<sup>5</sup>, and t<sup>-</sup> meaning a point in time immediately before time t

Finally, Merton's (1973, p. 873) stochastic volatility model reads<sup>6</sup>

$$d\sigma(t) = f\sigma(t)dt + g\sigma(t)dz_{\sigma}(t)$$
(4)

with f and g denoting per unit time mean and standard deviation of  $\frac{d \sigma(t)}{\sigma(t)}$ , the relative change in volatility

<sup>&</sup>lt;sup>5</sup> This specification of a jump probability does not contradict the statement that decision makers are unable to specify probabilities for a certain stock price process under model uncertainty. For this probability just states that a jump occurs with probability  $\lambda$  d t assuming that a combined jump/diffusion model is the correct specification of the stock price model. It should, thus, not be confused with an assertion on the probability that the combined jump/diffusion model itself is valid.

<sup>&</sup>lt;sup>6</sup> For simplicity, a stochastic drift is not considered. It does not produce any insights that exceed those of stochastic volatility.

Consequently, one obtains the following dynamic of the index under stochastic volatility (or  $g(t) = (\sigma(t) - \sigma) dz(t)$ ):

$$dS(t) = \alpha S(t) dt + \sigma(t) S(t) dz(t)$$
(5)

#### 2 Portfolio Insurance

Decision problems considered in this article, namely maximization of expected utility under a minimum investment goal constraint, belong to the category of so-called Portfolio Insurance strategies. It is necessary, however, to distinguish between Portfolio Insurance strategies under minimum wealth guarantee constraints (Option Based Portfolio Insurance) and those under a guaranteed income stream constraint (Constant Proportion Portfolio Insurance). Although wealth can be transformed into an annuity, a minimum wealth guarantee is unequal to an income stream guarantee because this annuity turned wealth may or may not last over the uncertain lifespan of an individual whereas a guaranteed income stream is due during the whole lifespan of an individual. Likewise, a guaranteed income stream differs from a minimum wealth guarantee; from the fact that it is possible to defend an income stream and the observation that an income stream can be accumulated to a wealth level cannot be concluded that it is possible to defend guaranteed minimum wealth with the help of this detour. This income stream needs some re-investment to be transferred into wealth wherefore the search for a non-trivial portfolio strategy to guarantee minimum wealth starts anew.

#### 2.2 Option Based Portfolio Insurance

A minimum wealth guarantee promises its buyer – to keep things simple, I assume that there is just one buyer – at the beginning of his retirement at T to pay back his investment in bad states and to capitalize on a positive wealth development in good state: the buyer receives wealth minus some administrative fees; formally:

$$\max\{(1-k)W(T), K\}$$
(6)

where W(T) denotes total wealth of the seller's portfolio, K the guaranteed minimum wealth (so-called floor), and k the percentage of (proportional) administrative fees

Having modeled the obligatory payoff for the seller of the guarantee in equation (6), he, i.e., the decision maker in my framework, faces the following decision problem

## $\underset{w(t)}{\text{Max}} E\{U[k \cdot W(T)]\}$

s.t.  $(1-k)W(T) \ge K$ 

wealth dynamic according to either (2) or (3) or (5)

In words, he maximizes his expected utility of (risky) payoffs from administrative fees at time T under an minimum wealth constraint by choosing an optimum portfolio strategy. The key element of the decision problem (7) is the constraint (6) which can be written as

$$\max\{(1-k)W(T), K\} = (1-k)W(T) + \underbrace{\max\{K-(1-k)W(T), 0\}}_{F}$$
(8)

that is, the constraint is identical with an investment in the index and the purchase of a put option with strike price K and maturity at T.

Since real-world options do not necessarily offer the desired strike price nor the adequate maturity, the implied put's payoff at time T has to be duplicated by a dynamic portfolio strategy. This means, the solution to the decision problem (7) involves as an integral part a well-known portfolio strategy, namely Option Based Portfolio Insurance, as Grossman/ Zhou (1996, p. 1379) have shown. Depending on the assets from which the duplication portfolio is constructed, there is: first, "classical" Option Based Portfolio Insurance (see Leland (1980) and Rubinstein/Leland (1981)), which uses a portfolio made of the option's underlying and the riskless asset; second, roll-over Portfolio Insurance with options (see Rubinstein (1985)). It employs traded options on the same underlying, but with shorter maturities to duplicate the implied put. After the maturity of the first set of options, it switches to the next set of options and so forth, until the implied put becomes due at time T.

#### 2.2.1 "Classical" Option Based Portfolio Insurance under different volatility scenarios

Leland (1980) has demonstrated under geometric Brownian motion that the price of the implied put option at every time t and, thus, also at maturity T, can be duplicated by trading in the index and the riskless asset because implied option and index depend linearly on the same single source of risk. The duplication portfolio consists of investing the number  $F_s$  in stocks (delta of the option calculated with the help of the Black/Scholes formula) and the amount  $F - F_s S$  in the riskless asset. However, both under combined jump/diffusion processes (see Merton (1976)) and stochastic volatility (see Johnson/Shanno (1987), and Hull/White (1987)) duplication fails. There are two types of risk, but just one risky asset.

(7)

Therefore, decision makers can eliminate one risk only, leaving them fully exposed with respect to the other type of risk.

#### 2.2.2 Roll-over Portfolio Insurance with options under geometric Brownian motion

The dynamic of every derivative F<sup>j</sup> in the market under geometric Brownian motion reads:

$$dF^{j}(t) = F_{t}^{j}dt + F_{s}^{j}\alpha S(t)dt + F_{s}^{j}\sigma S(t)dz(t) + \frac{1}{2}F_{ss}^{j}\sigma^{2}S^{2}(t)dt$$
(9)

Complete replication requires that the stochastic component of the implied put F must coincide with the one of another derivative  $F^{i}$ . This can be achieved (see Rubinstein (1985), p. 46) by acquiring the number

$$N^{i}(t) = \frac{F_{s}^{i}}{F_{s}}$$
(10)

of one arbitrary derivative F<sup>i</sup>.

#### 2.2.3 Roll-over Portfolio Insurance with options under combined jump and diffusion risk

$$dF^{j}(t) = F_{t}^{j} dt + F_{S}^{j} \alpha S(t) dt + F_{S}^{j} \sigma S(t) dz(t) + \frac{1}{2} F_{SS}^{j} \sigma^{2} S^{2}(t) dt$$
(11)

with probability  $1 - \lambda dt$  (diffusion case)

$$F^{j}(S(t^{-})(1+\phi)) - F^{j}(S(t^{-}))$$
  
with probability  $\lambda$  dt (jump case)

denotes the price dynamic of every derivative F under combined jump/diffusion risk. Since there are two sources of uncertainty, jump and diffusion risk, two derivatives are required to duplicate the implied put option. At exactly this point the advantage of roll-over Portfolio Insurance over "classical" Portfolio Insurance becomes visible. Following Ross (1976), increasing the number of derivatives makes the market "more complete", i.e., allows for more sources of risk to be duplicated, although the number of spot market instruments does not change. Consequently, the desired duplication portfolio under combined jump/diffusion risk first exists and, second, reads

$$N^{i}(t) = \frac{F_{s} \Delta F^{k} - F_{s}^{k} \Delta F}{F_{s}^{i} \Delta F^{K} - F_{s}^{k} \Delta F^{i}}$$
(12a)

$$N^{k}(t) = \frac{F_{S}^{i} \Delta F - F_{S} \Delta F^{i}}{F_{S}^{i} \Delta F^{K} - F_{S}^{k} \Delta F^{i}}$$
(12b)

Equations (12a) and (12b) put in place a widely found statement (see, e.g., Geman (1992), p. 187, and Zhou/Kavee (1988), p. 54) namely that Option Based Portfolio Insurance does not work in a jump/diffusion environment. These equations demonstrate that this statement is true for "classical" Portfolio Insurance only, but not for roll-over Portfolio insurance strategies. That way, equations (12a) and (12b) elaborate Leland's (1992, p. 155) idea of extending Option Based Portfolio Insurance to an environment of combined jump/diffusion risk and generalize a result of Rubinstein (1985, p. 49), who just uses one option  $F^i$  to duplicate the implied put. His result, however, holds just when both  $F_s = a \cdot F_s^i$  (with a an arbitrary constant) and  $\Delta F = a \cdot \Delta F^i$  are true.

#### 2.2.4 Roll-over Portfolio Insurance with options under stochastic volatility

The price of every derivative  $\vec{F}$  in a market under stochastic volatility has the following dynamic:

$$dF^{j}(t) = F_{t}^{j} dt + F_{s}^{j} \alpha S(t) dt + F_{s}^{j} \sigma(t) S(t) dz(t) + \frac{1}{2} F_{ss}^{j} \sigma^{2}(t) S^{2}(t) dt$$

$$+ F_{\sigma}^{j} f \sigma(t) dt + F_{\sigma}^{j} g \sigma(t) dz_{\sigma} + \frac{1}{2} F_{\sigma\sigma}^{j} g^{2} \sigma^{2}(t) dt$$
(13)

To eliminate both sources of uncertainty, diffusion and volatility risk, the duplication portfolio of the implied put consists of two derivatives, whose numbers can be obtained in a way similar to equations (12a) and (12b):

$$N^{i}(t) = \frac{F_{s} F_{s}^{k} - F_{s}^{k} F_{\sigma}}{F_{s}^{i} F_{\sigma}^{k} - F_{s}^{k} F_{\sigma}^{i}}$$
(14a)

$$N^{k}(t) = \frac{F_{S}^{i} F_{\sigma} - F_{S}^{k} F_{\sigma}^{i}}{F_{S}^{i} F_{\sigma}^{k} - F_{S}^{k} F_{\sigma}^{i}}$$
(14b)

Again, equations (14b) and (14b) put into practice an idea of Leland (1992, p. 155) to adapt Option Based Portfolio Insurance to stochastic volatility, specify Geman (1992, p. 187), namely that only "classical" Option Based Portfolio Insurance does not work under stochastic volatility, and generalize Rubinstein (1985, p. 49), who uses just one option  $F^i$  to achieve duplication; thereby he must assume that  $F_s = a \cdot F_s^i$  implies  $F_{\sigma} = a \cdot F_{\sigma}^i$ .

#### 2.3 Constant Proportion Portfolio Insurance

By selling a life annuity, the seller guarantees that the buyer can withdraw every period the amount K dt as long as he lives – as opposed to a minimum wealth guarantee, K denotes now withdrawal per unit time, i.e., a rate, and not an amount of money.<sup>7</sup>

With the obligation to pay K dt, the seller of this annuity faces the following decision problem

$$\max_{C(t),w(t)} E_0 \left\{ \int_0^\infty e^{-\rho t} \frac{C(t)^{\gamma}}{\gamma} dt \right\}$$
(15)

s.t.:  $C(t) \ge C_{min} = K$ 

wealth dynamic according to either (2) or (3) or (5)

#### where $1 - \gamma$ denotes decision maker's (constant) relative risk aversion

In words, the decision maker maximizes his utility from aggregate consumption subject to a minimum consumption constraint by optimizing his consumption and portfolio strategy. His planning horizon is assumed to be infinity because this trick circumvents the problem of specifying a date at which the buyer of the annuity will die and, thus, the obligation to pay the annuity will end. The consumption constraint contains the guaranteed income stream. Since both consumption and payment for the annuity are withdrawals, they are integrated in one constraint. However, only consumption, i.e., a withdrawal above K, makes a positive contribution to utility. Therefore, the utility function is normalized so that  $U[C_{min}] = 0$  holds.

Black/Jones (1987) label investment strategies that cope with minimum withdrawals per unit time Constant Proportion Portfolio Insurance.

<sup>&</sup>lt;sup>7</sup> Technically speaking, the seller of such a product offers its buyer a revolving forward contract with constant forward price Kdt and maturity after the period dt. Owing to these institutional reasons, an analysis of Constant Proportion Portfolio Insurance under a stochastic floor as in Grossman/Zhou (1993) is omitted.

#### 2.4.1 Constant Proportion Portfolio Insurance under geometric Brownian motion

Black/Perold (1992, p. 420, 425 n.) prove that the following portfolio strategy solves the decision problem (15) under geometric Brownian motion:

$$\mathbf{w}(t) \cdot \mathbf{W}(t) = \begin{cases} \frac{1}{1 - \gamma} \cdot \frac{\alpha - \mathbf{r}}{\sigma^2} \cdot \mathbf{W}(t) & \text{für } \mathbf{W}(t) \ge \mathbf{W}^+ \\ \frac{1}{1 - \gamma'} \cdot \frac{\alpha - \mathbf{r}}{\sigma^2} \cdot \left[ \mathbf{W}(t) - \frac{\mathbf{K}}{\mathbf{r}} \right] & \text{für } \mathbf{W}(t) < \mathbf{W}^+ \end{cases}$$
(16)

with  $W^+ = \frac{K}{r} \cdot \frac{\gamma - 1}{\gamma - \gamma'}$ 

According to equation (16), the optimum portfolio weight of the risky asset calls for a division of the portfolio strategy into two parts: above wealth level W<sup>+</sup>, decision makers follow the well-known portfolios strategy under geometric Brownian motion (see Merton (1969)). Below the critical wealth level W<sup>+</sup>, the strategy changes and becomes more conservative in two respects. First, the structural component of the optimum portfolio weight is no longer calculated from total wealth, but a portion of it, namely wealth minus the floor discounted at the riskfree rate. Second, the volume component, i.e., the factor with which the structural component is multiplied, depends on mean and variance of the index as well as the endogenously derived risk aversion parameter  $\gamma'^8$ , which deviates from the one ( $\gamma$ ) of the utility function. Since a geometric Brownian motion is a diffusion process, trading in continuous time assures that portfolio restructuring occurs fast enough to protect the floor and, thus, the volume component does not have an exogenous upper limit, but is determined model endogenously.

#### 2.3.2 Constant Proportion Portfolio Insurance under combined jump/diffusion risk

Under combined jump/diffusion processes, the optimum portfolio weight reads9

$$\mathbf{w}(t) \cdot \mathbf{W}(t) = -\frac{1}{\varphi_{\text{extr}}} \left( \mathbf{W}(t^{-}) - \frac{\mathbf{K}}{\mathbf{r}} \right)$$
(17)

As opposed to the situation under geometric Brownian motion, the optimum portfolio strategy under combined jump and diffusion processes (17) does not distinguish between a

<sup>&</sup>lt;sup>8</sup>  $\gamma$ ' can be calculated from equation (A.5a). This section, however, is interested in the fundamentals of the optimum solution and not in its details. Therefore, the explicit calculation of  $\gamma$ ' is omitted.

<sup>&</sup>lt;sup>9</sup> A proof can be found in Appendix A.

critical and an uncritical region. Instead, its structural component is dominated by the critical region wherefore it does not alter between a more and a less conservative portfolio strategy. The reasons for this behavior are quite easy to understand. First, because there are two completely different types of risks involved, it is not possible to define a critical wealth level that works under both types of risk. Consequently, one encounters the same argument that has prevented the duplication of an option under combined jump/diffusion risk. Second, choosing a conservative structural component of the optimum portfolio weight is not enough – the large price movements caused by jumps can nevertheless violate the floor. Therefore, the volume component is restricted from above by the maximum negative <sup>10</sup> jump amplitude ( $\phi_{extr}$ ); this upper limit for the volume component is exogenous to the model.

By adapting Constant Proportion Portfolio Insurance to jumps, equation (17) puts in place a statement by Black/Jones (1987, p. 49) namely that the performance of Constant Proportion Portfolio Insurance under jumps is inappropriate. Instead, Constant Proportion Portfolio Insurance can be modified to work under combined jump/diffusion risk.

#### 2.3.3 Constant Proportion Portfolio Insurance under stochastic volatility

The optimum portfolio weight under stochastic volatility can be determined as follows<sup>11</sup>

$$\mathbf{w}(t) = \left[\frac{1}{1-\delta} \cdot \frac{\alpha-r}{\sigma^2(t)} + \frac{1}{1-\delta} \frac{\mathbf{B}_{\sigma}(\sigma(t))}{\mathbf{B}(\sigma(t))} \cdot \frac{\mathbf{cov}_{\sigma W}(t)}{\sigma^2(t)}\right] \left[\mathbf{W}(t) - \frac{\mathbf{K}}{\mathbf{r}}\right]$$
(18)

Similar to the situation under combined jump/diffusion risk, the optimum portfolio strategy under stochastic volatility (18) does not distinguish between a critical and an uncritical region, i.e., does not alter between a more and a less conservative strategy. Instead, its structural component is determined from the critical region only. Since there are two completely different types of risks involved, it is not possible to define a critical wealth level that works under both types of risk – the same argument has prevented the duplication of an option under stochastic volatility. The volume component on the other hand, depends on

<sup>&</sup>lt;sup>10</sup> Concentrating on jumps with negative amplitude implies that the risky asset is not sold short in the optimum. Otherwise, maximum risk would be the maximum price increase of the risky asset.

Yet, selling the risky asset short would signify that the only risky asset in the market would be worse than the riskless asset. Since this would mean that stocks cannot be in positive net supply, this case is excluded from further analysis.

<sup>&</sup>lt;sup>11</sup> A proof can be found in Appendix B.

mean and variance of the index, as well as the endogenously derived risk aversion parameter  $\delta$  and terms evaluating the stochastic volatility risk. As the price process of the index remains a diffusion process even under stochastic volatility, trading in continuous time assures that portfolio restructuring occurs fast enough to protect the floor and, thus, the volume component does not have an exogenous upper limit, but is determined model endogenously.

In other words, equation (18) adapts Constant Portfolio Insurance strategies – to my knowledge for the first time in literature – to work in a stochastic volatility environment.

#### **3** Model uncertainty

#### 3.1 Interpreting Portfolio Insurance strategies in the light of model uncertainty

The analysis of "classical" Option Based Portfolio Insurance under several price process specifications and, thus (combined parameter risk and) model uncertainty, has shown that "classical" Option Based Portfolio Insurance can duplicate the desired option under geometric Brownian motion only. Or to formulate this statement in a more general way: "classical" Option Based Portfolio Insurance has to operate in an environment which is characterized by just one source of risk. Whenever there is a second source of risk (combined jump/diffusion processes, parameter risk, or model uncertainty) "classical" Option Based Portfolio Insurance duplicate there is a second source of risk (combined jump/diffusion processes, parameter risk, or model uncertainty) "classical" Option Based Portfolio Insurance duplicate duplicate in a more general way:

As opposed to "classical" Option Based Portfolio Insurance, roll-over Portfolio Insurance with options is able to duplicate options under all three classes of price processes considered. Therefore, model uncertainty seems to be absent at first sight. However, a closer look at the details of the respective duplication portfolios reveals that they require different numbers N(t) for different price processes. Under geometric Brownian motion, there is just one option involved (equation (10)), yet jumps need two options to finish duplication, for what equations (12a) and (12b) do not coincide with equation (10). The same is true under stochastic volatility. Again, two options are necessary, a fact which makes the duplication portfolios under geometric Brownian motion (equation (10)) and stochastic volatility (equations (14a) and (14b)) diverge. Moreover, the reaction of the option price to a change of volatility ( $F_{\sigma}$  linear change of the option price) does not coincide with its price movement due to a stock price jump ( $\Delta F$ , non-linear change of the option price). Therefore, the number of options held under stochastic volatility (equations (12a) and (12b)) are unequal.

One could argue, though, that increasing the number of options used for duplication purposes might circumvent these problems. That is, to cope with model uncertainty, the following strategy is applied. One uses three options to be able to manage three types of risks: normal risk, jump risk, and stochastic volatility (parameter) risk. – The problem with this argumentation is that this strategy takes action against a situation in which all three types of risk are present at the same time, but its duplication portfolio does not work when a maximum number of two risks is present at the same time; in other words, roll-over Portfolio Insurance with options cannot assure guaranteed minimum wealth under model uncertainty.

To sum up, roll-over Portfolio Insurance with options can solve the duplication problem within a specified class of price process and, thus, even under estimation risk. It is model uncertainty that makes roll-over Portfolio Insurance with options fail, an observation which delivers another proof of a statement made by Avramov (2001, p. 21) that model uncertainty seemed to be more important than parameter risk.

Constant Portfolio Insurance will be able to defend guaranteed income streams if a pure geometric Brownian motion (equation (16)), a pure combined jumps/diffusion risk (equation (17)), and a pure stochastic volatility (parameter risk; equation (18)) are considered separately – a result similar to that obtained for roll-over Portfolio Insurance with options. In addition, it can cope with model uncertainty. To see this, recall that the structural components  $W - \frac{K}{r}$  of all portfolio weights coincide. Choosing the volume component according to the maximum negative jump amplitude ( $\phi_{extr}$ ), (see equation (17)) assures the desired income stream even under model uncertainty. Nevertheless, Constant Proportion Portfolio Insurance is influenced by model uncertainty because it calls for a portfolio strategy that is too conservative under diffusion processes in general and in particular under geometric Brownian motion.

## 3.2 Confronting the Anderson/Hansen/Sargent (2002) solution of portfolio selection under model uncertainty with the findings from the analysis of Portfolio Insurance strategies

Anderson/Hansen/Sargent (2000) demonstrate that the best non-trivial portfolio strategy under model uncertainty, non-trivial in the sense that wealth is not completely invested in the riskless asset, is a portfolio strategy that performs best in a reasonable worst case capital market scenario. Reasonable is particularized as in a certain way close to a reference model (see Cagetti/Hansen/Sargent/Williams (2002), p. 374).

This section aims to examine both aspects of Anderson/Hansen/Sargent's (2000) solution, namely the existence of a reasonable worst case scenario and the existence of a non-trivial portfolio strategy, by confronting them with the findings from the analysis of Portfolio Insurance strategies.

The worst case scenario is the most extreme scenario under which the minimum investment goal for portfolio selection is met. Should the idea of a worst case scenario be useful for decision purposes, there would have to be several other scenarios, that permit compliance with the constraint, besides the worst one. In the case of defending a guaranteed minimum wealth this is not true which means the idea of a worst case scenario is reduced to absurdity. To see this, recall that portfolio selection under a minimum wealth guarantee calls for duplication of the implied put option. A duplication portfolio, though, is fitted to a certain price process and can handle this price process only. Therefore, there is just one price process that allows for duplication and this price process is the worst case scenario at the same time.

The concept of a worst case scenario is better suited to a guaranteed income stream. Under geometric Brownian motion (see equation (16)), the portfolio strategy is divided into a more conservative (within the critical region) and a more aggressive part (within the uncritical region); under stochastic volatility – estimation risk – (see equation (18)), just the more conservative investment style is appropriate. Thus, (the whole set of models under) stochastic volatility can be interpreted as a worst case scenario. However, within this class of stochastic processes no further worst case scenario can be identified. The structural components of the optimum portfolio weight under stochastic volatility,  $W - \frac{K}{r}$ , remain the same and different volume components due to, e.g., different  $cov_{\sigma W}(t)$ , do not endanger the guaranteed income stream because processes under stochastic volatility are characterized by infinitesimal prices changes. – The situation will change if jumps are added. Owing to the non-infinitesimal movement of stock prices, the volume component plays a crucial role in protecting the floor (see equation (17)), a fact which in turn means that there is a clear-cut worst scenario: the jump with the maximum negative amplitude.

With this evaluation of the (reasonable) worst case scenario approach in mind, its potential consequences for the existence of a non-trivial portfolio strategy, the second element of Anderson/Hansen/Sargent's (2002) solution, can be examined. Since duplication just

works for one specific price process, there is only one trivial strategy that is able to defend guaranteed minimum wealth: invest in the riskless asset guaranteed wealth discounted with the riskfree rate.<sup>12</sup> This results holds irrespective of whether there is homogenous or heterogonous model uncertainty.

In the case of a guaranteed income stream, things turn out to be slightly more sophisticated. Under homogenous model uncertainty the portfolio strategy is indeed non-trivial because it calculates a volume component which is not exogenously specified, but contains a risk aversion parameter that is determined model endogenously. However, this result breaks down in the case of heterogeneous model uncertainty. Then, the volume component is exclusively determined by the maximum negative jump amplitude and, thus, model exogenous. This fact makes the determination of the optimum portfolio weight rather trivial: invests in the riskless asset  $1 - \varphi_{extr}$  times guaranteed income per period discounted at the riskfree rate. To make things even worse, consider the situation when a stock exchange does not specify<sup>13</sup> an upper limit for stock price movements. The maximum negative amplitude reads  $\varphi_{extr} = 1$  making Constant Proportion Portfolio Insurance indistinguishable from a buy and hold strategy investing completely in the riskless asset.

Taking these findings together, there are two reasons as to why Anderson/Hansen/Sargent's (2000) solution does not work in general under model uncertainty combined with minimum investment goals. First, a preference for robustness in the objective function is less demanding for portfolio selection than minimum investment goals thereby failing to work out different consequences of minimum wealth guarantees or guaranteed income streams with respect to portfolio selection. In particular, it is unable to elaborate that Anderson/Hansen/Sargent's (2000) solution performs far better for guaranteed income streams than minimum wealth guarantees. Second, Anderson/Hansen/Sargent (2000) have always a meaningful worst case scenario in their model's setup because a penalty term (heavily) restricts the malevolent player in choosing the (worst case) state (see Anderson/Hansen/Sargent (2000), p. 9). Therefore, this approach to justify a preference for robustness cannot assure that all scenarios are captured that are critical for meeting minimum

<sup>&</sup>lt;sup>12</sup> Technically speaking, this strategy equals a super-replicating strategy in its extreme, i.e., most expensive form.

<sup>&</sup>lt;sup>13</sup> See Roll (1989, p. 54) for an overview over stock exchanges that have limits for maximum possible price changes per day.

investment goals. For example, it ignores the role of heterogeneous model uncertainty under Constant Proportion Portfolio Insurance.

To sum up, under minimum wealth guarantees (Option Based Portfolio Insurance) there is no reasonable worst case scenario and there is only a trivial portfolio strategy – both aspects (worst case scenario and non-trivial composition) of Anderson/Hansen/Sargent's (2000) portfolio strategy become inapplicable. This result holds irrespective of the degree (homogenous or heterogeneous) of model uncertainty. Under income stream guarantees the situation is different. Homogenous model uncertainty produces a non-trivial portfolio strategy, yet the idea of a worst case scenario turns out to be not very telling: the whole class of processes under stochastic volatility (parameter risk) must be interpreted as worst case scenario. Under heterogeneous model uncertainty there is a pronounced worst case scenario, but then, just a (rather) trivial portfolio strategy applies.

#### 4 Conclusion

The point of departure for this paper was that in real-world market financial markets insurance products are observable that offer their buyers a minimum wealth or income stream guarantee irrespective of capital market conditions, i.e., stock price processes assumed. Therefore, sellers of these products will be well-advised to pursue a portfolio strategy that is able to meet this requirement if they want to avoid additional cash payments. The literature under model uncertainty, in particular Anderson/Hansen/Sargent (2000), seems to offer a solution **b** this portfolio problem by proposing the following non-trivial portfolio strategy, non-trivial in the sense that wealth is not completely invested in the riskless asset: follow that portfolio strategy that performs best in a reasonable worst case capital market scenario. Reasonable is particularized as in a certain way close to a reference model (see Cagetti/Hansen/Sargent/Williams (2002), p. 374). It is the objective of this paper to show that this solution does not work in general under model uncertainty.

Defending a guaranteed minimum wealth (Option Based Portfolio Insurance) calls for duplication of the option implied by this minimum wealth guarantee. A duplication portfolio, though, is fitted to a concrete price process and can handle this price process only. Therefore, the idea of a non-trivial portfolio strategy that is able to defend guaranteed minimum wealth even under a worst case scenario is reduced to absurdity; the only portfolio strategy that is capable of defending guaranteed minimum wealth irrespective of the capital market environment reads: invest in the riskless asset guaranteed wealth discounted at the riskfree rate. In other words, both aspects (worst case scenario and non-trivial composition) of AnAnderson/Hansen/Sargent's (2000) portfolio strategy become inapplicable irrespective of the degree (homogenous or heterogeneous) of model uncertainty.

In the case of Constant Proportion Portfolio Insurance, things turn out to be slightly more sophisticated. Under homogenous model risk, there is a non-trivial portfolio strategy, but no pronounced worst case scenario, in that all portfolio weights have the same structure. Heterogeneous model risk, on the other hand, makes a clear-cut worst case scenario occur, the maximum possible negative jump amplitude  $\varphi_{extr}$  But then the optimum portfolio weight turns out to be trivial: it invests in the riskless asset  $(1 - \varphi_{extr})$  times guaranteed income per period discounted at the riskfree rate.

These results mean that confronting portfolio selection with a range of application often found in the pragmatic world (minimum investment goals) results in a pretty frustrating outcome. Either decision makers follow a trivial portfolio strategy, trivial in the sense that the investment goal is achieved by riskless investment, or they employ a sophisticated portfolio strategy which has to accept that there is model uncertainty and speculate on it. This speculation, though, differs from that of "normal" portfolio selection. Whereas "normal" portfolio selection tries to forecast in particular stocks' means, which is difficult as Merton (1980) has shown, a speculation on model uncertainty has to determine the probabilities that a certain class of price processes is the true one; this job might be an even tougher task.

#### Appendix

# A Derivation of Constant Proportion Portfolio Insurance strategies under combined jump/diffusion processes

After having particularized the wealth dynamic in the decision problem (15) with a combined jump and diffusion process, the following Hamilton/Jacobi/Bellman equation is obtained:

$$0 = \max_{C(t),w(t)} \left\{ J_{t} + J_{w}(\alpha - r)w(t)W(t) + J_{w}\left(rW(t) - \frac{C(t)}{dt}\right) + \frac{1}{2}J_{ww}w^{2}(t)\sigma^{2}W^{2} + \lambda E\{J[(1 + w(t)\phi(t))W] - J\} \right\}$$
s.t.:  $C(t) \ge C_{min} = K$ 
(A.1)

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Following the line of argumentation developed by Black/Jones (1992), and Merton (1993, p. 186 n.), a candidate solution of equation (A.1) can be found by splitting the determination of the optimum investment strategy w(t) into two related, but unconstrained problems: the optimum portfolio strategy for a critical and for an uncritical region.

To verify that this candidate is indeed the solution to equation (A.1), three steps are needed:

- 1. Derivation of the optimum portfolio weight for the critical region.
- 2. Derivation of the optimum portfolio weight for the uncritical region.
- 3. Determination of a critical wealth level that separates the critical from the uncritical region.

This third step serves in particular to verify that the two-step procedure delivers an admissible solution, i.e., keeps the indirect utility function J[.] continuous and twice differentiable.

- Step 1

The decision maker's budget equations reads

$$W(t^{-}) = E(t^{-}) + E_0(t^{-})$$
(A.2)

where E denotes the amount invested in the risky index and  $E_0$  the one invested in the riskless asset

Jumps entail a sudden and large change of wealth. Therefore, immediately after a jump wealth changes to (after having used the budget constraint (A.2) to substitute out  $E_0$ )

$$W(t) = E(t^{-}) \cdot (1 + \phi(t)) + [W(t^{-}) - E(t^{-})]$$
(A.3)

By assumption, the guaranteed income stream K, the so-called floor, must be defended. This means at the beginning of each period wealth must not fall below  $\frac{K}{r}$ , which calls for a multiple m of the amount  $W(t^-) - \frac{K}{r}$  to be invested in the risky asset. This multiple m can be determined by taking the worst environment, i.e., a jump with the maximum negative amplitude  $\varphi_{extr}$ , into account:

$$W(t) = \frac{K}{r} = m \left( W(t^{-}) - \frac{K}{r} \right) \cdot \left( 1 + \varphi_{extr} \right) + \left[ W(t^{-}) - m \left( W(t^{-}) - \frac{K}{r} \right) \right]$$
(A.4)

which yields

$$m = -\frac{1}{\phi_{extr}}$$

- Step 2

I omit the depiction of step 2 as will become clear shortly.

The third step involves checking whether J[.] is continuous and twice differentiable. To that end, the following boundary conditions are added:

$$\mathbf{J}^{c}\left[\mathbf{W}^{+}-\frac{\mathbf{K}}{\mathbf{r}}\right] = \mathbf{J}^{u}\left[\mathbf{W}^{+}\right]$$
(A.5a)

$$J_{W}^{c}\left[W^{+}-\frac{K}{r}\right] = J_{W}^{u}\left[W^{+}\right]$$
(A.5b)

$$\mathbf{J}_{WW}^{c} \left[ \mathbf{W}^{+} - \frac{\mathbf{K}}{\mathbf{r}} \right] = \mathbf{J}_{WW}^{u} \left[ \mathbf{W}^{+} \right]$$
(A.5c)

Without having to rely on an explicit calculation of J[.], such a  $W^+$  cannot exist under combined jump/diffusion processes. To see this, define the critical wealth level based on the diffusion component, i.e., an infinitesimal price movement. A jump, however, signifies a non-infinitesimal price movements for what this  $W^+$  does not keep J[.] continuous in a jump environment. Conversely, it does not work to define  $W^+$  based on the maximum negative jump amplitude. The jump amplitude is defined as a percentage of current wealth – a relative quantity – whereas  $W^+$  is an absolute quantity.

In other words, the only portfolio strategy that guarantees the floor under combined jump/diffusion processes employs the portfolio weight for the critical region and does not switch to a less "cautious" portfolio strategy. In particular, it does not split the optimum portfolio weight into a critical and an uncritical region. Therefore, the portfolio strategy depicted in equation (17) is obtained.

# **B** Derivation of Constant Proportion Portfolio Insurance strategies under stochastic volatility

After having particularized the wealth dynamic in the decision problem (15) with a diffusion process under stochastic volatility, the following Hamilton/Jacobi/Bellman equation is derived:

$$0 = \underset{C(t),w(t)}{\text{Max}} \left\{ J_{t} + J_{w} (\alpha - r) w(t) W(t) + J_{w} \left( r W(t) - \frac{C(t)}{d t} \right) + \frac{1}{2} J_{ww} w^{2}(t) \sigma^{2}(t) W^{2} + J_{\sigma} f \sigma(t) + \frac{1}{2} J_{\sigma\sigma} g^{2} \sigma^{2}(t) + J_{w\sigma} w(t) W \operatorname{cov}_{\sigma W}(t) \right\}$$
s.t.:  $C(t) \ge C_{\min} = K$ 

$$(B.1)$$

To solve problem (B.1), the same procedure as in Appendix A is utilized.

- Step 1

Since the guaranteed income stream K must be defended, at the beginning of each period wealth must not fall below  $\frac{K}{r}$ , which calls for a multiple m of the amount  $W(t) - \frac{K}{r}$  to be invested in the risky asset.

According to Merton (1973), the optimum portfolio weight (for both parts of the unconstrained problem (B.1)) reads:

$$\mathbf{w}^{c/u}(t) = -\frac{\mathbf{J}_{W}^{c/u}}{\mathbf{J}_{WW}^{c/u}W(t)} \cdot \frac{\alpha - r}{\sigma^{2}(t)} - \frac{\mathbf{J}_{\sigma W}^{c/u}}{\mathbf{J}_{WW}^{c/u}W(t)} \cdot \frac{\operatorname{cov}_{\sigma W}(t)}{\sigma^{2}(t)}$$
(B.2)

- Step 3:

To be able to figure out whether J[.] is continuous and twice differentiable, J[.] has to be determined explicitly. Substituting the portfolio weight (B.2) back into (B.1), yields the following differential equation for  $J^{c/u}$ 

$$0 = J_{t} + J_{W} \left( r W(t) - \frac{C(t)}{dt} \right) - \frac{1}{2} \left( \frac{\alpha - r}{\sigma(t)} \right)^{2} \frac{J_{W}^{2}}{J_{WW}} + J_{\sigma} f \sigma(t) + \frac{1}{2} J_{\sigma\sigma} g^{2} \sigma^{2}(t)$$
$$- \frac{J_{W} J_{\sigmaW}}{J_{WW}} \left( \frac{\alpha - r}{\sigma(t)} \right) \cdot cov_{\sigmaW}(t) - \frac{1}{2} \frac{J_{\sigmaW}^{2}}{J_{WW}} \cdot \frac{cov_{\sigmaW}^{2}(t)}{\sigma^{2}(t)}$$
(B.3)

s.t.:  $C(t) \ge C_{min} = K$ 

By analogy to Cox/Ingersoll/Ross (1985, p. 389), the solution of (B.3) reads

• for the uncritical region

$$J^{u}[W(t),\sigma(t),t] = A(\sigma(t)) \cdot e^{-ct} \frac{W^{\gamma}(t)}{\gamma} + G(\sigma(t))$$
(B.4)

with A and G arbitrary functions of  $\sigma(t)$ , and c a time preference rate, which is determined endogenously

• for the critical region

$$J^{c}[W(t),\sigma(t),t] = B(\sigma(t)) \cdot e^{-ct} \frac{\left[W(t) - \frac{K}{r}\right]^{\delta}}{\delta} + D(\sigma(t))$$
(B.5)

with B and D arbitrary functions of  $\sigma(t)$ , and  $\delta$  a risk aversion parameter, which is determined endogenously

Moreover, form boundary conditions (A.5b) and (A.5c), a candidate for the critical wealth level becomes sizeable:

$$W^{+} = \frac{K}{r} \frac{1 - \gamma}{\delta - \gamma}$$
(B.6)

where the only remaining unknown  $\delta$  might is accessible with the help of boundary condition (A.5a)

$$B(\sigma(t))\frac{\left(\frac{K}{r}\frac{1-\gamma}{\delta-\gamma}-\frac{K}{r}\right)^{\delta}}{\delta} + D(\sigma(t)) = A(\sigma(t))\frac{\left(\frac{K}{r}\frac{1-\gamma}{\delta-\gamma}\right)^{\gamma}}{\gamma} + G(\sigma(t))$$
(B.7)

Yet, equation (B.7) demonstrates the dependence of  $\delta$  on  $\sigma$ (t). Consequently, W<sup>+</sup>, as specified in equation (B.6), cannot be part of the solution of equation (B.1). However, (B.6) has been calculated with the help of the (Cox/Ingersoll/Ross-) solution of the unconstrained problem. This means that there is no wealth level W<sup>+</sup> that is both part of the

(Cox/Ingersoll/Ross-) solution for J[.] and fulfils (B.7) at the same time, i.e., allows for a split of J[.] into  $J^{c}$ [.] as well as  $J^{u}$ [.], and nevertheless keeps J[.] continuous and twice differentiable.

Putting all these arguments together, the optimum portfolio strategy under stochastic volatility employs the portfolio weight for the critical area as the sole optimum portfolio weight and does not split the optimum portfolio weight into a critical and an uncritical region. Therefore, the portfolio strategy depicted in equation (18) is obtained.

#### References

- Anderson E W, Hansen L P, Sargent T J (2000) Robustness, detection and the price of risk. Working Paper (March 2000), University of Chicago, Chicago
- Avramov D (2001) Stock return predictability and model uncertainty. Working Paper (April 2001), University of Maryland, Robert H. Smith School of Business Administration, Baltimore
- 3. Bates D S (2000) Post-'87 crash fears in the S&P 500 futures option markets. Journal of Econometrics 94: 181-238
- 4. Black F, Perold A F (1992) Theory of constant proportion portfolio insurance. Journal of Economic Dynamics and Control 16: 403-426
- 5. Black F, Jones (1987) Simplifying portfolio insurance. Journal of Portfolio Management 13 (Fall): 48-51
- 6. Cagetti M, Hansen L P, Sargent T, Williams N (2002) Robustness and pricing with uncertain growth. The Review of Financial Studies 15: 363-404.
- 7. Cox J C, Ingersoll J E, Ross S A (1985) A theory of the term structure of interest rates. Econometrica 53: 385-408
- Geman H (1992) Portfolio insurance and synthetic securities. Applied stochastic models and data analysis 8: 179-188
- Grossman S J, Zhou Z (1993) Optimal investment strategies for controlling drawdowns. Mathematical Finance 3: 241-276
- 10. Grossman S J, Zhou Z (1996) Equilibrium analysis of portfolio insurance. The Journal of Finance 51: 1379-1403

- 11. Hull J, White A (1987) The pricing of options on assets with stochastic volatility. The Journal of Finance 42: 281-300
- Johnson H, Shanno D (1987) The pricing of options when the variance is changing. Journal of Financial and Quantitative Analysis 22: 143-151
- Leland H E (1980) Who should buy portfolio insurance? The Journal of Finance 35: 581-596
- 14. Leland H E (1992) Portfolio insurance. In: Newman P, Milgate M, Eatwell J (eds) The new Palgrave dictionary of money & finance. Macmillan, London
- Maenhout P J (2001) Robust portfolio rules, hedging and asset pricing. Working Paper (July 2001), INSEAD, Finance Department, Fontainebleau
- Merton R C (1969) Lifetime portfolio selection under uncertainty: the continuous-time case. Review of Economics and Statistics 51: 247-257
- 17. Merton R C (1973) An intertemporal Capital Asset Pricing Model. Econometrica 41: 867-888
- Merton R C (1976) Option pricing when the underlying stock returns are discontinuous. Journal of Financial Economics 5: 125-144
- 19. Merton R C (1980) On estimating the expected return on the market: an exploratory investigation. Journal of Financial Economics 8: 323-361
- 20. Merton R C (1993) Continuous-time finance. Blackwell Publishers, Cambridge
- 21. Roll R (1989) The international crash of October 1987. In: Kamphuis R L, Kormendi R C, Watson J W (eds) Black monday and the future of financial markets. Dow Jones-Irwin, Homewood
- 22. Ross S A (1976) Options and efficiency. Quarterly Journal of Economics 90: 75-89
- 23. Rubinstein M (1985) Alternative path to portfolio insurance. Financial Analysts Journal 41 (July-August): 42-52
- 24. Rubinstein M, Leland H E (1981) Replicating options with positions in stock and cash. Financial Analysts Journal 37 (July-August): 63-72
- 25. Trojani F, Vanini P (2001) Risk, Robustness and Knightian uncertainty in continuous time, heterogeneous agents, financial equilibria. Working Paper, University of Southern Switzerland, Lugano

- 26. Uppal R, Wang T (2002) Model misspecification and under-diversification. Working Paper (January 2002), London Business School, London
- 27. Xia Y (2001) Learning about Predictability: The effects of parameter risk on dynamic asset allocation. The Journal of Finance 56: 205-246
- Zhu Y, Kavee R C (1988) Performance of portfolio insurance strategies. Journal of Portfolio Management 14 (Spring): 48-54