The Fair Premium of an Equity–Linked Life and Pension Insurance

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Abstract:

An equity linked life and pension insurance consists of an non-linear combination of a life and pension insurance with an investment strategy. In addition to the guaranteed payments the insured receives a bonus depending on the value of an investment strategy. The additional payment is similar to an Asian typ option. Since the insurance contract combines mortality and financial risks in a non-linear way, the value or premium of the contract must reflect these uncertainties. The paper shows the existence of a fair periodic premium defined so that the expected discounted premium is equal to the expected discounted payments. For two different pension policies an approximation of the fair periodic premium is derived, which imlies the approximation of long term Asian typ options.

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Introduction

A capital life insurance contract is equal to a linear combination of a term insurance with duration T and a pure endowment insurance. Due to the term insurance the insurer guarantees, in case of death of the insured, his or her heirs a fixed amount. The pure endowment insurance consists of a minimum guaranteed payment if the insured survives the maturity or duration T of the contract. By choosing the duration T equal to the intended retirement of the insured, the capital life insurance may be used to insure the retirement pension, at least partly.

The reason for the following discussion is neither to discuss different arguments in favour of or against the capital life insurance. Nor is it the intention to analyse the differences of and the problems associated with the pay-as-you-go and the capital funded pension system. Instead, the analysis will concentrate on a situation where the insurance of the pension payment is founded by a combination of both risk diversification principles. On the one hand, the arbitrage pricing principle, i.e. risk diversification by duplication, and on the other hand, the equivalence premium principle, i.e. risk diversification by large populations. The insurance contract is designed to combine financial market risk with pure insurance risk. Combining a life insurance contract with an investment strategy may imply a reduction of the total risk exposure for the insurer. Obviously, this reduction is based on a non-linear combination of the pure life insurance part and of the financial portfolio strategy.

A closely related example is the equity–linked life insurance contract. This contract combines a life insurance with a minimum guarantee on the outcome of an investment strategy. Ekern and Persson (1996) give an overview of different contract specifications. A first analysis within the context of the arbitrage pricing theory is given by Brennan and Schwartz (1976). They consider the situation of an equity–linked life insurance contract with one front premium and use a deterministic model for the term structure of interest rates. Bacinello and Ortu (1994), as well as Nielsen and Sandmann (1995, 1996) , extend the analysis to include the case of a stochastic interest rate development and periodic premium payments by the insured.

Instead of a pension plan the equity–linked life insurance contract results in a final payoff to the insured if he or she survives the duration T of the contract.

The discussion is organized as follows: The contract will be defined in Section 1. Section 2 introduces the underlying model of a financial market, which includes stochastic interest rates as well as a stochastic model of the mutual fund. The analysis of the contract refers to the fair premium principle, which is a combination of the equivalence premium principle usually applied in life insurance and the no-arbitrage pricing principle. The existence, the uniqueness and the properties of the fair premium are discussed in Section 3. Although the properties of the contract can be derived in a fairly general setup, no closed–form solution for the fair premium exists. Section 4 focuses on the numeric technique and includes an analytical approximation to the fair premium. To summarize the analysis, we report in Section 5 on some numerical results. Proofs are summarized in the appendix.

1 Equity–linked life and pension insurance

The basic principle of the contract is similar to a usual life insurance contract. The insured pays a periodic premium to the insurer until his or her death or the maturity or the duration T whichever comes first. The premium is assumed either to be constant or a deterministic function of time. In both cases the periodic premium is fixed at the beginning of the contract. As in the case of a pure term insurance, the heirs, receive a minimum guaranteed amount if the insured dies before the duration of the term insurance. Instead, if the insured survives the maturity date, T, he or she receives a periodic pension until his or her death. Unlike the usual capital life insurance contract, at each payment date a fraction of the periodic premium will be invested into a mutual fund. Furthermore, the payment of the insurer to the insured will depend on the portfolio value.

More precisely, let $t_N = T$ be the duration of the term insurance and suppose that, at each time $t_i \in \underline{\underline{T}} := \{0 = t_0 < t_1 < \cdots < t_{N-1}\}$, with $t_{N-1} < t_N = T$, the insurer receives a premium $K(t_i)$ if the insured is alive at time t_i. Let $S(t)$ be the market value of the mutual fund at time t and suppose that a fraction $\alpha \in [0,1]$ of the premium at each time $t_i \in \underline{\underline{T}}$ is invested into the mutual fund. α will be called the investment share. If the insured is alive at time $t \in]t_0, t_N]$ the value of his or her portfolio is equal to

$$\begin{aligned} P(t,\alpha,K) &:= \sum_{i=0}^{\min\{n^*(t),N-1\}} \alpha \cdot K(t_i) \cdot \frac{S(t)}{S(t_i)} \qquad (1.1) \\ \text{with } n^*(t) &:= \max\{j \in \mathbb{N}_0 | t_j < t\}. \end{aligned}$$

If the insured survives the duration T of the term insurance, the value of the portfolio is equal to

$$P(t,\alpha,K) = \sum_{i=0}^{N-1} \alpha \cdot K(t_i) \cdot \frac{S(t)}{S(t_i)} \qquad \forall t \geq T.$$

The premium function $K : \underline{\underline{T}} \to \mathbb{R}_{\geq 0}$ is assumed to be constant or deterministic, and α, the investment share of the premium is assumed to be constant. If the insured dies at time $t \in]t_0, T]$, his or her heirs receive the guaranteed amount $g_I(t)$ plus a bonus, which is proportional to the positive difference between the value of the portfolio at that time and the guaranteed amount. The total payment $G_I(t)$ in this situation is therefore given by

$$G_I(t) := g_I(t) + \eta_1 \cdot [P(t,\alpha,K) - g_I(t)]^+ \qquad \forall t \in]t_0, T], \qquad (1.2)$$

where we define $[x]^+ := \max\{x, 0\}$. For $\alpha = 0$, the payment coincides with a pure term insurance. The value $\eta_1 \in [0,1]$ defines the repayment level of the portfolio value in case of death of the insured before the duration T. For $\eta_1 = 1$, this payment is equal to the maximum between the guaranteed amount and the value of the portfolio. For $\eta_1 = 0$, the insured receives only the guaranteed amount. Obviously, the value of the insurance contract will increase in η_1.

If the insured survives the duration T of the pure term insurance, he or she receives a periodic minimum guaranteed pension until his or her death. In addition to this minimum guaranteed pension, the insured receives a non-negative payment which is related to the value of the mutual fund. Among the different possibilities to define this additional pension payment, we will consider two specific pension policies.

In pension policy A the insured will, during his remaining lifetime, receive at any time $t_j \geq T, j = N, N+1, \cdots$, a guaranteed periodic pension plus a bonus depending on the portfolio value at time T. At time T the insurer sells the total portfolio at the market value and divides the value by the number of pension periods L, which on average, seen from time t_0, have to be financed by the pension system. If, in a considered period, this fraction of the portfolio value is larger than the guaranteed pension, the insured will receive a bonus, if not, he or she will only receive the guaranteed amount. Therefore the bonus is only affected by the value of the portfolio at time T. The bonus is proportional to the excess of the fraction $\frac{P(T,\alpha.K)}{L}$ over the guaranteed payment at date t_j rolled over by the interest rate market from time T until t_j.

Defining by $\{r(t)\}_t$ the stochastic process of the instantaneous spot rate, the return of a roll-over strategy from time T to time $t_j > T$ is equal to:

$$\beta_{T,t_j} := \exp\left\{\int_T^{t_j} r(u)du\right\}.$$

Denote by $q(t_j)$ the time T present value of the guaranteed pension at time t_j. At each time $t_j, j \geq N$ the total periodic pension $Q(t_j)$ is defined by

$$Q(t_j) := \beta_{T,t_j} \cdot \left(q(t_j) + \eta_2 \cdot \left[\frac{1}{L} \cdot P(T, \alpha, K) - q(t_j) \right]^+ \right), \tag{1.3}$$

where $L \in \mathbb{N}$ is given at t_0 and equals the expected number of pension payments for a life aged x at t_0, if he or she survives the duration T of the contract.[1]

The constant $\eta_2 \in [0,1]$ can be interpreted as the participation rate of the pension. The guaranteed pension $q(t)$ serves as a floor for the total periodic pension.

In addition to the periodic pension, we have to define the possible payment $G_P(\cdot)$, when the insured dies at some time $t > T$. Under the regime of the pension policy A, we assume that the insurer pays an amount which is proportional to the positive difference of a fixed amount

[1] It should be noted that in case of survival, at time $t > t_{N+L}$ the insured still benefits from a non-negative bonus on the guaranteed pension.

$g_P(t)$ reduced by a sum related to the already payed guaranteed pension:

$$G_P(t) := \eta_3 \cdot \left[g_P(t) - \sum_{j=N}^{n^*(t)} q(t_j) \right]^+ \qquad \forall t > T = t_N. \tag{1.4}$$

The value $\eta_3 \in [0, 1]$ defines the repayment level after the duration T. As for the repayment level η_1 the value of the contract increases with η_3. The value $\eta_3 = 0$ corresponds to the case of no repayment if the insured dies after the duration of the term insurance, whereas $\eta_3 = 1$ indicates the full repayment level. Furthermore, the standard case of a capital life insurance with deterministic guaranteed amount $g_I(t)$ and deterministic guaranteed pension scheme $q(t)$ is given by $\alpha = 0$. Summing up, the pension policy A is defined by the periodic pension (1.3) and the repayment (1.4). The contract defined by the payoff functions (1.1) - (1.4) will be called an *equity–linked life and pension insurance with pension policy* A, investment share α, repayment levels η_1 and η_3 and participation rate η_2. Figure 1.1 gives a summary of the different payments under pension policy A.

In the situation of the pension policy B, the value of the mutual fund during the pension period is reflected in the different payments. At time T the number of shares in the portfolio is determined by

$$\sum_{i=0}^{N-1} \alpha K(t_i) \frac{1}{S(t_i)}.$$

Without any insurance, the holder of the portfolio could, to finance his pension, e.g. sell at L consecutive points in time, $t_N, t_{N+1}, ..., t_{N+L-1}$, the fraction $\frac{1}{L}$ of the number of shares in the portfolio at time T. With insurance, a floor on the periodic pension is introduced. The periodic pension to the insured under pension policy B is defined by

$$Q(t_j) := q(t_j) + \eta_2 \cdot \left[\frac{1}{L} P(t_j, \alpha, K) - q(t_j) \right]^+ \qquad \forall t_j = t_N, ...t_{N+L-1}, \tag{1.5}$$

$$Q(t_j) := q(t_j) \qquad \forall t_j \geq t_{N+L}, \tag{1.6}$$

where L is a given constant as under policy A. Unlike policy A, the periodic pension is now related to the fund dynamics beyond the duration T. Furthermore, if the insured dies within the pension period the payoff $G_P(\cdot)$ is defined by the value of the remaining shares in the

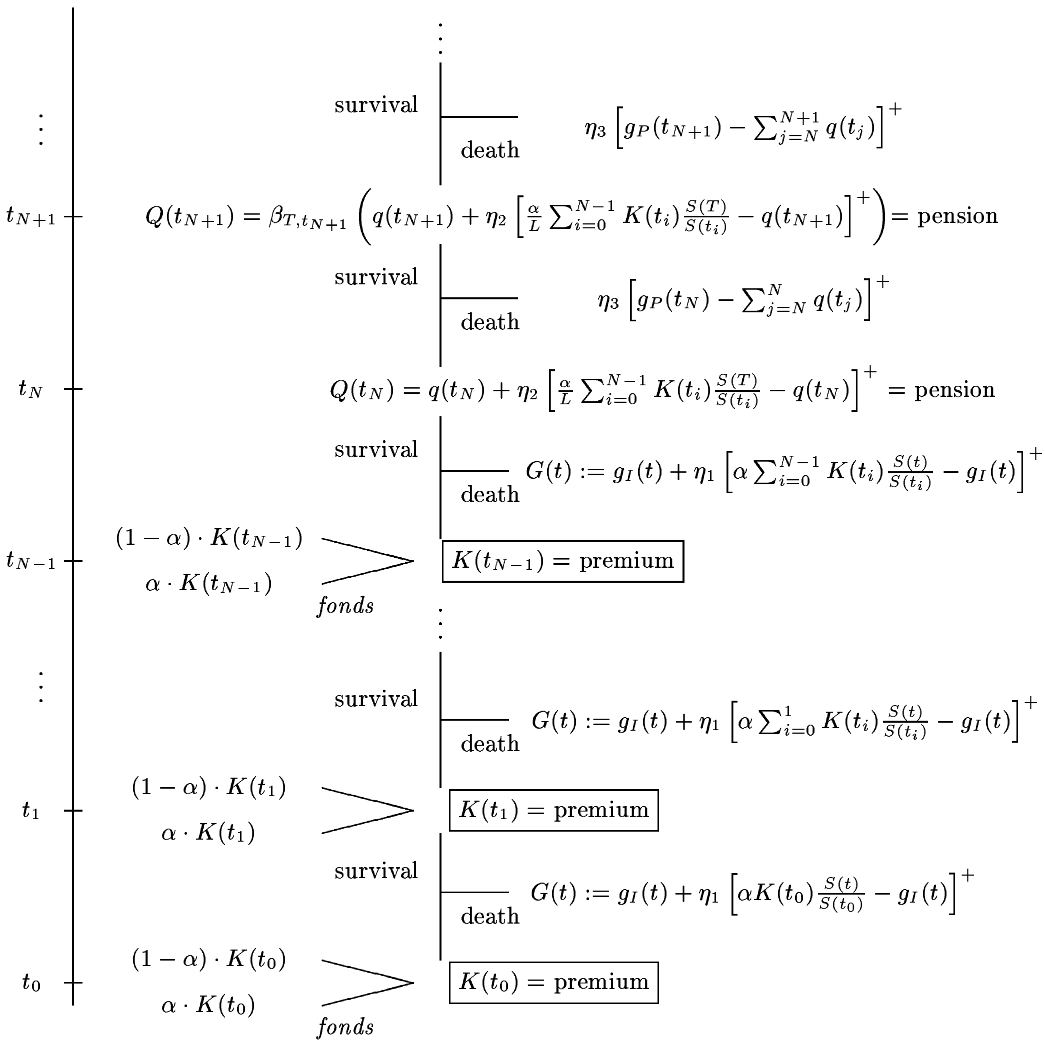

Figure 1.1: Contract specification of an equity linked life and pension insurance with pension policy A

portfolio, i.e

$$G_P(t) := \eta_3 \cdot P(t, \alpha, K) \cdot \left[\frac{L + N - 1 - n^*(t)}{L}\right]^+ \qquad \forall t > T. \tag{1.7}$$

The contract defined by the payments (1.2),(1.5) to (1.7) is called an *equity–linked life and pension insurance with pension policy B*, investment share α and repayment levels η_1 and η_3 and participation rate η_2.

Although it seems to be natural to assume that the guaranteed insurance amounts $g_I(\cdot)$ and $g_P(\cdot)$, and the guaranteed pension $q(\cdot)$ are deterministic functions, this is not necessary. Besides the obvious constant specification of these functions, one other example for a deterministic specification is:

$$\begin{aligned} g_I(t) &:= \sum_{i=0}^{n^*(t)} \exp\{\delta_1 \cdot (t-t_i)\} \cdot \beta_1 \cdot \alpha \cdot K \qquad \forall t \in]0,T[\\ q(t) &:= \exp\{\delta_2 \cdot t\} \cdot \beta_2 \cdot \alpha \cdot K \qquad \forall t \geq T, \\ g_P(t) &:= \sum_{i=N}^{N+L} q(t_i) \qquad \forall t > T \end{aligned} \tag{1.8}$$

where $K > 0$ is equal to the constant periodic premium, β_1 and β_2 are non-negative constants and $\delta_1 > 0$ and $\delta_2 > 0$ are internal guaranteed interest rates. Such a contract with pension policy A is actually offered by the Postbank in Germany. In this case, the parameters are chosen to be $\delta_1 = \delta_2 = 0$, $\eta_1 = \eta_3 = 1, \eta_2 = 0$ and $L = 180$ months.

A constant or deterministic specification of the guaranteed payments has the disadvantage that, in terms of present value, the guaranteed insurance amounts do not really cover the needs of the insured. One solution is to relate the size of the guaranteed payments to the interest rate dynamics. An example of an interest rate related specification is as follows:

$$\begin{aligned} g_I(t) &:= \exp\left\{\int_0^t r(u)du\right\} \cdot \bar{g_I} = \beta_{0,t} \cdot \bar{g_I} \qquad \forall t \in]0,T[\\ q(t) &:= \exp\left\{\int_0^t r(u)du\right\} \cdot \bar{q} = \beta_{0,t} \cdot \bar{q} \qquad \forall t \geq T, \\ g_P(t) &:= \exp\left\{\int_0^t r(u)du\right\} \cdot \bar{g_P} = \beta_{0,t} \cdot \bar{g_P} \qquad \forall t > T, \end{aligned} \tag{1.9}$$

where $\bar{g_I}, \bar{g_P}$ and $\bar{q}$ are positive constants defining the present value of the guaranteed payments and $g_P(.)$ applies only to the case of the pension policy A.

The analysis of the contract will mainly refer to the case with deterministic guarantees. One example of such a situation is given by equation (1.8). Nevertheless, the interest rate dependent situation can be analysed by basically the same arguments.

2 Insurance and Financial Risk

The analysis of an equity–linked life and pension insurance, as defined in Section 1, should include at least three different sources of uncertainty.

First, the payment of the contract is defined with respect to the survival or death of the insured. Denote by $\pi_x(t)$ the density function of the death distribution for a life aged x at time t_0. The death resp. survival distributions are defined by:

$$\begin{aligned}
\int_{t_0}^{t} \pi_x(u)du &=: prob[\text{ a life aged } x (\text{at time } t_0) \text{ dies within the period }]t_0, t]] \\
1 - \int_{t_0}^{t} \pi_x(u)du &=: prob[\text{ a life aged } x (\text{at time } t_0) \text{ survives time } t].
\end{aligned}$$

We assume that, at time t_0, the probability density functions of the death distribution for a life aged x are known $\forall x$. With this notation the expected number of periods L for a life aged x at t_0 under the condition that he or she survives the duration T is determined by:

$$\begin{aligned}
L := \frac{\int_T^{+\infty}(n^*(u) - (N-1))\pi_x(u)du}{1 - \int_{t_0}^{T} \pi_x(u)du} &= \frac{\sum_{i=N}^{+\infty} \int_{t_i}^{t_{i+1}} (i - (N-1))\pi_x(u)du}{1 - \int_{t_0}^{T} \pi_x(u)du} \\
&= \frac{\sum_{i=N}^{+\infty} \left(1 - \int_{t_0}^{t_i} \pi_x(u)du\right)}{1 - \int_{t_0}^{T} \pi_x(u)du}.
\end{aligned} \tag{2.1}$$

Second, the equity–linked life and pension insurance contract is influenced by two types of financial risk. Since the benefit to the insured is a function of the value of the investment portfolio, the dynamics of the underlying mutual fund are involved. In addition, the contract is of long term and therefore the interest rate risk must be considered. These financial aspects are included in the analysis by a complete and arbitrage–free model of the financial market. Let P^* be the unique equivalent martingale measure, such that, with respect to a filtered probability space $(\Omega, \mathbb{F}, P^*, \{\mathbb{F}_t\})$, the discounted price processes of the mutual fund $\{S(t)\}_t$ and the discounted price processes of all the zero coupon bonds $\{D(t,\tau)\}_{t\in[t_0,\tau]}$ with maturity $\tau \in \mathbb{R}_{\geq 0}$ are martingales:

$$\begin{aligned}
S(t) &= E_{P^*}\left[\beta_{t,\bar{t}}^{-1} S(\bar{t}) \middle| \mathbb{F}_t\right] \qquad \forall \bar{t} \geq t, \\
D(t,\tau) &= E_{P^*}\left[\beta_{t,\bar{t}}^{-1} D(\bar{t},\tau) \middle| \mathbb{F}_t\right] \qquad \forall \bar{t} \in [t,\tau], \forall \tau \geq t, \\
D(\tau,\tau) &= 1 \qquad P^* a.s.
\end{aligned} \tag{2.2}$$

Furthermore, we assume enough regularity, so that for each $\tau \in \mathbb{R}_{>0}$ there exists a unique τ-forward risk adjusted measure P^τ defined by

$$\left.\frac{dP^\tau}{dP}\right|_t = \frac{\beta_{t_0,t}^{-1} D(t,\tau)}{E_{P^*}\left[\beta_{t_0,t}^{-1} D(t,\tau) | \mathbb{F}_{t_0}\right]}. \tag{2.3}$$

Applying this change of measure implies that the forward prices of the financial assets are martingales under the appropriate forward risk adjusted measure, i.e.

$$\begin{aligned} S(t) &= D(t,\tau)E_{P^\tau}\left[\left.\frac{S(\tau)}{D(\tau.\tau)}\right|\mathbb{F}_t\right] \qquad \forall \tau \geq t \geq t_0, \\ D(t,T) &= D(t,\tau)E_{P^\tau}\left[\left.\frac{D(\tau,T)}{D(\tau,\tau)}\right|\mathbb{F}_t\right] \qquad \forall T \geq \tau \geq t \geq t_0. \end{aligned} \tag{2.4}$$

This is the usual and standard setup of a complete and arbitrage–free financial market model including interest rate risk. At this point in the discussion we do not need to specify the volatility structure. The general results, with respect to the premium of the equity–linked life and pension insurance, are not depending on any more specific assumptions about the volatility structure. Nevertheless, for the numerical analysis to be performed, we will assume a special framework with deterministic volatilities. For further details concerning the financial market model we refer to Geman, El Karoui and Rochet (1995).

The following Proposition summarizes some useful results with respect to the expected value of the mutual fund and the death distribution.

Proposition 2.1 *Consider a financial market model satisfying the relationships (2.2) to (2.4) and a deterministic premium function* $K(t_i), i = 0, ..., N-1$. *For* $t_j \geq t_N = T > t > t_0$ *and* $M := \min\{n^*(u), N-1\}$

$$\begin{aligned} E_{P^t}\left[\left.\sum_{i=0}^{n^*(t)} \frac{S(t)}{S(t_i)}\right| \mathbb{F}_{t_0}\right] &= \sum_{i=0}^{n^*(t)} \frac{D(t_0,t_i)}{D(t_0,t)}, \\ E_{P^{t_j}}\left[\left.\beta_{T,t_j} \cdot \sum_{i=0}^{N-1} \frac{S(T)}{S(t_i)}\right| \mathbb{F}_{t_0}\right] &= \sum_{i=0}^{N-1} \frac{D(t_0,t_i)}{D(t_0,t_j)}, \\ \int_{t_0}^{T}\left(\sum_{i=0}^{n^*(u)} K(t_i)D(t_0,t_i)\right)\pi_x(u)du &+ \left(\sum_{i=0}^{N-1} K(t_i)D(t_0,t_i)\right)\cdot\left(1-\int_{t_0}^{T}\pi_x(u)du\right) \\ &= \sum_{i=0}^{N-1}\left(K(t_i)D(t_0,t_i)\left(1-\int_{t_0}^{t_i}\pi_x(u)du\right)\right), \\ \int_{t_0}^{+\infty}\left(\sum_{i=0}^{M} K(t_i)D(t_0,t_i)\right)\pi_x(u)du &= \sum_{i=0}^{N-1}\left(K(t_i)D(t_0,t_i)\left(1-\int_{t_0}^{t_i}\pi_x(u)du\right)\right). \end{aligned}$$

Proof: See the Appendix.

3 The Fair Premium Principle

The equity–linked life and pension insurance is related to three sources of uncertainty. With respect to the insurance risk we assume that the equivalence principle prevails. That is, we assume that the insurance risk can be diversified by the insurer within the population of the insured. Obviously, the financial risk cannot be diversified by this risk management technique. The assumption of a complete market implies that any payoff at a given time t can be perfectly hedged by a self-financing portfolio strategy on the financial market. Furthermore, the initial value of this portfolio strategy is equal to the expected discounted payoff under the unique martingale measure. Since the equity–linked life and insurance contract does not allow the separation of the payoff into a pure financial and a pure insurance contract, we cannot apply these two principles separately. Therefore the premium principle needed has to combine both risk management strategies. As Brennan and Schwartz (1976), Bacinello and Ortu (1994) as well as Nielsen and Sandmann (1995, 1996), we assume that the insurance risk and the financial risks are independent.

A premium is called a fair premium if the expected discounted payments are equal to the expected discounted payoffs. More precisely we define:

Definition 1 *Consider an equity–linked life and pension insurance as defined in Section 1. A non-negative periodic premium $K^*(t_i), i = 0, ..., N-1$ is called a fair premium, if $K^*(t_i)$ is a solution to*

$$\begin{aligned}
&\sum_{i=0}^{N-1} K^*(t_i) \cdot D(t_0, t_i) \cdot \left(1 - \int_{t_0}^{t_i} \pi_x(u) du\right) \qquad (3.1)\\
= \quad &\int_{t_0}^{T} D(t_0, u) \cdot E_{P^u}\left[G_I(u)|\mathbb{F}_0\right] \cdot \pi_x(u)\ du \\
&+ \sum_{j=N}^{\infty} D(t_0, t_j) \cdot E_{P^{t_j}}\left[Q(t_j)|\mathbb{F}_0\right] \cdot \left(1 - \int_{t_0}^{t_j} \pi_x(u) du\right) \\
&+ \int_{T}^{\infty} D(t_0, u) \cdot E_{P^u}\left[G_P(u)|\mathbb{F}_0\right] \cdot \pi_x(u)\ du.
\end{aligned}$$

To derive the existence of a fair premium, two extreme cases have to be considered. Proposition 3.1 covers the situation without any guaranteed payments by the insurer. The second situation is given for the choice $\alpha = 0$. This latter case corresponds to a life and pension insurance without any portfolio strategy. For simplicity's sake, suppose that the guarantees by the insurer are constant, i.e. $g_I(t_i) = \bar{g_I}, \quad q(t_i) = \bar{q}$ and $g_P(t_i) = \bar{g_P} \forall i$. Furthermore, suppose that the periodic premium is constant, i.e. $K(t_i) = \bar{K} \ \forall \ i$. The fair premium of a life and pension insurance under pension policy A with investment share $\alpha = 0$ is given by:

$$\begin{aligned} \bar{K} &= \left[\bar{g_I} \cdot \int_{t_0}^{T} D(t_0,u)\pi_x(u)du + \bar{q} \cdot \sum_{j=N}^{+\infty} D(t_0,T) \left(1 - \int_{t_0}^{t_j} \pi_x(u)du\right) \right. \\ &\quad \left. \eta_3 \cdot \int_{T}^{+\infty} D(t_0,u) \left[\bar{g_P} - (n^*(u) - (N-1))\bar{q}\right]^+ \pi_x(u)du \right] \\ &\quad \cdot \left[\sum_{i=0}^{N-1} D(t_0,t_i) \left(1 - \int_{t_0}^{t_i} \pi_x(u)du\right) \right]^{-1} . \end{aligned} \tag{3.2}$$

The unique solution in the case of policy B with investment share $\alpha = 0$ is equal to:

$$\begin{aligned} \bar{K} &= \left[\bar{g_I} \cdot \int_{t_0}^{T} D(t_0,u)\pi_x(u)du + \bar{q} \cdot \sum_{j=N}^{+\infty} D(t_0,t_j) \left(1 - \int_{t_0}^{t_j} \pi_x(u)du\right) \right] \\ &\quad \cdot \left[\sum_{i=0}^{N-1} D(t_0,t_i) \left(1 - \int_{t_0}^{t_i} \pi_x(u)du\right) \right]^{-1} . \end{aligned} \tag{3.3}$$

The premium determined by equation (3.2) and (3.3) respectively can be used as a benchmark for the situation with investment, i.e. $\alpha > 0$.

Even though the investment share α has been defined as a constant $\in [0,1]$, we will in the following proposition allow that $\alpha \in \mathbb{R}^+$.

Proposition 3.1 *Consider an equity–linked life and pension insurance with no guaranteed payments by the insurer, i.e.* $g_I = g_P = q = 0$. *Assume that* L *is equal to the expected number of pension periods for a life aged* x *at* t_0 *if he or she survives the duration* T *of the contract, i.e.* L *is given by equation (2.1).*

a) Suppose that both repayment levels and the participation rate are equal to one, i.e. $\eta_1 = \eta_2 = \eta_3 = 1$. *Under this assumption a non-negative premium policy* $\{K(t_i)\}_{i=0}^{N-1}$ *is a fair premium sequence for both pension systems if and only if the investment share is equal to one, i.e.* $\alpha = 1$.

b) Suppose that the repayment levels and the participation rate are positive and less than one, then there exists for both pension systems an investment share $\alpha > 1$ *uniquely* *determined by the periodic premium $K(t_i)$, so that this premium is a fair premium of the equity–linked life and pension insurance.*

Proof: See the Appendix.

Without any financial guarantees, both pension systems are only affected by the insurance risk, i.e. the death probability of the insured. In this situation, the fair premium principle coincides with the equivalence principle. Therefore Proposition 3.1 is based on the ability of the insurer to diversify the insurance risk within the population of the insured. The same situation arises if we consider an investment share α equal to zero. Although the insurer now guarantees a deterministic amount in the case of the death of the insured and otherwise a periodic pension after the duration T, these payments do not represent a guarantee with respect to the return of a financial strategy. Therefore the fair premium principle again coincides with the equivalence principle. As a consequence, the periodic premium can be understood as pure insurance premium.

3.1 The Fair Premium and the Share Function

So far the premium policy has not been restricted to any specific functional form. In the remaining part of the analysis we will, however, assume that the periodic premium is determined by

$$K(t_i) = K \cdot F(t_i) \qquad \forall i = 0, \ldots, N-1, \tag{3.4}$$

where the function $F : \{t_0, t_1, \ldots, T_{N-1}\} \to \mathbb{R}^+$ covers the case of a deterministic change in the premium payment. With this assumption, the existence of the fair premium is simplified to the existence of a premium K which satisfies Definition 1.

Theorem 3.2 *Consider an equity–linked life and pension insurance with investment share $\alpha \in]0,1[$, repayment levels $\eta_i \in [0,1]$, and a periodic premium policy satisfying Assumption 3.4. Assume that for any t the value of the investment fund, $S(t)$, is continuously distributed. Furthermore, suppose that $\forall t_i$ the guaranteed amounts under policy A or B divided by the* *premium value K are strictly decreasing and continuous in K with*

$$\lim_{K\to\infty} \frac{g_I(t_i)}{K} = 0 \quad , \quad \lim_{K\to 0} \frac{g_I(t_i)}{K} = \infty,$$
$$\lim_{K\to\infty} \frac{g_P(t_i)}{K} = 0 \quad , \quad \lim_{K\to 0} \frac{g_P(t_i)}{K} = \infty,$$
$$\lim_{K\to\infty} \frac{q(t_i)}{K} = 0 \quad , \quad \lim_{K\to 0} \frac{q(t_i)}{K} = \infty.$$

Then there exists a unique and positive fair premium K^*.

Proof: See the Appendix.

In particular, the assumptions are satisfied for constant guarantees. In addition, the result applies indirectly to the situation with guarantees homogeneous of degree one in K. Suppose that K^* is the unique solution greater than zero with constant guarantees g_I, g_P, and q. For any $\gamma > 0$ the value $\gamma \cdot K^*$ is the unique fair premium for the contract with guarantees $\gamma \cdot g_I, \gamma \cdot g_P$, and $\gamma \cdot q$. In the case of homogeneous guarantees, Theorem 3.2 implies the existence and uniqueness of the fair shapes (the coefficients to K) of the guarantees. A homogeneous specification of the guarantees was discussed in Section 1. In a more abstract setting, the different guarantee functions can be formulated as

$$\begin{aligned} g_I(t) &= K \cdot \alpha \cdot b \cdot F_I(t) \quad \forall 0 \le t \le T, \\ g_P(t) &= K \cdot \alpha \cdot b \cdot \theta_P \cdot F_P(t) \quad \forall t \ge T, \\ q(t) &= K \cdot \alpha \cdot b \cdot \theta_q \cdot F_q(t) \quad \forall t \ge T, \end{aligned} \tag{3.5}$$

where $F_I(\cdot), F_P(\cdot)$ and $F_q(\cdot)$ are non-negative and bounded deterministic functions and θ_P and θ_q are non-negative contract parameters. In a simple case, the functions $F_I(\cdot), F_P(\cdot)$ and $F_q(\cdot)$ are equal to one. In this situation, θ_P is equal to the ratio between the guarantee, $g_P(t)$, (Policy A) and the guaranteed life insurance amount, $g_I(t)$. θ_q is equal to the ratio between the guaranteed periodic pension, $q(t)$, and the guaranteed life insurance amount, $g_I(t)$. Setting e.g. $g_I(t)$ to 200.000, $q_P(t)$ to 100.000 and $q(t)$ to 5.000 would imply $\theta_P = 0.5$ and $\theta_q = 0.025$. In general, non-negative and time dependent functions $F_I(\cdot), F_P(\cdot)$ and $F_q(\cdot)$ arise if we consider contract specifications with a time dependent change in the periodic premium and/or the contractual conditions.

The main advantage of the guarantee specification (3.5) is a simplification of the fair premium problem. Inserting (3.5) into Definition 1, the equation can be solved with respect to α. This implies that

$$\alpha(b) =: \frac{\sum_{i=0}^{N-1} F(t_i) \cdot D(t_0, t_i) \cdot \left(1 - \int_{t_0}^{t_i} \pi_x(u)du\right)}{R(b)}, \tag{3.6}$$

where $R(\cdot)$ is determined by the pension policy. Furthermore, the function $\alpha(b)$ characterizes the fair premium completely. Considering an equity–linked life and pension insurance with an investment share α and supposing that b is given with $\alpha = \alpha(b)$, then $\bar{K}$ is the fair premium for the contract with guarantees equal to:

$$\begin{aligned} g_I(t) &= \bar{K} \cdot \alpha \cdot b \cdot F_I(t) \quad \forall 0 \le t \le T, \\ g_P(t) &= \bar{K} \cdot \alpha \cdot b \cdot \theta_P \cdot F_P(t) \quad \forall t \ge T, \\ q(t) &= \bar{K} \cdot \alpha \cdot b \cdot \theta_q \cdot F_q(t) \quad \forall t \ge T. \end{aligned}$$

Furthermore, $\alpha(b) \cdot b$ is the shape of the guaranteed amount during the life insurance period, $\alpha(b) \cdot b \cdot \theta_P$ during the pension period and $\alpha(b) \cdot b \cdot \theta_q$ of the pension payment. Vice versa, for an investment share $\alpha = \alpha(b)$ and constant guarantees $\bar{g_I}, \bar{g_P}$ and $\bar{q}$ (i.e setting $F_I(t) = F_P(t) = F_q(t) = 1$) with $\theta_P = \frac{\bar{g_P}}{\bar{g_I}}$ and $\theta_q = \frac{\bar{q}}{\bar{g_I}}$ the unique fair premium equals

$$K^* = \frac{\bar{g_I}}{\alpha(b) \cdot b} = \frac{\bar{g_P}}{\alpha(b) \cdot b \cdot \theta_P} = \frac{\bar{q}}{\alpha(b) \cdot b \cdot \theta_q}.$$

For this reason $\alpha(b)$ will be called the fair share of the life and pension insurance contract. The remaining problem is therefore the computation of the function $R(\cdot)$. The precise form is depending on the pension policy. For pension policy A we have

$$
\begin{aligned}
& R_A(b) && (3.7)\\
:= \;& b \cdot \int_{t_0}^{T} D(t_0,u) F_I(u) \pi_x(u) du \\
& + \eta_3 \cdot b \cdot \int_{T}^{+\infty} D(t_0,u) \left[\theta_P F_P(u) - \theta_q \sum_{j=N}^{n^*(u)} F_q(t_j) \right]^+ \pi_x(u) du \\
& + b \cdot \sum_{j=N}^{+\infty} D(t_0,T) \cdot \theta_q \cdot F_q(t_j) \cdot \left(1 - \int_{t_0}^{t_j} \pi_x(u) du \right) \\
& + \eta_1 \cdot \int_{t_0}^{T} D(t_0,u) E_{P^u} \left[\left[\sum_{i=0}^{n^*(u)} F(t_i) \frac{S(u)}{S(t_i)} - b \cdot F_I(u) \right]^+ \right] \pi_x(u) du \\
& + \eta_2 \cdot \sum_{j=N}^{+\infty} D(t_0,T) E_{P^T} \left[\left[\frac{1}{L} \sum_{i=0}^{N-1} F(t_i) \frac{S(T)}{S(t_i)} - b \cdot \theta_q \cdot F_q(t_j) \right]^+ \right] \left(1 - \int_{t_0}^{t_j} \pi_x(u) du \right).
\end{aligned}
$$

Similarly, in case of the pension policy B the functional form of the function $R(b)$ is equal to

$$
\begin{aligned}
R_B(b) \;:=\; & b \cdot \int_{t_0}^{T} D(t_0,u) F_I(u) \pi_x(u) du && (3.8)\\
& + \eta_3 \cdot \int_{T}^{t_{N+L}} \sum_{i=0}^{N-1} D(t_0,t_i) F(t_i) \left[\frac{L+N-1-n^*(u)}{L} \right]^+ \pi_x(u) du \\
& + b \cdot \sum_{j=N}^{+\infty} D(t_0,t_j) \cdot \theta_q \cdot F_q(t_j) \cdot \left(1 - \int_{t_0}^{t_j} \pi_x(u) du \right) \\
& + \eta_1 \cdot \int_{t_0}^{T} D(t_0,u) E_{P^u} \left[\left[\sum_{i=0}^{n^*(u)} F(t_i) \frac{S(u)}{S(t_i)} - b \cdot F_I(u) \right]^+ \right] \pi_x(u) du \\
& + \eta_2 \cdot \sum_{j=N}^{N+L-1} D(t_0,t_j) E_{P^{t_j}} \left[\left[\frac{1}{L} \sum_{i=0}^{N-1} F(t_i) \frac{S(t_j)}{S(t_i)} - b \cdot \theta_q \cdot F_q(t_j) \right]^+ \right] \\
& \cdot \left(1 - \int_{t_0}^{t_j} \pi_x(u) du \right).
\end{aligned}
$$

At first glance the function $R(\cdot)$ is complicated. But as a function of the parameter b

$$R : \mathbb{R}^+ \to \mathbb{R}$$

is increasing and convex (see proof of Theorem 3.3). Assuming continuous distribution functions for the fund and the interest market, $R(\cdot)$ is strictly increasing and convex. This implies that the fair share defined by equation (3.6) is decreasing in b. Furthermore, the limit behaviour of the function $\alpha(b)$ is intuitive because:

- In the case of $\eta_i = 1, i = 1, 2, 3$, we have $\alpha(0) = 1$, which implies that the shape of all guaranteed payments is equal to $\alpha(0) \cdot 0 = 0$. In this case any premium is a fair premium of the contract.

- In the case of $\eta_i \leq 1, i = 1, 2, 3$, we have $\alpha(0) \geq 1$, which again implies that the insurance policy offers no guaranteed payment. In contrast to the first situation, the payment of the life insurance is larger than the portfolio value, and the pension will be smaller than the fraction of the portfolio (if $\eta_1 \cdot \alpha(0) > 1$ and $\eta_2 \cdot \alpha(0) < 1$) or vice versa.

- Since $\lim_{b\to\infty} \alpha(b) = 0$, this coincides with the situation of no investment. In this case the fair premium is given by equation (3.2), which implies that the shape of the guarantees is determined by

$$\lim_{b\to\infty} \alpha(b) \cdot b = \frac{\sum_{i=0}^{N-1} F(t_i) \cdot D(t_0, t_i) \cdot \left(1 - \int_{t_0}^{t_i} \pi_x(u)du\right)}{\lim_{b\to\infty} \frac{R(b)}{b}}.$$

Theorem 3.3 *Consider a continuously distributed mutual fund and suppose that the guarantees of an equity–linked life and pension insurance are given by (3.5), then the fair periodic premium K^* as a function of the investment share α is strictly increasing and convex.*

Proof: See the Appendix.

As an immediate consequence of the theorem, the fair premium for an equity–linked life and pension insurance is larger than the one for an identical insurance without investment component, i.e. the premium determined by equation (3.2) and (3.3) are lower bounds.

4 Pricing the Embedded Options

All the options in the expressions $R_A(b)$ and $R_B(b)$ are closely related to Asian options. To evaluate these options, we will determine their upper and lower bounds applying the technique developed in Nielsen and Sandmann (2002). With this in mind we need to be specific in our choice of the stochastic processes for the underlying stock and term structure of interest rates. The volatilities are all assumed to be deterministic. Written under the τ-forward risk adjusted measure, the stochastic differentials are

$$
\begin{aligned}
d\left(\frac{S(t)}{D(t,\tau)}\right) &= \frac{S(t)}{D(t,\tau)}\left([\sigma_1(t)-\sigma(t,\tau)]\,dW_1^\tau(t) \;+\; \sigma_2(t)\,dW_2^\tau(t)\right),\\
d\left(\frac{D(t,t')}{D(t,\tau)}\right) &= \frac{D(t,t')}{D(t,\tau)}\;[\sigma(t,t')-\sigma(t,\tau)]\,dW_1^\tau(t),
\end{aligned}
$$

where $\sigma(t,t)=0$. The stochastic processes $W_1^\tau(t)$ and $W_1^\tau(t)$ are defined by

$$(dW_1^\tau,\, dW_2^\tau) \;:=\; ((dW_1^*(t)-\sigma(t,\tau))dt, dW_2^*(t)),$$

where $(W_1^*(t))$ and $W_2^*(t))$ are independent Wiener processes under the equivalent martingale measure P^*. The measure change to the forward risk adjusted measure is determined by the Radon-Nikodym derivative

$$\left.\frac{dP^\tau}{dP^*}\right|_t = \exp\left\{\int_{t_0}^{t}\sigma(u,\tau)dW_1^*(u) - \frac{1}{2}\int_{t_0}^{t}\sigma^2(u,\tau)du\right\}.$$

The arbitrage–free price at time t_0 of an Asian type option is defined through

$$C(t_0,t_M,t_j,K_j) = D(t_0,t_j)\cdot E_{P^{t_j}}\left[\left[\sum_{i=0}^{M}F(t_i)\frac{S(t_j)}{S(t_i)} - K_j\right]^+\right], \tag{4.1}$$

where $t_j > t_M$ with $M := \min\{N-1, j-1\}$. Under the above assumptions, the solution of the stochastic process $\left(\frac{S(t_j)}{S(t_i)}\right)$ is equal to

$$\frac{S(t_j)}{S(t_i)} = H(t_0,t_i,t_j)\cdot\exp\{Z(t_0,t_i,t_j)\},$$

where $H(t_0,t_i,t_j)$ is a deterministic function defined by

$$
\begin{aligned}
&H(t_0,t_i,t_j)\\
&:= \tfrac{D(t_0,t_i)}{D(t_0,t_j)}\quad \exp\left\{-\frac{1}{2}\int_{t_0}^{t_i}(\sigma(u,t_i)-\sigma(u,t_j))^2du - \frac{1}{2}\int_{t_i}^{t_j}\left((\sigma_1(u)-\sigma(u,t_j))^2+\sigma_2(u)^2\right)du\right\}
\end{aligned}
$$

and $Z(t_0,t_i,t_j)$ is a normally distributed random variable with expectation equal to zero under P^τ. $Z(t_0,t_i,t_j)$ is determined by

$$
\begin{aligned}
&Z(t_0,t_i,t_j)\\
:=\; &\int_{t_0}^{t_i}(\sigma(u,t_i)-\sigma(u,t_j))dW_1^{t_j}(u) + \int_{t_i}^{t_j}(\sigma_1(u)-\sigma(u,t_j))dW_1^{t_j}(u) + \int_{t_i}^{t_j}\sigma_2(u)dW_2^{t_j}(u).
\end{aligned}
$$

As the sum of the random variables, $Z(t_0, t_i, t_j)$ is normally distributed, for any $j = 1, 2, ...$ a standardized normal distributed variable $Z_{M,j}$ is defined by

$$Z_{M,j} := \frac{1}{\Omega_{M,j}} \sum_{i=0}^{M} Z(t_0, t_i, t_j), \tag{4.2}$$

where $\Omega_{M,j}$ is determined so that $Var_{P^{t_j}}[Z_{M,j}] = 1$ and $M := \min\{N-1, j-1\}$.

The lower bound of the Asian type option (4.1) is then established through

$$\begin{aligned} & D(t_0, t_j) \cdot E_{P^{t_j}} \left[E_{P^{t_j}} \left[\left[\sum_{i=0}^{M} F(t_i) \frac{S(t_j)}{S(t_i)} - K_j \right]^+ \middle| Z_{M,j} \right] \right] \\ \geq \quad & D(t_0, t_j) \cdot E_{P^{t_j}} \left[E_{P^{t_j}} \left[\sum_{i=0}^{M} F(t_i) \frac{S(t_j)}{S(t_i)} - K_j \middle| Z_{M,j} \right]^+ \right] \\ =: \quad & C^l(t_0, t_M, t_j, K_j). \end{aligned}$$

The inner expectation can be evaluated as

$$\begin{aligned} & E_{P^{t_j}} \left[\sum_{i=0}^{M} F(t_i) \frac{S(t_j)}{S(t_i)} - K_j \middle| Z_{M,j} = z \right] \\ = \quad & \sum_{i=0}^{M} F(t_i) H(t_0, t_i, t_j) \exp\{m_{M,j}(t_i) \cdot z + \frac{1}{2} v_{M,j}^2(t_i, t_i)\} - K_j, \end{aligned}$$

where

$$m_{M,j}(t_i) \quad := \quad E_{P^{t_j}} \left[Z_{M,j} \cdot Z(t_0, t_i, t_j) \right] \tag{4.3}$$

$$v_{M,j}^2(t_i, t_k) \quad := \quad cov_{P^{t_j}} \left[Z(t_0, t_i, t_j), Z(t_0, t_k, t_j) | \, Z_{M,j} \right]. \tag{4.4}$$

As each of the above terms

$$\begin{aligned} f_i(z) \quad := \quad & F(t_i) H(t_0, t_i, t_j) \exp\{m_{M,j}(t_i) \cdot z + \frac{1}{2} v_{M,j}^2(t_i, t_i)\} \\ = \quad & F(t_i) \frac{D(t_0, t_i)}{D(t_0, t_j)} \exp\{m_{M,j}(t_i) \cdot z - \frac{1}{2} m_{M,j}^2(t_i)\} \end{aligned}$$

is a convex function, the equation $\sum_{i=0}^{M} f_i(z) - K_j = 0$ has infinitely many, zero, one or two solutions.

Definition 2

- *If* $\sum_{i=0}^{M} f_i(z) - K_j > 0 \quad \forall z$, *define* $z^* = z^{**} := 0$,
- *if* $\sum_{i=0}^{M} f_i(z) - K_j \leq 0 \quad \forall z$, *which could only be the case if* $m_{M,j}(t_i) = 0 \quad \forall i$, *we define* $z^* := -\infty$ *and* $z^{**} := \infty$,

- *if* $\sum_{i=0}^{M} f_i(z) - K_j = 0$ *has one solution and* $m_{M,j}(t_i) \not< 0 \quad \forall i$, *we define* $z^* := -\infty$ *and denote the solution by* z^{**},
- *if* $\sum_{i=0}^{M} f_i(z) - K_j = 0$ *has one solution and* $m_{M,j}(t_i) \not> 0 \quad \forall i$, *we denote the solution by* z^* *and define* $z^{**} := \infty$,
- *if* $\sum_{i=0}^{M} f_i(z) - K_j = 0$ *has two solutions, we denote these by* z^* *and* z^{**} *and let* $z^* < z^{**}$.

Returning to the expression of the lower bound, the implication of Definition 2 is that

$$\begin{aligned} C^l(t_0, t_M, t_j, K_j) &= D(t_0, t_j) \left[\sum_{i=0}^{M} E_{P^{t_j}} \left[(f_i(z) 1_{\{z \le z^*\}} \right] - K_j E_{P^{t_j}} \left[1_{\{z \le z^*\}} \right] \right. \\ &\quad \left. + \sum_{i=0}^{M} E_{P^{t_j}} \left[(f_i(z) 1_{\{z \ge z^{**}\}} \right] - K_j E_{P^{t_j}} \left[1_{\{z \ge z^{**}\}} \right] \right] \\ &= \left[\sum_{i=0}^{M} D(t_0, t_i) \cdot \Phi(z^* - m_{M,j}(t_i)) - D(t_0, t_j) \cdot K_j \cdot \Phi(z^*) \right. \\ &\quad \left. + \sum_{i=0}^{M} D(t_0, t_i) \cdot \Phi(-z^{**} + m_{M,j}(t_i)) - K_j \cdot D(t_0, t_j) \cdot \Phi(-z^{**}) \right], \end{aligned} \tag{4.5}$$

where $\Phi(\cdot)$ denotes the cumulative standard normal distribution. The proof of a statement equivalent to the one in Equation 4.5 can be found in Nielsen and Sandmann (2002). Applying this closed form solution for the options, the lower bound for the function $R_A(b)$ equals

$$\begin{aligned} R_A^l(b) &:= b \cdot \int_{t_0}^{T} D(t_0, u) F_I(u) \pi_x(u) du \\ &\quad + \eta_3 \cdot b \cdot \int_{T}^{+\infty} D(t_0, u) \left[\theta_P F_P(u) - \theta_q \sum_{j=N}^{n^*(u)} F_q(t_j) \right]^+ \pi_x(u) du \\ &\quad + b \cdot \sum_{j=N}^{+\infty} D(t_0, T) \cdot \theta_q \cdot F_q(t_j) \cdot \left(1 - \int_{t_0}^{t_j} \pi_x(u) du \right) \\ &\quad + \eta_1 \cdot \int_{t_0}^{T} D(t_0, u) \cdot C^l(t_0, t_{n^*(u)}, u, b \cdot F_I(u)) \cdot \pi_x(u) du \\ &\quad + \frac{\eta_2}{L} \cdot \sum_{j=N}^{\infty} D(t_0, T) \cdot C^l(t_0, t_{N-1}, t_N, b \cdot L \cdot \theta_q \cdot F_q(t_j)) \cdot \left(1 - \int_{t_0}^{t_j} \pi_x(u) du \right) \end{aligned}$$

Similarly in case of the pension policy B, the functional form of the lower bound is equal to

$$
\begin{aligned}
R_B^l(b) \quad := \quad & b \cdot \int_{t_0}^{T} D(t_0,u)F_I(u)\pi_x(u)du \\
& + \eta_3 \cdot \int_{T}^{t_{N+L}} \sum_{i=0}^{N-1} D(t_0,t_i)F(t_i)\left[\frac{L+N-1-n^*(u)}{L}\right]^+ \pi_x(u)du \\
& + b \cdot \sum_{j=N}^{\infty} D(t_0,t_j)\cdot\theta_q\cdot F_q(t_j)\cdot\left(1-\int_{t_0}^{t_j}\pi_x(u)du\right) \\
& + \eta_1 \cdot \int_{t_0}^{T} D(t_0,u)\cdot C^l(t_0,t_{n^*(u)},u,b\cdot F_I(u))\cdot\pi_x(u)du \\
& + \frac{\eta_2}{L}\cdot\sum_{j=N}^{N+L-1} D(t_0,t_j)\cdot C^l(t_0,t_{N-1},t_N,b\cdot L\cdot\theta_q\cdot F_q(t_j))\cdot\left(1-\int_{t_0}^{t_j}\pi_x(u)du\right).
\end{aligned}
$$

Denote by $\varepsilon(t_0,t_M,t_j)$ the error made applying the conditioning method, then an upper bound consists of

$$C^u(t_0,t_M,t_j,K_j) := C^l(t_0,t_M,t_j,K_j) + \varepsilon(t_0,t_M,t_j).$$

Conditioning by $Z_{M,j}$ is equivalent to the conditioning by the geometric average, i.e. for $M := \min\{N-1, j-1\}$

$$
\begin{aligned}
Z_{M,j} \quad &= \quad \frac{\ln(G(M,j)) - E_{P^{t_j}}[\ln(G(M,j))]}{(V_{P^{t_j}}[\ln(G(M,j))])^{\frac{1}{2}}}, \qquad (4.6)\\
G(M,j) \quad &:= \quad \left(\prod_{i=0}^{M}\frac{S(t_j)}{S(t_i)}\right)^{\frac{1}{M+1}}.
\end{aligned}
$$

By using that the arithmetic average is no smaller than the geometric average, we obtain, applying the notation

$$d := \frac{\ln\left(\frac{K_j}{M+1}\right) - E_{P^{t_j}}[\ln(G(M,j))]}{(\mathrm{Var}_{P^{t_j}}[\ln(G(M,j))])^{\frac{1}{2}}} = \frac{M+1}{\Omega_{M,j}}\cdot\ln\left[\frac{K_j}{(M+1)(\prod_{i=0}^{M}F(t_i)H(t_0,t_i,t_j))^{\frac{1}{M+1}}}\right],$$

the error term by conditioning on the subset $\{Z_{M,j} \leq d\}$. Denoting by $\phi(\cdot)$ the standard normal density function for $Z_{M,j}$ a bound can therefore be expressed as

$$\begin{aligned}
0 &\leq E_{P^{t_j}}\left[E_{P^{t_j}}\left[\left[\sum_{i=0}^{M} F(t_i)H(t_0,t_i,t_j)\cdot\exp\{Z(t_0,t_i,t_j)\}-K_j\right]^+ \middle| Z_{M,j}\right]\right. \\
&\quad \left. -\left[E_{P^{t_j}}\left[\sum_{i=0}^{M} F(t_i)H(t_0,t_i,t_j)\cdot\exp\{Z(t_0,t_i,t_j)\}-K_j \middle| Z_{M,j}\right]\right]^+\right] \\
&= \int_{-\infty}^{d}\left(E_{P^{t_j}}\left[\left[\sum_{i=0}^{M} F(t_i)H(t_0,t_i,t_j)\cdot\exp\{Z(t_0,t_i,t_j)\}-K_j\right]^+ \middle| Z_{M,j}\right]\right. \\
&\quad \left. -E_{P^{t_j}}\left[\sum_{i=0}^{M} F(t_i)H(t_0,t_i,t_j)\cdot\exp\{Z(t_0,t_i,t_j)\}-K_j \middle| Z_{M,j}\right]^+\right)\phi(z)dz \\
&\leq \frac{1}{2}\int_{-\infty}^{d}\left(\mathrm{Var}_{P^{t_j}}\left[\sum_{i=0}^{M} F(t_i)H(t_0,t_i,t_j)\cdot\exp\{Z(t_0,t_i,t_j)\} \middle| Z_{M,j}\right]\right)^{\frac{1}{2}}\phi(z)dz \\
&= \frac{1}{2}E_{P^{t_j}}\left[\left(\mathrm{Var}_{P^{t_j}}\left[\sum_{i=0}^{M} F(t_i)H(t_0,t_i,t_j)\cdot\exp\{Z(t_0,t_i,t_j)\} \middle| Z_{M,j}\right]1_{\{Z_{M,j}<d\}}\right)^{\frac{1}{2}}\right. \\
&\quad \left.\left(1_{\{Z_{M,j}<d\}}\right)^{\frac{1}{2}}\right] \\
&\leq \frac{1}{2}\left(E_{P^{t_j}}\left[\mathrm{Var}_{P^{t_j}}\left[\sum_{i=0}^{M} F(t_i)H(t_0,t_i,t_j)\cdot\exp\{Z(t_0,t_i,t_j)\} \middle| Z_{M,j}\right]1_{\{Z_{M,j}<d\}}\right]\right)^{\frac{1}{2}} \\
&\quad \left(E_{P^{t_j}}\left[1_{\{Z_{M,j}<d\}}\right]\right)^{\frac{1}{2}} \quad ,
\end{aligned}$$

where Hölder's inequality has been applied in the last inequality.

The bound on the pricing error of the conditional approach is therefore given by

$$\begin{aligned}
&\varepsilon(t_0,t_M,t_j) = \frac{1}{2}\cdot\Phi(d)^{\frac{1}{2}} \\
&\cdot\left(\sum_{i=0}^{M}\sum_{k=0}^{M} D(t_0,t_i)\cdot D(t_0,t_k)\cdot e^{m_{M,j}(t_i)\cdot m_{M,j}(t_k)}\cdot\left(e^{\nu^2_{M,j}(t_i,t_k)}-1\right)\cdot\Phi(d_{M,j}(i,k))\right)^{\frac{1}{2}},
\end{aligned}$$

where $d_{M,j}(i,k) := d-(m_{M,j}(t_i)+m_{M,j}(t_k))$.

To compute the approximation to the solution of the fair premium problem the coefficients $m_{M,j}(t_i), \nu^2_{M,j}(t_i,t_k)$ and $\Omega^2_{M,j}$ for $i,k=0,\ldots,j-1, j=1,\ldots,N+L$ and $M := \min\{N-1, j-1\}$ have to be calculated. For $0 \leq i \leq k \leq M$ the computation of these coefficients is closely related to the following expectation

$$\begin{aligned}
E_{P^{t_j}}[Z(t_0,t_i,t_j),Z(t_0,t_k,t_j)] &= \int_{t_0}^{t_i}(\sigma(u,t_i)-\sigma(u,t_j))\cdot(\sigma(u,t_k)-\sigma(u,t_j))du \\
&+\int_{t_i}^{t_k}(\sigma_1(u)-\sigma(u,t_j))\cdot(\sigma(u,t_k)-\sigma(u,t_j))du \\
&+\int_{t_j}^{t_k}(\sigma_1(u)-\sigma(u,t_j))^2du+\int_{t_k}^{t_j}\sigma_2^2(u)du \\
&= \int_{t_0}^{t_i}(\sigma(u,t_i)-\sigma_1(u))(\sigma_1(u)-\sigma(u,t_j))du \\
&-\int_{t_0}^{t_i}(\sigma(u,t_i)-\sigma_1(u))(\sigma_1(u)-\sigma(u,t_k))du \\
&+\int_{t_0}^{t_k}(\sigma(u,t_k)-\sigma_1(u))(\sigma_1(u)-\sigma(u,t_j))du \\
&-\int_{t_0}^{t_j}(\sigma(u,t_j)-\sigma_1(u))(\sigma_1(u)-\sigma(u,t_j))du \\
&+\int_{t_k}^{t_j}\sigma_2^2(u)du.
\end{aligned}$$

This expression can be simplified to

$$E_{P^{t_j}}[Z(t_0,t_i,t_j)Z(t_o,t_k,t_j] = a_{i,j}-a_{i,k}+a_{k,j}-a_{j,j}+\int_{t_k}^{t_j}\sigma_2^2(u)du \quad \forall\ 0\le i\le j\le M,$$

where the $a_{i,k}$'s are defined by

$$a_{i,k} := \int_{t_0}^{t_i}(\sigma(u,t_i)-\sigma_1(u))(\sigma_1(u)-\sigma(u,t_k))du.$$

As an intermediate value define $\hat{m}_{M,j}(t_i)$ by

$$\begin{aligned}
\hat{m}_{M,j}(t_i): &= \sum_{k=0}^{M}E_{P^{t_j}}\left[Z(t_0,t_i,t_j)\cdot Z(t_0,t_i,t_j)\right] \\
&= (M+1)(a_{i,j}-a_{j,j})+\sum_{k=0}^{M}\left(a_{k,j}-a_{\min\{k,i\},\max\{k,i\}}+\int_{\max\{t_i,t_k\}}^{t_j}\sigma_2^2(u)du\right).
\end{aligned}$$

With this notation the computation of the coefficients $m_{M,j}(t_i),\nu_{M,j}^2(t_i,t_k)$ and $\Omega_{M,j}^2\forall\ i,k=0,\dots,M$ and $\forall j=1,\dots,N+L$ is reduced to

$$\begin{aligned}
\Omega_{M,j}^2 &= \sum_{i=0}^{M}\hat{m}_{M,j}(t_i), \\
m_{M,j}(t_i) &= \frac{\hat{m}_{M,j}(t_i)}{\Omega_{M,j}}, \\
\nu_{M,j}^2(t_i,t_k) &= a_{i,j}-a_{j,j}+a_{k,j}-a_{\min\{k,i\},\max\{k,i\}}+\int_{\max\{t_i,t_k\}}^{t_j}\sigma_2^2(u)du \\
&\quad -m_{M,j}(t_i)\cdot m_{M,j}(t_k).
\end{aligned}$$

5 Numerical Results

The numerical results are derived for a Vasicek (1977) model of the term structure of interest rates, i.e.

$$\sigma(u,t) = \frac{\sigma}{\alpha}\left(1 - \exp\{-\alpha(t-u)\}\right) \quad \forall 0 \le u \le t,$$

with speed factor $\alpha = 0,25$ and volatility $\sigma = 15\%$. The initial term structure is assumed to be flat with an interest rate equal to 4%. The volatilities of the mutual fund are set equal to $\sigma_1 = 0$ and $\sigma_2 = 25\%$, i.e. the instantaneous correlation between the interest rate market and the fund is set to zero.

For the death distribution, we assume a mortality table adjusted with the Makeham formula

$$\begin{aligned} l_x &:= b \cdot s^x \cdot g^{c^x} \quad \text{with} \\ s &:= 0.99949255, \quad g := 0.99959845, \\ c &:= 1.10291509, \quad b := 1000401.71, \end{aligned}$$

which leads to

$$\begin{aligned} \pi_x(\tau_i) &= \frac{l_{x+\tau_i} - l_{x+\tau_i+\Delta\tau}}{l_x} \\ &\hat{=} \text{ the probability that a life-aged-}x\text{ will survive } \tau_i \text{ years and die within} \\ &\quad\ \text{the following } \Delta\tau \text{ years.} \end{aligned}$$

All payments connected with an insurance event in the time period $]t_j, t_{j+1}], j = 0,1,2,...$ are assumed to take place at the end of the period, i.e. at time t_{j+1}. It means e.g. that a term like

$$b \cdot \int_{t_0}^{T} D(t_0,u) F_I(u) \pi_x(u) du$$

in $R(\cdot)$ is replaced by

$$b \cdot \sum_{j=1}^{N} D(t_0,t_j) F_I(t_j) \pi_x(\tau_{j-1})$$

with $\Delta\tau = \frac{1}{12}$ year.

In the numerical analysis we fix $x = 35$ years and $T = 30$ years. The functions $F_I(\cdot)$ and

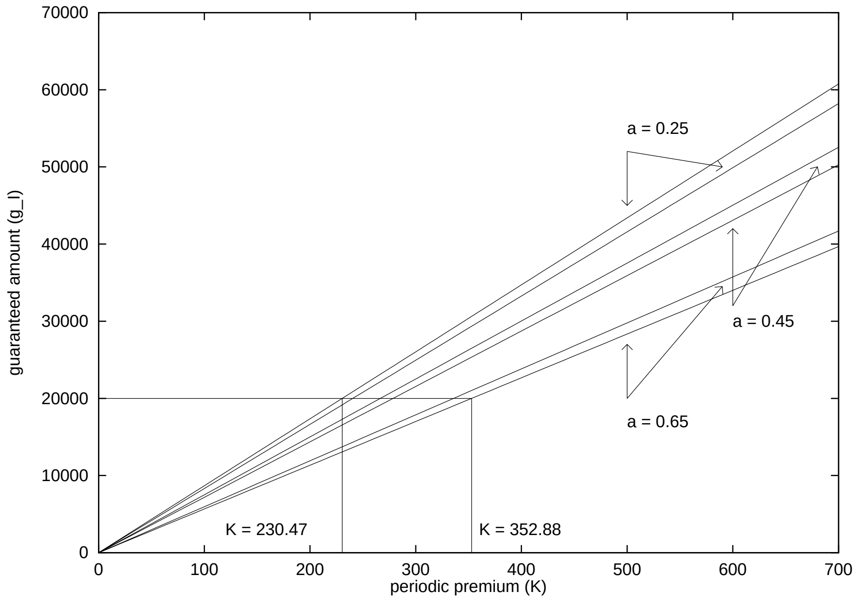

Figure 5.1: Upper and lower guaranteed insurance amount g_I for an equity-linked life and pension insurance with monthly premium, $\theta_I = 1, \theta_q = 0.05, x = 35, T = 30$, and pension policy A.

$F_q(\cdot)$ are chosen to be constant and equal to one. The parameters θ_P and θ_q we fix as $\theta_P = 1$ and $\theta_q = 0.05$. Finally, we set $g_I(t) = 20,000, q(t) = 1,000$ and $g_P(t) = 20,000$.

The results of the numerical analysis are illustrated in several tables and figures. The assumed homogeneity of $g_I(\cdot)$ leads to a linear relationship between the guaranteed amount, g_I, and the periodic premium. This is shown in Figure 5.1 and 5.2, for $\alpha-$values equal to $0.25, 0.45$ and 0.65, for pension policy A and B, respectively. The upper as well as the lower bound are shown for these $\alpha-$values. Explicitly, the fair monthly premium intervals can be seen for the contract with a guarantee of the size $20,000$. For $\alpha = 0.25$ the lower bound on the fair monthly premium is e.g. for pension policy A found to be $K = 230.47$. For the same value of g_I it is observed that the monthly premium is smaller for pension policy B than for the policy A.

The lower and upper bounds of the fair premium of an equity-linked life and pension insurance

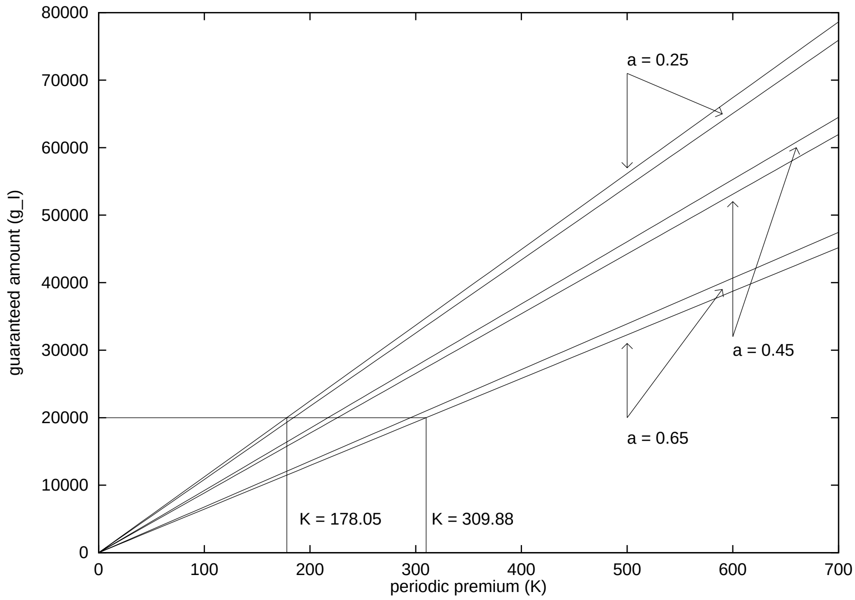

Figure 5.2: Upper and lower guaranteed insurance amount g_I for an equity-linked life and pension insurance with monthly premium, $\theta_I = 1, \theta_q = 0.05, x = 35, T = 30$, and pension policy B.

are determined by applying the arbitrage free bounds for the embedded Asian type options. For $x \in \{A, B\}$ and $\rho \in \{l, u\}$ the lower and upper bound respectively of the fair premium under pension policy A and B are denoted by $K_x^{*,\rho}$. Assume that the periodic premium, the guaranteed payments in the case of death, and the guaranteed pension all are constant over time, i.e.

$$K(t_i) = K,\ g_I = g_i(t_i),\ g_P = g_P(t_i) \text{ and } q = q(t_i)\ \forall i.$$

Applying Definition 1 the upper and lower bound under pension policy A are the unique solution to:

$$K = h_1 + h_2^A + h_3^A + h_4^\rho(K) + h_5^{\rho,A}(K)\ , \tag{5.1}$$

where the factors h_i are associated with the different components of the insurance contract. In particular, h_1 denotes the monthly premium for the term insurance, i.e. the guaranteed payment in case of death before T :

$$\begin{aligned} h_1 &:= \frac{g_I}{h_0}\int_{t_0}^{T} D(t_0,u)\pi_x(u)du \ , \\ h_0 &:= \sum_{i=0}^{N-1} \cdot D(t_0,t_i)\cdot\left(1-\int_{t_0}^{t_i}\pi_x(u)du\right) \ . \end{aligned}$$

In the case of pension policy A the remaining factors are defined by:

$$\begin{aligned} h_2^A &:= \frac{\eta_3}{h_0}\int_{T}^{+\infty} D(t_0,u)\left[g_P - q\cdot(n^*(u)+1-N)\right]^+\pi_x(u)du \\ &= \frac{\eta_3\cdot q}{h_0}\sum_{i=0}^{\frac{g_P}{q}}\left(\frac{g_P}{q}-i\right)\int_{t_{N+i}}^{t_{N+1+i}} D(t_0,u)\pi_x(u)du \ , \\ h_3^A &:= \frac{q\cdot D(t_0,T)}{h_0}\sum_{j=N}^{+\infty}\left(1-\int_{t_0}^{t_j}\pi_x(u)du\right) = \frac{q\cdot D(t_0,T)\cdot L}{h_0}\left(1-\int_{t_0}^{T}\pi_x(u)du\right) \ , \\ h_4^\rho(K) &:= \frac{\eta_1\cdot\alpha\cdot K}{h_0}\left(\int_{t_0}^{T} C^\rho\left(t_0,t_M,u,\frac{g_I}{\alpha\cdot K}\right)\pi_x(u)du\right) \ , \\ h_5^{\rho,A}(K) &:= \frac{\eta_2\cdot\alpha\cdot K}{h_0}\sum_{j=N}^{+\infty} D(t_0,T)E_{P^T}\left[\left[\frac{1}{L}\sum_{i=0}^{N-1}\frac{S(T)}{S(t_i)}-\frac{q}{\alpha\cdot K}\right]^+\right]\left(1-\int_{t_0}^{t_j}\pi_x(u)du\right) \\ &= \frac{\eta_2\cdot\alpha\cdot K}{h_0}\cdot C^\rho\left(t_0,t_{N-1},T,\frac{L\cdot q}{\alpha\cdot K}\right)\cdot\left(1-\int_{t_0}^{T}\pi_x(u)du\right) \ . \end{aligned}$$

With this notation h_3^A is equal to the monthly fair premium to be payed until time t_{N-1} for the guaranteed pension received after the duration T. Under policy A the insurer guarantees, in case of death within the pension period, that the aggregated pension payments are no less than the guaranteed amount g_P. The monthly premium for this insurance guarantee is equal to h_2^A. These premium parts are independent of the embedded options. The payment in case of death before the pension period is equal to the guaranteed amount g_P plus an option on the portfolio value. The monthly premium for this option equals $h_4^\rho(K)$. Finally, $h_5^{\rho,A}(K)$ equals the premium on a monthly basis for the bonus in addition to the guaranteed periodic pension, q. Under policy A the premium parts h_1, h_2^A and h_3^A are independent of the investment share, which obviously is not the case for the parts containing the options. Therefore these premiums are depending on the upper and lower approximation of the option values. Adding $h_4^\rho(K)$ and $h_5^{\rho,A}(K)$ to one premium part, Table 5.1 shows some results for the decomposition of the fair premium.

The decomposition in case of pension policy B is similar to the one under policy A, i.e. the

Table 5.1: Decomposition of the monthly fair premium for an equity-linked life and pension insurance contract with $T = 30$ years, constant guarantees $g_I = g_P = 20,000, q = 1,000$ for pension policy A and B for a life aged 35.

Pension policy A with $\eta_1 = 0.5, \eta_2 = 0.5, \eta_3 = 0.5$

	fair premium		term	aggregated	single	portfolio guarantee	
α	low	up	insurance	pension	pension	low	up
	$K_A^{*,l}$	$K_A^{*,u}$	h_1	h_2^A	h_3^A	$h_4^l(.) + h_5^{l,A}(.)$	$h_4^u(.) + h_5^{u,A}(.)$
0.8	273.79	279.98	9.72	0.25	201.49	62.33	68.52
0.7	260.14	265.84	9.72	0.25	201.49	48.68	54.38
0.6	248.52	253.88	9.72	0.25	201.49	37.06	42.42
0.5	238.63	243.74	9.72	0.25	201.49	27.17	32.28
0.4	230.24	235.14	9.72	0.25	201.49	18.78	23.68
0.3	223.21	227.92	9.72	0.25	201.49	11.75	16.46
0.2	217.37	221.86	9.72	0.25	201.49	5.91	10.40
0	211.46	211.46	9.72	0.25	201.49	0	0

Pension policy B with $\eta_1 = 0.5, \eta_2 = 0.5, \eta_3 = 0.5$

	fair premium		term	aggregated pension		single	portfolio guarantee	
α	low	up	insurance	low	up	pension	low	up
	$K_B^{*,l}$	$K_B^{*,u}$	h_1	$h_2^B(.)$		h_3^B	$h_4^l(.) + h_5^{l,B}(.)$	$h_4^u(.) + h_5^{u,B}(.)$
0.8	216.97	221.76	9.72	16.28	16.64	144.81	46.16	50.59
0.7	204.01	208.31	9.72	13.39	13.67	144.81	36.09	40.11
0.6	192.98	196.84	9.72	10.86	11.07	144.81	27.59	31.23
0.5	183.48	187.02	9.72	8.60	8.77	144.81	20.35	23.72
0.4	175.31	178.60	9.72	6.58	6.70	144.81	14.20	17.37
0.3	168.25	171.25	9.72	4.73	4.82	144.81	8.99	11.90
0.2	162.17	164.88	9.72	3.04	3.09	144.81	4.60	7.26
0	154.53	154.53	9.72	0	0	144.81	0	0

lower and upper bounds $K_B^{*,l}$ and $K_B^{*,u}$ of the fair premium are given by the solution of

$$K = h_1 + h_2^B(K) + h_3^B + h_4^\rho(K) + h_5^{\rho,B}(K) \ , \tag{5.2}$$

where the new coefficients $h_2^B, h_3^B(K)$ and $h_5^{\rho,B}(K)$ are defined as follows:

$$\begin{aligned} h_2^B(K) &:= \frac{\eta_3 \cdot \alpha \cdot K}{h_0} \left(\sum_{i=0}^{N-1} D(t_0, t_i) \right) \left(\sum_{j=1}^{\bar{L}} \int_{t_N}^{t_{N+j}} \pi_x(u) du \right) \ , \\ h_3^B &:= \frac{q}{h_0} \sum_{j=N}^{+\infty} D(t_0, t_j) \left(1 - \int_{t_0}^{t_j} \pi_x(u) du \right) < h_3^A \ , \\ h_5^{\rho,B} &:= \frac{\eta_2 \cdot \alpha \cdot K}{h_0 \cdot \bar{L}} \sum_{j=N}^{N+\bar{L}} C^\rho \left(t_0, t_{N-1}, t_j, \frac{\bar{L} \cdot q}{\alpha \cdot K} \right) \cdot \left(1 - \int_{t_0}^{t_j} \pi_x(u) du \right) \ , \end{aligned}$$

with $\bar{L} := \max\{i \in \mathbb{N} | i \leq L\}$ and L equal to the expected number of pension periods defined by equation (2.1). The same interpretation as under policy A applies. Since both policies coincide with respect to the premium frequency and the payment in case of death before the pension period, h_0, h_1 and $h_4^{\varrho}(K)$ are not different to those under policy A. The payment of the insurer under pension system B in case of death during the pension period now depends on the remaining value of the investment portfolio. Therefore the premium part h_2^B does depend on the investment share, α, and on the monthly premium, K. Another important difference between the pension policies deals with the size of the guaranteed pension and the bonus connected to the pension. The guaranteed pension payoff under policy A is equal to the guaranteed pension q multiplied by the roll-over return. In difference, under policy B the insured receives only the guaranteed pension q, which implies that h_3^A is larger than h_3^B. Furthermore, the additional pension payment due to the portfolio value is different under the two policies. Considering policy A, this payment is determined at time T and then increased through time by the roll-over return. In particular, the insured receives the bonus even if he or she survives the expected number of pension periods L. Under policy B, the periodic pension will not exceed the guaranteed payment q after $\bar{L}$ periods. The size of the pension payment at $t_j, j < N + L$ is affected by the portfolio value at exactly that point in time. With respect to the differences in the pension period, policy A is more expensive than policy B.

Although the decomposition of the fair premium, given by equation (5.1) and equation (5.2), separates the different premium parts, this does not allow to distinguish between the premium for the financial and the pure insurance risk. Moreover, the derived decomposition seems to suggest that the different parts of the contract are related in a linear way. Yet, this view does not consider the non-linear relationship between the portfolio insurance and the life and pension insurance.

In case of policy B with equal repayment and participation rates, i.e. $\eta_i = \eta, \forall i$ the insured receives no less than η multiplied by the value of the investment portfolio. From the viewpoint of the insured the decomposition of the fair premium into an investment equivalent $\eta \cdot \alpha \cdot K$

Table 5.2: Monthly fair premium for an equity-linked life and pension insurance contract with $T = 30$ years, constant guarantees $g_I = g_P = 20,000, q = 1,000$ and pension policy B for a life aged 35.

$\eta_1 = 1, \eta_2 = 1, \eta_3 = 1$

	lower premium			upper premium		
		investment	risk premium		investment	risk premium
α	K	$\alpha \cdot K$	$(1-\alpha) \cdot K$	K	$\alpha \cdot K$	$(1-\alpha) \cdot K$
0.8	435.10	348.08	87.02	464.10	371.28	92.82
0.7	328.25	229.77	98.47	346.04	242.23	103.81
0.6	269.16	161.50	107.67	281.87	169.12	112.75
0.5	231.33	115.67	115.67	241.23	120.61	120.61
0.4	205.01	82.00	123.01	213.15	85.26	127.89
0.3	185.79	55.74	130.05	192.66	57.80	134.86
0.2	171.50	34.30	137.20	177.29	35.46	141.83
0	154.53	0	154.53	154.53	0	154.53

$\eta_1 = 0.5, \eta_2 = 0.5, \eta_3 = 0.5$

	lower premium			upper premium		
		investment	risk premium		investment	risk premium
α	K	$0.5\alpha \cdot K$	$(1-0.5\alpha) \cdot K$	K	$0.5\alpha \cdot K$	$(1-0.5\alpha) \cdot K$
0.8	216.97	86.79	130.18	221.76	88.71	133.06
0.7	204.01	71.40	132.61	208.31	72.91	135.40
0.6	192.98	57.89	135.08	196.84	59.05	137.79
0.5	183.48	45.87	137.61	187.02	46.75	140.26
0.4	175.31	35.06	140.25	178.60	35.72	142.88
0.3	168.25	25.24	143.01	171.25	25.69	145.56
0.2	162.17	16.22	145.95	164.88	16.49	148.39
0	154.53	0	154.53	154.53	0	154.53

and an insurance equivalent $(1 - \eta \cdot \alpha) \cdot K$ is therefore natural. Furthermore, the insurer benefits from the portfolio value. If the insured dies before the pension period the payoff from the insurer in excess to η multiplied by the value of the portfolio is equal to $(1-\eta) \cdot g_I$ added to η multiplied by the positive difference between the guaranteed amount g_I and the portfolio value, i.e.

$$G_I(t) - \eta \cdot P(t, \alpha, K) = (1-\eta) \cdot g_I + \eta \cdot [g_I - P(t, \alpha, K)]^+ \; \forall t < T \; .$$

The payment in excess of the portfolio value is equal to a short position in a fixed amount and an Asian type put option. A high investment share reduces the excess payoff in case of death before the pension period of the insurer. Obviously the payoff reduction for the insurer

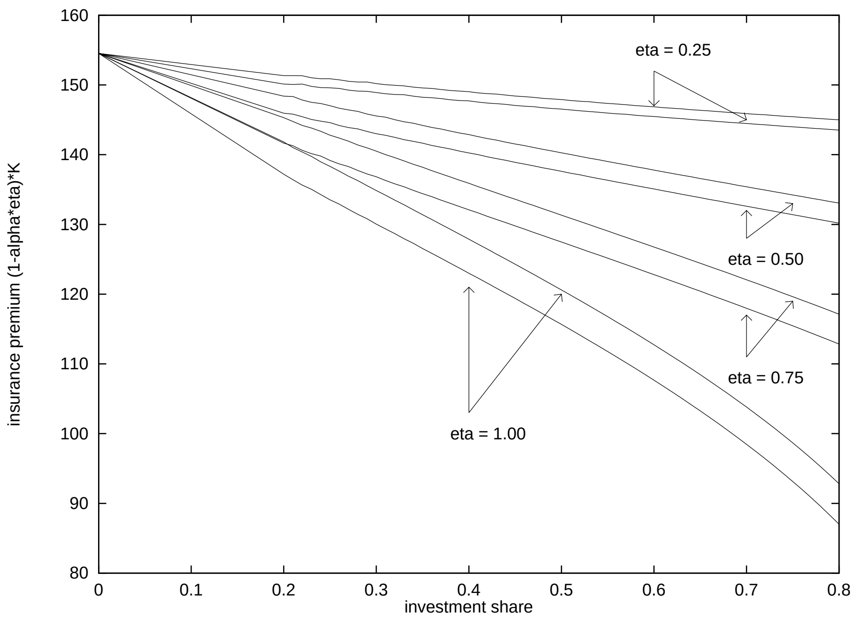

Figure 5.3: Monthly insurance premium $(1-\eta\cdot\alpha)\cdot K$ for an equity-linked life and pension insurance with monthly premium, $g_I = g_P = 20,000, q = 1,000, x = 35, T = 30$, and pension policy B.

is increasing in η. For $\eta = 1$ the position of the insurer is equal to a short position in the put option. The same argument applies during the pension period, i.e. the excess payment of the insurer is equal to a short position in a fixed amount and a put option, i.e.

$$Q(t_i) - \frac{\eta}{L}\cdot P(t_i,\alpha,K) = (1-\eta)\cdot q + \eta\cdot[g_I - \frac{1}{L}P(t_i,\alpha,K)]^+ \ \forall i = N,\cdots,N+L\ .$$

In both cases the monetary obligation of the insurer is decreasing in the value of the investment portfolio. In addition, the payment, in case of death during the time period $t\in[T,t_{N+L}]$, is equal to the remaining value of the portfolio. Therefore the interpretation of $(1-\eta\cdot\alpha)\cdot K$ as the insurance premium equivalent under pension policy B is valid. Table 5.2 and Figure 5.3 show the relationship between the insurance premium equivalent and the investment share.

Basically, the same arguments can be applied under pension policy A. Since the payments for both policies coincide for $t\in[0,T[$, it is sufficient to consider the pension period. If the insured survives the average number of pension periods the aggregated pension is not less

Table 5.3: Monthly fair premium for an equity-linked life and pension insurance contract with $T = 30$ years, constant guarantees $g_I = g_P = 20,000, q = 1,000$ and pension policy A for a life aged 35.

$\eta_1 = 1, \eta_2 = 1, \eta_3 = 1$

	lower premium			upper premium		
		investment	risk premium		investment	risk premium
α	K	$\alpha \cdot K$	$(1-\alpha) \cdot K$	K	$\alpha \cdot K$	$(1-\alpha) \cdot K$
0.8	447.17	357.73	89.43	476.36	381.09	95.27
0.7	363.68	254.57	109.10	383.36	268.35	115.01
0.6	313.37	188.02	125.35	328.59	197.16	131.44
0.5	279.52	139.76	139.76	292.35	146.17	146.17
0.4	255.34	102.14	153.20	266.73	106.69	160.04
0.3	237.58	71.27	166.30	247.96	74.39	173.57
0.2	224.40	44.88	179.52	234.06	46.81	187.25
0	211.71	0	211.71	211.71	0	211.71

$\eta_1 = 0.5, \eta_2 = 0.5, \eta_3 = 0.5$

	lower premium			upper premium		
		investment	risk premium		investment	risk premium
α	K	$0.5\alpha \cdot K$	$(1-0.5\alpha) \cdot K$	K	$0.5\alpha \cdot K$	$(1-0.5\alpha) \cdot K$
0.8	273.79	109.52	164.27	279.98	111.99	167.99
0.7	260.14	91.05	169.09	265.84	93.04	172.79
0.6	248.52	74.56	173.97	253.88	76.17	177.72
0.5	238.63	59.66	178.97	243.74	60.94	182.81
0.4	230.24	46.05	184.19	235.14	47.03	188.11
0.3	223.21	33.48	189.73	227.92	34.19	193.73
0.2	217.37	21.74	195.63	221.86	22.19	199.67
0	211.46	0	211.46	211.46	0	211.46

than the portfolio value at time T. This implies that the payment in excess of the portfolio fraction is equal to a short position in a fixed amount and a put option, i.e.

$$Q(t_i) - \beta_{T,t_i} \cdot \frac{\eta}{L} \cdot P(t_i, \alpha, K) = \beta_{T,t_i} \cdot \left((1-\eta) \cdot q + \eta \cdot [g_I - \frac{1}{L} P(t_i, \alpha, K)]^+ \right) \ \forall i \geq N \ .$$

In contradiction to to pension policy B, the payment of the insurer can exceed the guaranteed pension q even at $t > t_{N+L}$. In this situation the portfolio value is less than the aggregated pension, which again suggests to define the investment equivalent of the monthly premium as $\eta \cdot \alpha \cdot K$. A problem with this interpretation arises if instead the insured survives less than the average number of pension period L. Under pension policy A the aggregated pension

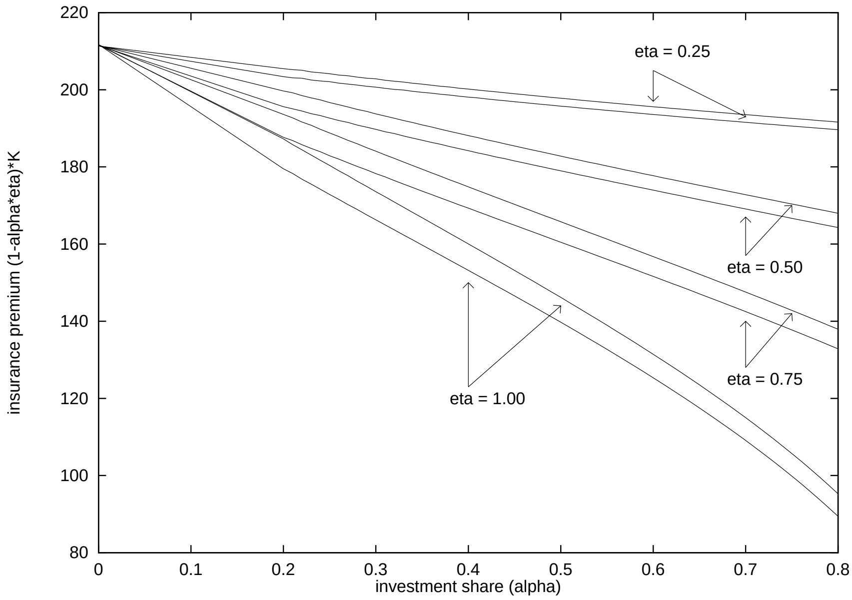

Figure 5.4: Monthly insurance premium $(1 - \eta \cdot \alpha) \cdot K$ for an equity-linked life and pension insurance with monthly premium, $g_I = g_P = 20,000, q = 1,000, x = 35, T = 30$, and pension policy A.

can now be less than the portfolio value at the beginning of the pension period. This is the only situation under pension policy A where the insured may receive less than the value of the portfolio. The monthly premium h_2 for the aggregated pension insurance in Table 5.1 under pension policy A is independent of the total fair premium for the contract and due to the low guarantee $g_P = 20,000$ very low. Under pension policy B the monthly premiums for this insurance parts are substantially higher. In other words, the value of the insurance with respect to the aggregated pension under policy B exceeds the one under policy A. This implies that a long position in η times the portfolio is on average sufficient to finance the payment of the insurer in the case the insured survives less than the average number of pension periods. A high portfolio value has a positive effect on the payment obligation of the insurer. Although the distinction between the investment and the insurance fraction of the fair premium is not as clear as under pension policy B, the derived relationship can be used as an approximation to this effect. At least from the viewpoint of the insurer the

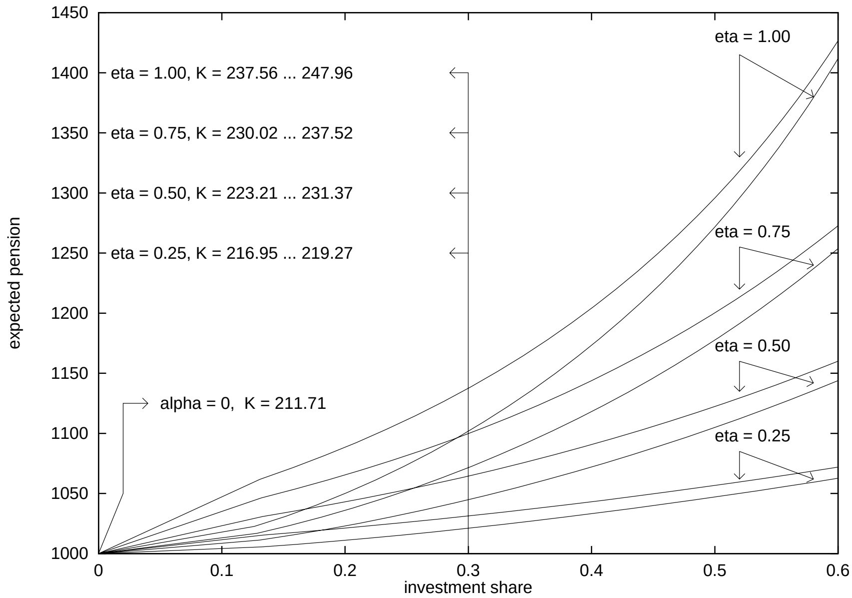

Figure 5.5: Expected monthly pension for an equity-linked life and pension insurance with monthly premium, $g_I = g_P = 20,000, q = 1,000, x = 35, T = 30$, and pension policy A.

monthly premium fraction $\eta \cdot \alpha \cdot K$ as an investment policy has the same effect under both policies. With this interpretation Table 5.3 and Figure 5.4 show the insurance equivalent under pension policy A as a function of the repayment and participation level η.

Finally we present in Figure 5.5 the expected monthly pension as a function of the investment share, α, for pension policy A. In case of pension policy A the present value at time T of the expected monthly pension is determined by the guaranteed pension plus the forward value of the bonus, i.e.

$$q + \frac{\eta_2 \cdot \alpha \cdot K}{h_0} E_{P^T} \left[\left[\frac{1}{L} \sum_{i=0}^{N-1} \frac{S(T)}{S(t_i)} - \frac{q}{\alpha \cdot K} \right]^+ \right]. \tag{5.3}$$

Since the periodic premium is increasing and convex as a function of the investment share α the expected monthly pension shares the same property.

Appendix

Proposition 2.1

Proof: The first expectation under the forward measure P^t is $\forall i$ with $t_i \leq t$ given by:

$$
\begin{aligned}
E_{P^t}\left[\left.\frac{S(t)}{S(t_i)}\right|\mathbb{F}_{t_0}\right] &= \frac{1}{D(t_0,t)}E_{P^*}\left[\beta_{t_0,t}^{-1}\cdot\left.\frac{S(t)}{S(t_i)}\right|\mathbb{F}_{t_0}\right] \\
&= \frac{1}{D(t_0,t)}E_{P^*}\left[\beta_{t_0,t_i}^{-1}\cdot\frac{1}{S(t_i)}E_{P^*}\left[\beta_{t_i,t}^{-1}\cdot S(t)\middle|\mathbb{F}_{t_i}\right]\middle|\mathbb{F}_{t_0}\right] \\
&= \frac{1}{D(t_0,t)}E_{P^*}\left[\beta_{t_0,t_i}^{-1}\cdot\left.\frac{S(t_i)}{S(t_i)}\right|\mathbb{F}_{t_0}\right] = \frac{D(t_0,t_i)}{D(t_0,t)} \quad \forall t \geq t_i \geq t_0
\end{aligned}
$$

The proof of the second equation follows the same line of argumentation. For each $t_j \geq T > t_i$ the expectation can be rewritten as follows:

$$
\begin{aligned}
E_{P^{t_j}}\left[\beta_{T,t_j}\cdot\left.\frac{S(T)}{S(t_i)}\right|\mathbb{F}_{t_0}\right] &= \frac{1}{D(t_0,t_j)}E_{P^*}\left[\beta_{t_0,t_j}^{-1}\cdot\beta_{T,t_j}\cdot\left.\frac{S(T)}{S(t_i)}\right|\mathbb{F}_{t_0}\right] \\
&= \frac{1}{D(t_0,t_j)}E_{P^*}\left[{\beta_{t_0,T}}^{-1}\cdot\left.\frac{S(T)}{S(t_i)}\right|\mathbb{F}_{t_0}\right] \\
&= \frac{D(t_0,T)}{D(t_0,t_j)}E_{P^T}\left[\left.\frac{S(T)}{S(t_i)}\right|\mathbb{F}_{t_0}\right] = \frac{D(t_0,T)}{D(t_0,t_j)}\cdot\frac{D(t_0,t_i)}{D(t_0,T)}
\end{aligned}
$$

The third equality is justified by the following calculation:

$$
\begin{aligned}
&\int_{t_0}^{T}\left(\sum_{i=0}^{n^*(u)} K(t_i)D(t_0,t_i)\right)\pi_x(u)du \\
=\; &\sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}}\left(\sum_{i=0}^{j} K(t_i)D(t_0,t_i)\right)\pi_x(u)du \\
=\; &\sum_{j=0}^{N-1}\left(\left(\sum_{i=0}^{j} K(t_i)D(t_0,t_i)\right)\int_{t_j}^{t_{j+1}}\pi_x(u)du\right) \\
=\; &\sum_{j=0}^{N-1}\left(K(t_j)D(t_0,t_j)\int_{t_j}^{T}\pi_x(u)du\right) \\
=\; &\sum_{j=0}^{N-1}\left(K(t_j)D(t_0,t_j)\left(1-\int_{t_0}^{t_j}\pi_x(u)du - 1 + \int_{t_0}^{T}\pi_x(u)du\right)\right)
\end{aligned}
$$

Applying this result we can now derive the last equation.

$$
\begin{aligned}
& \int_{t_0}^{+\infty} \left(\sum_{i=0}^{M} K(t_i) D(t_0, t_i) \right) \pi_x(u) du \\
= & \int_{t_0}^{T} \left(\sum_{i=0}^{n^*(u)} K(t_i) D(t_0, t_i) \right) \pi_x(u) du + \left(\sum_{i=0}^{N-1} K(t_i) D(t_0, t_i) \right) \cdot \int_{T}^{+\infty} \pi_x(u) du \\
= & \sum_{i=0}^{N-1} \left(K(t_i) D(t_0, t_i) \cdot \left(1 - \int_{t_0}^{t_i} \pi_x(u) du \right) \right) \\
& - \left(\sum_{i=0}^{N-1} K(t_i) D(t_0, t_i) \right) \left(1 - \int_{t_0}^{T} \pi_x(u) du \right) + \left(\sum_{i=0}^{N-1} K(t_i) D(t_0, t_i) \right) \int_{T}^{+\infty} \pi_x(u) du \\
= & \sum_{i=0}^{N-1} \left(K(t_i) D(t_0, t_i) \cdot \left(1 - \int_{t_0}^{t_i} \pi_x(u) du \right) \right).
\end{aligned}
$$

□

Proposition 3.1

Proof: In the case of pension system A with $\eta_1, \eta_2 \in]0, 1]$ the fair premium is a solution of the following problem:

$$
\begin{aligned}
& \sum_{i=0}^{N-1} K(t_i) D(t_0, t_i) \left(1 - \int_{t_0}^{t_i} \pi_x(u) du \right) \\
= & \int_{t_0}^{T} D(t_0, u) E_{P^u} \left[\eta_1 \cdot \alpha \cdot \sum_{i=0}^{n^*(u)} K(t_i) \frac{S(u)}{S(t_i)} \middle| \mathbb{F}_{t_0} \right] \pi_x(u) du \\
& + \sum_{j=N}^{+\infty} D(t_0, t_j) E_{P^{t_j}} \left[\frac{\eta_2 \cdot \alpha}{L} \cdot \beta_{T, t_j} \sum_{i=0}^{N-1} K(t_i) \frac{S(T)}{S(t_i)} \middle| \mathbb{F}_{t_0} \right] \left(1 - \int_{t_0}^{t_j} \pi_x(u) du \right) \\
= & \eta_1 \cdot \alpha \cdot \int_{t_0}^{T} \left(\sum_{i=0}^{n^*(u)} K(t_i) D(t_0, t_i) \right) \pi_x(u) du \\
& + \frac{\eta_2 \cdot \alpha}{L} \cdot \left(\sum_{i=0}^{N-1} K(t_i) D(t_0, t_i) \right) \cdot \sum_{j=N}^{+\infty} \left(1 - \int_{t_0}^{t_j} \pi_x(u) du \right) \\
= & \eta_1 \cdot \alpha \cdot \int_{t_0}^{T} \left(\sum_{i=0}^{n^*(u)} K(t_i) D(t_0, t_i) \right) \pi_x(u) du \\
& + \eta_2 \cdot \alpha \cdot \left(\sum_{i=0}^{N-1} K(t_i) D(t_0, t_i) \right) \cdot \left(1 - \int_{t_0}^{T} \pi_x(u) du \right),
\end{aligned}
$$

where we have applied Proposition 2.1 and the definition of L in (2.1). For a given premium sequence $K(t_i)$, the above equation implies the existence of a unique value of α such that this premium sequence is a solution of the fair premium problem. Moreover, for $\eta_1 = \eta_2 = \eta$ Proposition 2.1 yields that $\alpha = \frac{1}{\eta}$ is a necessary and sufficient condition for the premium to be fair. For $\eta = 1$, this implies that, if $K(t_i)$ is a fair premium, then α must be equal to one and vice versa. In addition if $\eta_1, \eta_2 \in]0,1[$, then α is larger than one.

The fair premium under pension policy B is defined as a solution of the following problem:

$$
\begin{aligned}
& \sum_{i=0}^{N-1} K(t_i)D(t_0,t_i)\left(1-\int_{t_0}^{t_i}\pi_x(u)du\right) \\
= & \int_{t_0}^{T} D(t_0,u)E_{P^u}\left[\eta_1\cdot\alpha\cdot\sum_{i=0}^{n^*(u)}K(t_i)\frac{S(u)}{S(t_i)}\middle|\mathbb{F}_{t_0}\right]\pi_x(u)du \\
& +\sum_{j=N}^{N+L-1} D(t_0,t_j)E_{P^{t_j}}\left[\frac{\alpha\cdot\eta_2}{L}\cdot\sum_{i=0}^{N-1}K(t_i)\frac{S(t_j)}{S(t_i)}\middle|\mathbb{F}_{t_0}\right]\left(1-\int_{t_0}^{t_j}\pi_x(u)du\right) \\
& +\int_{T}^{+\infty} D(t_0,u)E_{P^u}\left[\eta_3\cdot\alpha\cdot\sum_{i=0}^{N-1}K(t_i)\frac{S(u)}{S(t_i)}\middle|\mathbb{F}_{t_0}\right]\left[\frac{L+N-1-n^*(u)}{L}\right]^+\pi_x(u)du \\
= & \ \eta_1\cdot\alpha\cdot\int_{t_0}^{T}\left(\sum_{i=0}^{n^*(u)}K(t_i)D(t_0,t_i)\right)\pi_x(u)du \\
& +\frac{\alpha\cdot\eta_2}{L}\cdot\left(\sum_{i=0}^{N-1}K(t_i)D(t_0,t_i)\right)\cdot\sum_{j=N}^{L+N-1}\left(1-\int_{t_0}^{t_j}\pi_x(u)du\right) \\
& +\frac{\alpha\cdot\eta_3}{L}\left(\sum_{i=0}^{N-1}K(t_i)D(t_0,t_i)\right)\cdot\sum_{j=N}^{L+N-1}\int_{t_j}^{t_{j+1}}(L+N-j)\pi_x(u)du \\
= & \ \eta_1\cdot\alpha\cdot\int_{t_0}^{T}\left(\sum_{i=0}^{n^*(u)}K(t_i)D(t_0,t_i)\right)\pi_x(u)du \\
& +\frac{\alpha\cdot\eta_2}{L}\cdot\left(\sum_{i=0}^{N-1}K(t_i)D(t_0,t_i)\right)\cdot\sum_{j=N}^{L+N-1}\left(1-\int_{t_0}^{t_j}\pi_x(u)du\right) \\
& +\frac{\alpha\cdot\eta_3}{L}\left(\sum_{i=0}^{N-1}K(t_i)D(t_0,t_i)\right)\cdot\sum_{j=N}^{L+N-1}\left[\int_{t_0}^{t_j}\pi_x(u)du-\int_{t_0}^{t_N}\pi_x(u)du\right],
\end{aligned}
$$

where $\eta_1, \eta_2, \eta_3 \in]0,1]$. As before, for a given premium sequence $K(t_i)$ the above equation implies the existence and uniqueness of α so that this premium sequence solves the fair premium problem. In the special case $\eta_1 = \eta_2 = \eta_3 = 1$ the investment share must be equal to one. □

Theorem 3.2

Proof: Define for both policies

$$\begin{aligned} f(K) &:= \sum_{i=0}^{N-1} K \cdot F(t_i) \cdot D(t_0, t_i) \cdot \left(1 - \int_{t_0}^{t_i} \pi_x(u) du\right) \\ &- \int_{t_0}^{T} D(t_0, u) \cdot E_{P^u} \left[G_I(u) | \mathbb{F}_0\right] \cdot \pi_x(u)\ du \\ &- \sum_{j=N}^{\infty} D(t_0, t_j) \cdot E_{P^{t_j}} \left[Q(t_j) | \mathbb{F}_0\right] \cdot \left(1 - \int_{t_0}^{t_j} \pi_x(u) du\right) \\ &- \int_{T}^{\infty} D(t_0, u) \cdot E_{P^u} \left[G_P(u) | \mathbb{F}_0\right] \cdot \pi_x(u)\ du. \end{aligned} \tag{5.4}$$

A fair premium is a solution K^* to $f(K^*) = 0$. For any $\alpha \in]0,1[$, Proposition 3.1 implies that for $\eta_i \in [0,1]$

$$\lim_{K \to \infty} \frac{f(K)}{K} > 0.$$

Furthermore, for $K \to 0$, we have

$$\lim_{K \to 0} \frac{f(K)}{K} = -\infty.$$

Since the ratio of the guarantees and the premium are strictly decreasing functions of the premium, the expected values

$$E_{P^u}\left[\frac{G_I(u)}{K} | \mathbb{F}_0\right], E_{P^u}\left[\frac{G_P(u)}{K} | \mathbb{F}_0\right] \text{ and } E_{P^u}\left[\frac{Q(u)}{K} | \mathbb{F}_0\right]$$

are strictly decreasing functions of the premium K. This implies that $\frac{f(K)}{K}$ is strictly increasing and continuous in K, and by the mean value theorem, there exists a unique premium K^* with $\frac{f(K^*)}{K^*} = 0$. □

Theorem 3.3

Proof: Denote by $w(y)$ and $v(y)$ the density functions for

$$\begin{aligned} \hat{Y}(s) &= \sum_{i=0}^{n^*(s)} F(t_i) \cdot \frac{S(s)}{S(t_i)} \\ Y(s) &= \frac{1}{L} \cdot \sum_{i=0}^{N-1} F(t_i) \cdot \frac{S(s)}{S(t_i)} \end{aligned}$$

respectively. Differentiating with respect to b yields in the case of policy A

$$\begin{aligned} R'_A(b) &= \Bigg[\int_{t_0}^{T} D(t_0,u)F_I(u)\pi_x(u)du \\ &+\eta_3 \cdot \int_{T}^{+\infty} D(t_0,u) \left[\theta_P F_P(u) - \theta_q \sum_{j=N}^{n^*(u)} F_q(t_j)\right]^+ \pi_x(u)du \\ &+ \sum_{j=N}^{+\infty} D(t_0,T)\cdot\theta_q\cdot F_q(t_j)\cdot\left(1-\int_{t_0}^{t_j}\pi_x(u)du\right)\Bigg] \\ &-\eta_1\cdot\int_{t_0}^{T} D(t_0,u)F_I(u)\int_{bF_I(u)}^{\infty} w(y)dy\pi_x(u)du \\ &-\eta_2\cdot\sum_{j=N}^{\infty} D(t_0,T)\theta_q\cdot F_q(t_j)\int_{b\cdot\theta_q\cdot F_q(t_j)}^{\infty} v(y)dy\cdot\left(1-\int_{t_0}^{t_j}\pi_x(u)du\right). \end{aligned}$$

Differentiating once again leads to

$$\begin{aligned} R''_A(b) &= \eta_1\cdot\int_{t_0}^{T} D(t_0,u)F_I(u)^2 w(bF_I(u))\pi_x(u)du \\ &+\eta_2\cdot\sum_{j=N}^{\infty} D(t_0,T)\theta_q^2\cdot F_q(t_j)^2 v(b\cdot\theta_q\cdot F_q(t_j))\cdot\left(1-\int_{t_0}^{t_j}\pi_x(u)du\right) \quad , \end{aligned}$$

Similarly for pension policy B we obtain

$$\begin{aligned} R'_B(b) &= \Bigg[\int_{t_0}^{T} D(t_0,u)F_I(u)\pi_x(u)du \\ &+ \sum_{j=N}^{\infty} D(t_0,t_j)\cdot\theta_q\cdot F_q(t_j)\cdot\left(1-\int_{t_0}^{t_j}\pi_x(u)du\right)\Bigg] \\ &-\eta_1\cdot\int_{t_0}^{T} D(t_0,u)F_I(u)\int_{bF_I(u)}^{\infty} w(y)dy\pi_x(u)du \\ &-\eta_2\cdot\sum_{j=N}^{N+L-1} D(t_0,t_j)\theta_q\cdot F_q(t_j)\int_{b\cdot\theta_q\cdot F_q(t_j)}^{\infty} v(y)dy\cdot\left(1-\int_{t_0}^{t_j}\pi_x(u)du\right) \end{aligned}$$

$$\begin{aligned} R''_B(b) &= \eta_1\cdot\int_{t_0}^{T} D(t_0,u)F_I(u)^2 w(bF_I(u))\pi_x(u)du \\ &+\eta_2\cdot\sum_{j=N}^{N+L-1} D(t_0,t_j)\theta_q^2\cdot F_q(t_j)^2 v(b\cdot\theta_q\cdot F_q(t_j))\cdot\left(1-\int_{t_0}^{t_j}\pi_x(u)du\right) \quad . \end{aligned}$$

Since $\eta_i \in [0,1]$ the first and second derivative under both pension policies are positive and the function $R(\cdot)$ is convex.

To analyse the fair premium as a function of α we should analyse $\frac{1}{\alpha(b)\cdot b}$:

$$\frac{1}{\alpha(b)\cdot b} = \frac{1}{\sum_{i=0}^{N-1} F(t_i)\cdot D(t_0,t_i)\cdot\left(1-\int_{t_0}^{t_i}\pi_x(u)du\right)}\cdot\frac{R(b)}{b}.$$

We want to show, that this function is decreasing and convex. For this purpose we can and will disregard from the coefficient $\frac{1}{\sum_{i=0}^{N-1} F(t_i)\cdot D(t_0,t_i)\cdot\left(1-\int_{t_0}^{t_i}\pi_x(u)du\right)}$, Denoting by $H(b)$ the fraction of $R(b)$ to b, the derivatives

$$\begin{aligned} H'(b) &= \frac{b\cdot R'(b)-R(b)}{b^2}, \\ H''(b) &= \frac{R''(b)\cdot b^2-2R'(b)\cdot b+2R(b)}{b^3} \end{aligned}$$

are, by inserting the developed expressions for $R'(\cdot)$ and $R''(\cdot)$, immediately seen to satisfy $H'(b)<0$ and $H''(b)>0$. That is, the fair premium as a function of α is strictly increasing and convex. □

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