Valuation of Bond Illiquidity: An Option-Theoretical Approach

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Abstract

In this paper, we present an easy-to-apply option-theoretical approach to quantifying liquidity spreads of bonds. We model illiquidity in the spirit of Longstaff (1995) who describes the value of liquidity as that of an exotic option. We extend this model in two directions: First, we introduce interest rate uncertainty of the extended Vasicek type to model the dynamics of zero bonds. Second, we allow for an arbitrary distribution of trading dates rather than one single non-trading period. This results in liquidity spreads arising from the values of both continuously and discretely monitored lookback options written on a zero bond. The liquidity spreads show several meaningful and plausible properties; they are humped-shaped functions of the maturity and increase with the interest rate volatility. Furthermore, the liquidity spreads are not only influenced by the number of possible trading dates, but also by their distribution over time. In contrast to the total value of illiquid zero bonds, the theoretical liquidity spreads are independent of the short rate level.

When we regard German Jumbo Pfandbrief market data, we find several parallels between the theoretical liquidity spreads and the empirically observed ones. The main challenge for a practical implementation of our model is the determination of the possible trading dates for illiquid bonds. Nevertheless, we can provide some evidence for the empirical relevance of our model.

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1 Introduction

The liquidity of a bond measures the ease with which this security can be converted into money or other assets. Holding illiquid assets imposes additional risks to an investor since her portfolio cannot be adjusted without cost or delay. Consequently, these additional risks should be reflected in asset allocation decisions and market prices of bonds with different degrees of liquidity.

In recent years, there has been a growing interest in analyzing the asset pricing consequences of illiquidity outlined above. The interest was partly triggered by some spectacular losses of finance firms due to vanishing market liquidity, with LTCM being certainly the most prominent victim. Another even more important reason is the growing importance of the non-government sectors of fixed income markets. For example, the amount of outstanding US-$-denominated non-government bonds nearly doubled from 1995 to 2000. In the same period, the outstanding amount of the corresponding €-denominated debt issues has risen by over 17 per cent. Trading securities like mortgage bonds, corporate bonds or debt securities issued by financial institutions requires appropriate valuation models and risk management tools. Besides credit risk, the ease of trading is a very important determinant of the price and the risk of these securities since these bonds typically trade in markets with low liquidity.

Several studies examine the effects of liquidity on bond prices. For example, Amihud and Mendelson (1991) and Kamara (1994) show that illiquid Treasury Notes trade at a discount to more liquid Treasury Bills of the same maturity. The studies of Warga (1992), Redding (1999), and Krishnamurthy (2000) show that there is a significant yield difference between U.S. government bonds of the latest issuance and comparable old bonds. This "on-the-run" effect is attributed to the superior liquidity of the newly-issued bonds. The analysis of Boudhouk and Whitelaw (1991) shows similar benchmark effects for the Japanese government bond market. For the German market, Kempf and Uhrig-Homburg (2000) find a significant illiquidity discount for government bonds with lower liquidity.[1]

There is also a large body of literature that theoretically examines the impact of liquidity on assets prices. The studies can be roughly divided by the used description of liquidity. E.g., Amihud and Mendelson (1986) examine in their seminal paper the price dimension of liquidity by incorporating transaction costs. Other studies like Grossman and Miller (1988) focus on the time dimension of liquidity and identify liquidity risk as immediacy risk; i.e. an asset is regarded as more liquid than an other one if it can be traded more often. This is also the view of liquidity taken in this paper.

The aim of our paper is to provide an easy-to-apply method to calculate the discount for bonds due to liquidity differences. We derive a valuation methodology for illiquid bonds by extending a model by Longstaff (1995) originally designed to value the marketability of equities. Longstaff argues that liquidity offers a valuable option to trade. The option analogy enables Longstaff to determine the maximum discount for a stock that is restricted from trading for one period of time.

We extend the approach of Longstaff (1995) in two ways. First, we introduce stochastic interest rates of the extended Vasicek type[2] into the model to be able to tackle bond valuation problems. Second, we give up the assumption that the security is restricted

[1]Further empirical studies of liquidity effects include Elton and Green (1998), Shen and Starr (1998), Charkravarty and Sarkar (1999), and Longstaff (2001) among others.

[2]See Hull and White (1990).

from trading only for one time span and is perfectly liquid afterwards. In our model, the illiquid bond can be traded on any desired set of dates during its lifetime. The specification enables us to model time-varying liquidity of bonds among other things.

Furthermore, we give some rationale that our extended version of the Longstaff model enables us to calculate an estimate for the actual liquidity discount rather than to obtain only an upper bound for it. We analyze this question further in an empirical analysis of the German Jumbo Pfandbrief market.

We find that our model of liquidity spreads shows several meaningful and plausible properties; liquidity spreads are humped-shaped functions of maturity and increase with interest rate volatility. Furthermore, liquidity spreads are not only influenced by the number of possible trading dates, but also by their distribution over time. In particular, a bond which is traded more often at the beginning of its lifetime bears a lower liquidity spread compared with another otherwise identical bond with the same number of trading dates. Although illiquid zero bond values depend on the current short rate level, the liquidity spread is invariant of the short rate.

The main challenge for the practical implementation of our model consists in the determination of the possible trading dates for illiquid bonds. By applying an implicit estimation method to German Jumbo Pfandbrief market data, we can provide some evidence for the empirical relevance of our model.

The paper is organized as follows. Section 2 briefly sketches the model of Longstaff (1995) and presents our extensions to represent the value of bond liquidity as certain exotic interest rate options. Section 3 derives valuation formulae for the options set out in the previous section. Section 4 analyzes the influence of bond and market variables on the value of liquidity. Section 5 describes the data and the estimation methodology for an empirical study of the model. The results of the empirical implementation using German bond market data are presented in section 6. Section 7 concludes. All technical developments are shown in the appendix.

2 Option-Theoretical Valuation of Illiquidity

In this section, we describe our notion of liquidity. We further show that the liquidity advantage of perfectly liquid bonds can be identified by the fair price of an exotic interest rate option. Therefore, we briefly sketch the model of Longstaff (1995) who has developed this analogy between liquidity of a risky asset and option values for the case without interest rate risk and only one time period where the security is non-traded. We thereafter present our extensions of the Longstaff (1995) model. We integrate stochastic interest rates into the model and allow for multiple non-trading periods. I.e., we give up the assumption by Longstaff (1995) that the illiquid security becomes perfectly liquid after the first period of total illiquidity and rather assume that the bond possibly might stay illiquid. This specification also allows us to model empirically observed bond aging effects on liquidity[3] by assuming time-dependent degrees of liquidity. A critical appraisal of the developed analogy between liquidity and option prices concludes the section.

2.1 Notion of Liquidity

Liquidity of assets is a diverse concept which covers several aspects. Keynes (1930), p. 67, refers to the time dimension of liquidity by considering an asset as more liquid if it is "more certainly realizable at short notice without a loss". There is also a price dimension of liquidity. Hirshleifer (1972) defines illiquid assets in this spirit as "characterized by a relatively large discount for 'premature' realization." This at least two-dimensional nature of liquidity poses severe difficulties to develop a generally accepted model of liquidity effects.

Many studies in the literature define liquidity by its price dimension. Typically, these models impose exogenous bid-ask spreads or transaction costs that cause portfolio revisions to be costly. Examples for models of this class include the work of Amihud and Mendelson (1986), Vayanos and Vila (1999), Huang (2001), and many others.[4] The degree of illiquidity is measured by the size of the transaction costs.

Illiquid fixed income markets are often characterized by longer periods without any trading. Therefore, we define liquidity in our paper by the possible frequency of trading. Thus, we concentrate on the time dimension of liquidity. In our sense, a perfectly liquid asset can be traded continuously by investors throughout its lifetime. This is no longer true for bonds in thin markets. It is only possible to trade illiquid assets at certain points in time. Related models with similar specifications of liquidity include Rogers and Zane (1998), Subramanian and Jarrow (2001), and Longstaff (1995). In our approach, the total information about the liquidity of a bond is contained in the set of possible (not necessarily equidistantly distributed) trading dates $0 \leq t_1 < t_2 < ... < t_N = T$. If the trading dates are for example uniformly allocated over time, the magnitude of liquidity can be quantified by the distance $\Delta t = t_{i+1} - t_i$ between two adjacent trading dates. In other cases, additional variability parameters together with the average time distance can be used to measure liquidity.

2.2 One Period Marketability Discount without Interest Rate Risk

We now briefly review the Longstaff (1995) model. Longstaff abstracts from stochastic interest rates and regards a single risky asset whose equilibrium value from $t = 0$ to T

[3]See for example Warga (1992).

[4]The literature on portfolio choice and equilibrium under transaction costs is vast. Thus, we are forced not to mention a lot of relevant and insightful work in this area. Some references can be found in the survey of Sundaresan (2000) and the textbook of LeRoy and Werner (2001).

is described by some stochastic process V_t. Longstaff derives an upper bound for the additional value of this security in case that it is perfectly liquid in $[0, T]$ compared with the case when it cannot be traded at all in $[0, T]$. He assumes, in contrast to our model, that in both cases the security is perfectly liquid after T. Thus, the price V_T of the security at $t = T$ must be equal for both alternatives.

The additional value of the liquid security over the illiquid one is calculated by regarding the optimal strategy of a hypothetical investor with perfect market timing who wants to sell the security during $[0, T]$. In the second case, the investor is restricted to sell the security at the end of the non-trading period at $t = T$. The proceeds of this strategy are V_T. If the security is liquid, the investor can sell the security at the optimal time t^* when the sale price together with the interest earned on the sale price from t^* to T reaches its maximum. The proceeds of this strategy amount to $\max_{0 \leq t \leq T}\{V_t \exp(r(T-t))\}$ given a constant riskless interest rate r. The difference of both proceeds at time T is exactly the payoff of a floating strike lookback put. This exotic derivative gives the holder the right to receive the difference between the maximum price of the underlying $V_t \exp(r(T-t))$ within the option's lifetime and the underlying settlement price V_T at expiry. Consequently, Longstaff (1995) interprets the value of this option as the value of marketability for the hypothetical investor with perfect market timing ability. Longstaff argues that all other investors with inferior market timing ability can only derive smaller liquidity advantages.[5]

2.3 Illiquidity Discount under Stochastic Interest Rates

In this section, we present the basic ideas behind our option-theoretical approach to calculating illiquidity discounts for thinly traded bonds. Our goal is to establish a similar linkage between option payoffs and benefits of liquidity as in the previous section. To achieve this aim, we describe the benefits of zero bonds with different degrees of liquidity in a way similar to section 2.2 by regarding the potential trading gains of an investor with perfect market timing ability. However, our notion of illiquidity differs from the concept of non-marketability over a single period that is valued by Longstaff (1995). Since we assume that an illiquid bond is tradeable only on a certain set of dates during its lifetime, we have to take a modified approach which accounts for multiple periods where trading the illiquid bond is not possible. Our approach enables us to identify the benefits of liquidity as exotic interest rate option payoffs. The difference of the option prices yields the illiquidity discount.

We consider a financial market with a stochastic short rate r_t. Two zero bonds with the same maturity T are traded on this market. One bond is perfectly liquid in the sense

[5]E.g., it is straightforward to show that the additional value of perfect liquidity in this model is zero for an investor without any market timing ability. This follows from the martingale property of discounted security prices under the risk-neutral measure equipped with the usual filtration. Any feasible strategy involves a sale at a stopping time with regard to this filtration. By the theorem of optional sampling, no such strategy can create a payoff which is more valuable than selling the asset at time T.

outlined in section 2.1, i.e. it can be traded at any time in $[0, T]$. Its market value at time t is denoted by $P(t, T)$. The other bond is illiquid. Transactions comprising this bond can only take place on a set of dates $0 \leq t_1 < t_2 < ... < t_N = T$. At time T, both bonds pay their face value (normalized to 1) with certainty, i.e. there is no difference in credit risk.

Next, we consider an investor with perfect market timing. We characterize for each bond the investor's proceeds at time T if she sells the security optimally in $[0, T]$.

First, we investigate the investor's proceeds if she holds a perfectly liquid bond at time $t = 0$. At a time t^*, she chooses to sell the bond and invests the received sale price $P(t^*, T)$ in short term securities with the variable interest rate r_t from t^* to T. Because of her perfect market timing ability, the investor will choose the optimal t^* for switching from long-term bonds to short-term money market instruments, such that her proceeds at time T will equal

$$\max_{0 \leq t \leq T} \left\{ P(t, T) \cdot e^{\int_t^T r_s ds} \right\}. \tag{1}$$

Obviously, the gains of this strategy can be identified as the payoff of a fixed strike (with strike price equal to zero) lookback call option[6] on the fictive underlying $P(t, T)e^{\int_t^T r_s ds}$ which equals the price of the liquid bond including short term interests accrued from t to T.

Second, we repeat the analysis for the case of the illiquid bond. At first glance, one is tempted to replicate exactly the arguments along the lines of the liquid bond case. In this case one would conclude that the investor's proceeds would be equal to

$$\max_{i=1,...,N} \left\{ P(t_i, T) \cdot e^{\int_{t_i}^T r_s ds} \right\}$$

because of her perfect foresight. This expression also describes the payoff of a fixed strike (also with strike price equal to zero) lookback call option on the same underlying as outlined above. The only difference to (1) consists in the fact that the maximum of the underlying is now monitored discretely rather than continuously as in the case of the perfectly liquid bond.

However, we claim that the above given description of the proceeds in the illiquid bond case is inappropriate because it considers the effects of the bond illiquidity only insufficiently. While the above sketched approach takes the trading date restrictions into account, it neglects the fact that the bond stays illiquid till T. It assumes in fact that the investor is able to sell the illiquid bond at the price $P(t, T)$ whenever trading is possible. This contains the implicit assumption of zero liquidity spreads for the illiquid bond at all trading dates after $t = 0$. But this would contradict to the model's own predictions when it is applied at $t = t_i < T$.

Therefore, we argue that an appropriate approach should also incorporate a discount on the illiquid bond in later periods. We express this potentially state-dependent difference

[6] A (European) fixed strike lookback call pays the difference between the maximum price of the underlying during the monitoring period and the strike price at maturity given that the difference is positive.

at time t as an annualized yield which we term liquidity spread $\gamma(t,T)$; i.e. the value of an illiquid bond is given by:

$$e^{-\gamma(t,T)\cdot(T-t)}P(t,T)$$

The investor's actual proceeds at time T of an optimal sale strategy for the illiquid bonds are then given by

$$\max_{i=1,\dots,N}\left\{e^{\gamma(t_i,T)(T-t_i)}P(t_i,T)\cdot e^{\int_{t_i}^{T} r_s ds}\right\}. \tag{2}$$

After determining the options' payoffs, the calculation of the illiquidity discount $\gamma(0,T)$ at time $t=0$ is straightforward. As a matter of fact, the value of the proceeds in the liquid bond case exceed the corresponding value in the illiquid bond case. The difference stems from the fact that the better liquidity of the first bond offers valuable trading opportunities for investors holding this bond. Therefore, we claim that the relative difference of the value in the illiquid case to the value in the liquid case gives the discount for illiquidity. We will discuss this point in more detail in section 2.4 where we conduct a careful appraisal of our approach.

Equation (2) implicitly defines a recursive relation of the discounts. Thus, we need to know the levels of the future liquidity spreads in order to determine the current liquidity spread. Fortunately, because of the one-directional dependence of the spreads (future spreads affect present spreads and not vice versa), a simple iterative algorithm can be applied to derive the spreads endogenously. The method proceeds as follows:

Since both bonds repay their face values at $t_N = T$ the corresponding liquidity spread is $\gamma(T,T) = 0$. Next, regard t_{N-1}, the last trading date of the illiquid bond before T. Since the current liquidity spread $\gamma(t_{N-1},T)$ has not been determined yet, we calculate the value of the liquidity adjusted discretely monitored lookback option payoff given in equation (2) dependent on $\gamma(t_{N-1},T)$. But the difference of this option value to the option value in the liquid bond case expressed as an annualized yield difference equals the current liquidity spread, which is again $\gamma(t_{N-1},T)$ itself. Thus, the current liquidity spread $\gamma(t_{N-1},T)$ turns out to be the solution of a fixed point problem of the following form:

$$\frac{1}{T-t_{N-1}}\ln\left(\frac{\text{option value liquid case}}{\text{option value illiquid case}(\gamma(t_{N-1},T),\gamma(T,T))}\right) = \gamma(t_{N-1},T) \tag{3}$$

The existence of a positive solution to this fixed point problem can be easily shown via the following considerations: $\gamma(t_{N-1},T) = 0$ cannot be a solution to equation (3). In this case, the value of the continuously monitored lookback option (liquid case) exceeds the value of its discretely monitored counterpart (illiquid case).[7] Thus, the left hand side of equation (3) exceeds the right hand side. A potential solution to equation (3) therefore has to be greater than zero. On the other hand, a solution cannot exceed the value

$$\gamma^{\max}(t_{N-1},T) = \frac{1}{T-t_{N-1}}\ln\left(\frac{\text{option value liquid case}}{P(t_{N-1},T)}\right).$$

[7] Note that both options have the same underlying since we consider the case $\gamma(t_{N-1},T) = 0$.

This upper bound for the solution to (3) represents the liquidity spread for total illiquidity, i.e. the illiquid bond can only be sold at $t = T$. The risk-neutral value of such a strategy is simply $P(t_{N-1}, T)$. However, the hypothetical investor with perfect market timing ability can in some cases achieve higher payoffs since she has the additional opportunity to sell the illiquid bond directly in t_{N-1}. Therefore, the left hand side of equation (3) is smaller than the right hand side if we plug in $\gamma^{\max}(t_{N-1}, T)$. Because of continuity considerations, there has to be a solution for the fixed-point problem (3) within the interval $[0, \gamma^{\max}(t_{N-1}, T)]$.

Given $\gamma(t_{N-1}, T)$ obtained as shown above, the liquidity spread at the trading date just before t_{N-1} can be obtained in the same manner. Thus, all liquidity spreads up to $t = 0$ can be calculated iteratively. The solutions of the consecutive fixed point problems have to be computed numerically.

The presented algorithm requires quite a number of iterations steps. The steps themselves contain again many iterations because we are forced to apply numerical methods like Monte Carlo simulation or numerical integration in each step.[8] The fast-growing complexity of our model works against our goal to develop an easy-to-apply method for the determination of illiquidity discounts. Therefore, we decided to reduce the complexity by assuming a time-independent liquidity spread $\gamma(\cdot, T) = \gamma(T)$. I.e. we assume that the liquidity spread of a bond is the same at $t_1, t_2, ..., T$. The dependence on the maturity does not vanish, so that bonds with different maturities can still have different liquidity spreads. The time-T proceeds of the investor with perfect foresight are now of the form

$$\max_{i=1,\ldots,N} \left\{ e^{-\gamma(T)(T-t_i)} P(t_i, T) \cdot e^{\int_{t_i}^{T} r_s ds} \right\}.$$

With these simplifications, the use of an iterative procedure to determine the liquidity spread is no longer necessary. Since we only have one unknown variable, we now only have to solve one fixed-point problem of the form:

$$\gamma(T) = \frac{1}{T} \ln \left(\frac{\text{option value liquid case}}{\text{option value illiquid case}} \right) \tag{4}$$

2.4 A Critical Appraisal

After presenting the baselines of our illiquidity discount model, we find it appropriate to review once again the basic ideas of our approach before we solve the option valuation problems posed by it. Our model uses the similarity between option payoffs and liquidity benefits first explored by Longstaff (1995) to calculate liquidity spreads. Although its clearness and simplicity is very appealing, the approach can be questioned. The trigger

[8] E.g., the examination of an illiquid zero bond which matures in ten years and that can be traded once a week would require the solution of 500 different fixed-point problems. Our experience shows that we have to compute about ten different discrete-lookback option values for the solution of one particular fixed point problem with an appropriate precision. As it is necessary to apply a numerical method for the determination of each option value itself, the computations for every individual zero bond are rather costly.

for this concern rests on the assumption of perfect market timing ability. We argue in this section that though it is true that perfect market timing is an unrealistic assumption our model can nevertheless be applied to value illiquidity.

First of all, we want to stress that we do not believe in the existence of perfect market timing ability in real markets. To illustrate why the results of our model are still relevant for actual markets, we have to analyze the role of perfect foresight in our theoretical model more thoroughly. The basic structure of our liquidity discount model is in principle the same as the basic structure of any equilibrium model of illiquidity. It is a formal description of the tradeoff between the two major forces that drive liquidity spreads: trading motives and trading frictions. In other words, equilibrium liquidity spreads describe the balance between the desire to trade and the difficulty of trading.

Typically, it is a tough task to create such an equilibrium model with realistic properties. This is mainly due to the fact that the widely-used and well-accepted equilibrium models of capital markets explain trading volume very poorly despite of their impressing capability of explaining prices. We see our model as a possibility to develop a tradeoff between trading motives and trading frictions in a simple way. The trading frictions are the trading time constraints. The trading motives are the possession of superior information or the belief to have such.

Although the assumption about the perfect private information itself is unrealistic, we claim that its implications for the trading motives are quite reasonable. E.g., in our model, the value of perfect foresight and therefore the desire to trade rises with increasing market volatility. This parallels the situation in markets without perfect foresight. If a market is more volatile, the investor has to adjust her portfolio more often, i.e. her desire to trade rises also.

Furthermore, we do not calculate the value of superior information since our model does not compare investors with different information. It rather derives the liquidity spread from the difference of the possible gains of two investors who both possess the same superior information.

One more objection that could be put forward against our model concerns the magnitude of the trading motive. Though the reaction of our trading motive to other economic variables is reasonable, the ability of perfect market timing seems to create very strong trading motives which lead to exaggerated liquidity spreads. Longstaff (1995) therefore declares perfect foresight to be an extreme case which allows to derive upper bounds for illiquidity discounts. A first reply states that the strength of the trading motive is mitigated by restrictions on the investor's strategies.[9] But there is no reason why the overestimation of the strength of the motive by the perfect foresight assumption should be annihilated by the restrictions on the investor's strategy space. In order to correct for that possible misspecification of the magnitude of the trading motive we have to modify our model

[9] The investor's strategy is a buy-and-hold strategy. Obviously, the investor could earn a lot more if she is allowed to apply truly dynamic strategies. This was already mentioned by Kempf (1999), p. 74.

for practical implementation. Therefore, we introduce an information correction factor c^* into the model which basically measures the degree of deviation of the theoretically assumed and the real trading motives. Due to its origin, c^* is a systematic factor and is not bond specific. This allows its identification. Details about the introduction and the estimation of the factor c^* are set out in the empirical study of the model in section 6.

In summary, we think that though the assumption of perfect market timing ability does not hold in reality it leads (maybe after corrections) to reasonable pricing implications. Ultimately, this claim can only be verified by an empirical test of the model with real market data which is carried out in section 6. Before, we present the solutions to the valuation problems for the exotic options shown above in section 3 and analyze the behavior of the theoretical liquidity spread in section 4.

3 Determination of the Liquidity Spread

The liquidity spread $\gamma(T)$ arises from the values of two interest rate lookback options. In particular, we have to evaluate both a continuously and a discretely monitored lookback call option with strike price zero written on a zero bond plus earned interest. The valuation of these exotic interest rate options itself is an interesting and challenging issue from a theoretical perspective. However, we cannot obtain the required option values from the literature. This is the reason why we regard it as necessary to sketch the derivation of the required lookback options first. Then, we show how to determine $\gamma(T)$ from these option values.

3.1 Valuation of Interest Rate Lookback Derivatives

In this section, we determine the value of lookback derivatives having a payout as specified in equations (1) and (2). Therefore, we consider a complete and frictionless market for fixed income securities. The short term interest rate r_t follows an extended Vasicek process with a time-dependent mean-reversion $\theta(t)$ that has been introduced by Hull and White (1990). The increment of r_t is given by

$$dr_t = \kappa(\theta(t) - r_t) \cdot dt + \sigma \cdot dz_t, \tag{5}$$

where z_t stands for the value of a standard Wiener process under the risk-neutral (short-rate) measure and κ denotes the speed of mean-reversion. $\theta(t)$ is assumed to be an integrable function of time. This specification uniquely determines the prices of zero bonds $P(t,T)$ with a face value of unity and expiration date T as

$$P(t,T) = e^{A(t,T) - r_t \cdot B(t,T)}, \tag{6}$$

where

$$\begin{aligned} B(t,T) &= \frac{1-e^{-\kappa(T-t)}}{\kappa}, \\ A(t,T) &= \left(\frac{\sigma^2}{2\kappa^2}\right)(T-t) - \int_t^T \theta(s)\,ds - \frac{\sigma^2}{\kappa^2}B(t,T) \\ &+ \int_t^T e^{-\kappa(T-s)}\cdot\theta(s)\,ds + \left(\frac{\sigma^2}{4\kappa^3}\right)\left(1-e^{-2\kappa(T-t)}\right). \end{aligned}$$

In the first step, we consider the continuously monitored lookback call option with strike price zero written on the zero bond plus earned interest. We denote the value of this exotic option by $\overline{S}(t,T)$. The payoff from this contract at maturity T reads:

$$\overline{S}(T,T) = \max_{0\le t\le T}\left\{P(t,T)\cdot e^{\int_t^T r_s ds}\right\}$$

Recall that $\overline{S}(T,T)$ is the value obtained from selling a liquid zero bond maturing at time T at the optimal time. Therefore, $\overline{S}(t,T)$ stands for the value at time t obtained with one zero bond when its holder has perfect market timing ability. In order to find a representation for the value of $\overline{S}(0,T)$ at time 0, we apply the risk-neutral valuation. This leads us to the following representation:

$$\overline{S}(0,T) = \mathbb{E}_0\left(e^{-\int_0^T r_s ds}\max_{0\le t\le T}\left\{P(t,T)\cdot e^{\int_t^T r_s ds}\right\}\right) = \mathbb{E}_0\left(\max_{0\le t\le T}\left\{P(t,T)\cdot e^{-\int_0^t r_s ds}\right\}\right)$$

For the ease of notation, we define:

$$y_T := \ln\left(\max_{0\le t\le T}\left\{P(t,T)\cdot e^{-\int_0^t r_s ds}\right\}\right) - \ln\left(P(0,T)\right)$$

The corresponding density function $f(y_T)$ is given by

$$f(y_T) = \frac{1}{\sqrt{\xi(T)}}\Phi'\left(\frac{y_T+\frac{1}{2}\xi(T)}{\sqrt{\xi(T)}}\right) + e^{-y_T}\cdot\Phi\left(-\frac{y_T-\frac{1}{2}\xi(T)}{\sqrt{\xi(T)}}\right) + e^{-y_T}\cdot\Phi'\left(\frac{y_T-\frac{1}{2}\xi(T)}{\sqrt{\xi(T)}}\right),$$

where $\Phi'(\cdot)$ stands for the density function of the standard normal distribution and $\Phi(\cdot)$ for the corresponding distribution function, respectively. $\xi(\tau)$ is defined as:

$$\xi(\tau) = \sigma^2\frac{4e^{-T\kappa}\left(1-e^{\kappa\tau}\right) - e^{-2T\kappa}\left(1-e^{2\kappa\tau}\right) + 2\tau\kappa}{2\kappa^3}$$

Therewith, the value of $\overline{S}(0,T)$ amounts to:

$$\overline{S}(0,T) = \mathbb{E}_0\left(\max_{0\le t\le T}\left\{P(t,T)\cdot e^{-\int_0^t r_s ds}\right\}\right) = \int_0^\infty e^{y_T+\ln P(0,T)}\cdot f(y_T)\,dy_T$$

The evaluation of the integral leads us to a closed-form solution for $\overline{S}(0,T)$:

$$\begin{aligned} \overline{S}(0,T) &= \int_0^\infty e^{y_T+\ln P(0,T)}\cdot f(y_T)\,dy_T \qquad (7) \\ &= P(0,T)\cdot\left(2\cdot\Phi\left(\frac{1}{2}\sqrt{\xi(T)}\right) + e^{-\frac{\xi(T)}{8}}\sqrt{\frac{\xi(T)}{2\pi}} + \frac{\xi(T)}{2}\Phi\left(\frac{1}{2}\sqrt{\xi(T)}\right)\right) \end{aligned}$$

For the complete derivation, see appendix A.

It follows from representation (7) that $\overline{S}(0,T)$ is proportional to the zero bond price. In particular, the proportionality rate $\overline{S}(0,T)/P(0,T)$ is strictly increasing in σ and T and is decreasing with κ; this can be verified by taking the corresponding derivative. Hence, in line with our intuition, the proportionality rate benefits from a higher interest rate volatility.[10] The reason for this is analogous to the typical argument used for plain vanilla options. This argument states that a higher volatility is associated with better chances of higher final payoffs, while the loss-potential is limited. Note that a higher volatility does not only correspond to a higher σ but also to a lower κ. When κ rises, r_t approaches $\theta(t)$ more severely; this leads to a decline of the volatility.

The property that a longer time to maturity T implies a higher proportionality rate $\overline{S}(0,T)/P(0,T)$ is also consistent with our intuition. This is because the right embedded in $\overline{S}(t,T)$ to sell a zero bond optimally is related to a longer time period.

Since $\xi(T)$ does not depend on the short-rate r_t and on the mean-reversion factor $\theta(t)$, the proportionality rate is also independent of r_0 and $\theta(t)$ and the relative change of $\overline{S}(0,T)$ with a change of r_0 or $\theta(t)$ coincides with the corresponding relative change of the zero bond price $P(0,T)$.

In the next step, we determine the value of a similar lookback option $\underline{S}(t,\alpha;t_1,...,t_N)$ that monitors the underlying at discrete dates $0 \leq t_1,...,t_N = T$. Additionally, this derivative takes into account that the monitored underlying $P(t,T)\cdot e^{\int_t^T r_s ds}$ experiences a deterministic discount before maturity by the the factor $e^{-\alpha\cdot(T-t)}$, $\alpha \geq 0$. Hence, we can classify $\underline{S}(t,\alpha;t_1,...,t_N)$ as the value of a discrete lookback call option written on the fictive underlying $e^{-\alpha\cdot(T-t)}\cdot P(t,T)\cdot e^{\int_t^T r_s ds}$ with strike price zero. The payoff from this contract at time T reads:

$$\underline{S}(T,\alpha;t_1,...,t_N) = \max_{i=1,...,N}\left\{e^{-\alpha\cdot(T-t_i)}\cdot P(t_i,T)\cdot e^{\int_{t_i}^T r_s ds}\right\}$$

Following the arguments in the previous section, we can interpret $\underline{S}(t,\alpha;t_1,...,t_N)$ as the wealth an investor with perfect market timing ability can obtain with one zero bond when trading takes place subject to two restrictions; i.e. trading only occurs at $t_1,...,t_N$ and the trading price at time t equals $e^{-\alpha\cdot(T-t)}\cdot P(t,T)$.

The value of $\underline{S}(t,\alpha;t_1,...,t_N)$ at time 0 is given by:

$$\begin{aligned}\underline{S}(0,\alpha;t_1,...,t_N) &= \mathbb{E}_0\left(e^{-\int_0^T r_s ds}\cdot e^{-\alpha\cdot(T-t_i)}\cdot \max_{i=1,...,N}\left\{P(t_i,T)\cdot e^{\int_{t_i}^T r_s ds}\right\}\right) \\ &= \mathbb{E}_0\left(\max_{i=1,...,N}\left\{P(t_i,T)\cdot e^{-\alpha\cdot(T-t_i)-\int_0^{t_i} r_s ds}\right\}\right)\end{aligned} \quad (8)$$

[10] We define interest rate volatility as the conditional standard deviation of r_τ given r_t (with $\tau > t$).

In appendix B, we show that

$$\underline{S}\left(0,\alpha;t_1,...,t_N\right) = P\left(0,T\right)\cdot e^{-\alpha\cdot T} \cdot \sum_{i=1}^{N} e^{\alpha\cdot t_i}\cdot\Phi\left(d_{i,1}^1,d_{i,2}^1,...,d_{i,i-1}^1;\Theta_i^1\right)\cdot\Phi\left(d_{i,i+1}^2,...,d_{i,N-1}^2,d_{i,N}^2;\Theta_i^2\right), \tag{9}$$

with

$$d_{i,j}^1 := \frac{\alpha\cdot\left(t_i-t_j\right)+\frac{1}{2}\cdot\left(\xi\left(t_i\right)-\xi\left(t_j\right)\right)}{\sqrt{\xi\left(t_i\right)-\xi\left(t_j\right)}},$$

$$d_{i,j}^2 := -\frac{\alpha\cdot\left(t_j-t_i\right)-\frac{1}{2}\cdot\left(\xi\left(t_j\right)-\xi\left(t_i\right)\right)}{\sqrt{\xi\left(t_j\right)-\xi\left(t_i\right)}},$$

$$\Theta_i^1 := \begin{pmatrix} \phi_{1,1}^i & \cdots & \phi_{1,i-1}^i \\ & \ddots & \\ \vdots & \phi_{j,k}^i & \vdots \\ & \ddots & \\ \phi_{i-1,1}^i & \cdots & \phi_{i-1,i-1}^i \end{pmatrix},$$

$$\Theta_i^2 := \begin{pmatrix} \psi_{i+1,i+1}^i & \cdots & \psi_{i+1,N}^i \\ & \ddots & \\ \vdots & \psi_{j,k}^i & \vdots \\ & \ddots & \\ \psi_{N,i+1}^i & \cdots & \psi_{N,N}^i \end{pmatrix},$$

$$\phi_{j,k}^i := \frac{\xi\left(t_i\right)-\max\left(\xi\left(t_j\right),\xi\left(t_k\right)\right)}{\sqrt{\xi\left(t_i\right)-\xi\left(t_j\right)}\sqrt{\xi\left(t_i\right)-\xi\left(t_k\right)}},$$

$$\psi_{j,k}^i := \frac{\min\left(\xi\left(t_j\right),\xi\left(t_k\right)\right)-\xi\left(t_i\right)}{\sqrt{\xi\left(t_j\right)-\xi\left(t_i\right)}\sqrt{\xi\left(t_k\right)-\xi\left(t_i\right)}},$$

where $\Phi\left(\cdot;\cdot\right)$ denotes the corresponding multivariate cumulative standard normal distribution function.

Regarding representation (9), we can easily see that $\underline{S}\left(0,\alpha;t_1,...,t_N\right)$ is proportional to the zero bond price $P\left(0,T\right)$. The proportionality rate $\underline{S}\left(0,\alpha;t_1,...,t_N\right)/P\left(0,T\right)$ increases with a higher volatility. The intuition for this property is — same as for $\overline{S}\left(0,T\right)/P\left(0,T\right)$ — that a higher volatility is associated with a higher upside potential, while the loss potential is restricted. It is obvious that $\underline{S}\left(0,\alpha;t_1,...,t_N\right)/P\left(0,T\right)$ benefits from each additional trading date t_i. The higher α, the lower the proceeds from selling the bond prematurely; this results in a lower value of $\underline{S}\left(0,\alpha;t_1,...,t_N\right)$. Moreover, $\underline{S}\left(0,\alpha;t_1,...,t_N\right)/P\left(0,T\right)$ does not depend on r_0 and $\theta\left(t\right)$. This is because $P\left(0,T\right)$ is the only term of the right-hand-side of equation (9) that is impacted by r_0 or $\theta\left(t\right)$.

3.2 Computation of the Liquidity Spread

The values of $\overline{S}(t,T)$ and $\underline{S}(t,\alpha;t_1,...,t_N)$ allow us to quantify the liquidity spread described by equation (4) in section 2. Thereto, we link $\overline{S}(t,T)$ and $\underline{S}(t,\alpha;t_1,...,t_N)$ to the previously defined "option value liquid case" and the "option value illiquid case" respectively. This enables us to rewrite the liquidity spread for a zero bond which only trades at $t_1,...,t_N$ as follows:

$$\begin{aligned}\gamma(T) &= \frac{1}{T}\ln\left(\frac{\text{option value liquid case}}{\text{option value illiquid case}}\right)\\ &= \frac{1}{T}\ln\left(\frac{\overline{S}(0,T)}{\underline{S}(0,\gamma(T);t_1,...,t_N)}\right)\end{aligned} \tag{10}$$

The computation of $\gamma(T)$ poses two problems. First, $\underline{S}(0,\gamma(T);t_1,...,t_N)$ depends on the value of a multidimensional cumulative standard normal distribution function. As there exists no general closed form solution for this function, we have to employ a numerical procedure like numerical integration or Monte Carlo simulation to get the value of the distribution function. Alternatively, we could apply a Monte Carlo simulation to the expectation in equation (8). At this point, we decide for the second opportunity: i.e. all computations will be made by applying the Monte Carlo simulation to equation (8) throughout the paper.

Hereby, we have to draw one random number for each trading date $t_1,...,t_N$ to get one value for $\max_{i=1,...,N}\left\{P(t_i,T)\cdot e^{-\alpha\cdot(T-t_i)-\int_0^{t_i} r_s ds}\right\}$. As we work with $100,000$ repetitions, a total of $N\cdot 100,000$ random variables are required to get one value for $\underline{S}(0,\gamma(T);t_1,...,t_N)$.[11]

Second, recall that $\gamma(T)$ requires the value of $\underline{S}(0,\gamma(T);t_1,...,t_N)$ which depends on $\gamma(T)$ itself. As mentioned in the previous section, we have to solve a fixed-point problem to obtain $\gamma(T)$. In particular, we use a binary search algorithm with a relative accuracy of one percent[12] to quantify the solution of the fixed point problem $\gamma(T)$.

4 Properties of the Liquidity Spread

In this section, we analyze the properties of the liquidity spread $\gamma(T)$.

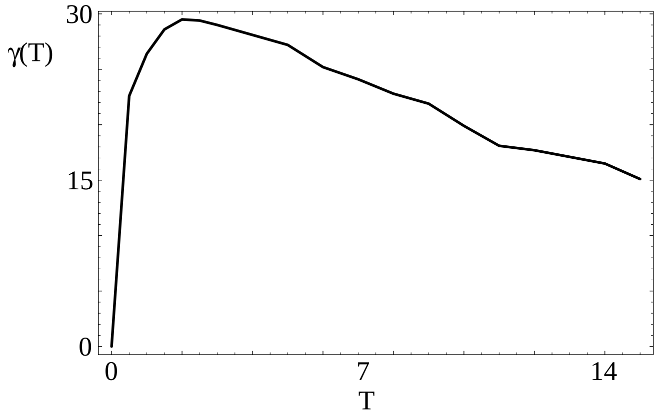
As figure 1 shows, the liquidity spread lies between 15.1 and 29.5 basis points (bp) for a time to maturity from 0.5 to 15 years. For a lower or a higher time to maturity, the corresponding liquidity spread is closer to zero. The interest rate parameters used for

[11] To find the appropriate number of repetitions, we have analyzed this number in detail. Our finding is that for more than $50,000$ repetitions the Monte Carlo simulation provides stable option values; i.e. different sets of random numbers lead to a deviation of less than one basis point of $\gamma(T)$. As this accuracy is good enough for our examination, we regard it as sufficient to deal with twice that number of repetitions namely $100,000$.

[12] This means $\left|x-\frac{1}{T}\ln\left(\frac{\overline{S}(0,T)}{\underline{S}(0,x;t_1,...,t_N)}\right)\right|/x<0.01$.

Figure 1: Liquidity Spread as a Function of Time to Maturity

The figure shows the liquidity spread measured in bp as a function of the time to maturity T. The distance between two consecutive trading dates is a constant equal to Δt. The used parameter values are: $\kappa = 0.16643$, $\sigma = 0.01097$, $\Delta t = 0.1$.



this example are those that are obtained by empirical estimations presented in section 5. Moreover, the distance between each pair of consecutive trading dates $t_{i+1} - t_i$, for all $i = 1, ..., N-1$, is equal to $\Delta t = \frac{1}{10}$. This value of Δt corresponds to the case that trading occurs ten times a year. Without regarding the empirical liquidity spreads yet, we find that the easy-to-apply option-theoretical approach provides us with proper values for the liquidity spread; $\gamma(T)$ is of a significant size and is not of an unexpected high magnitude. Next, we examine how $\gamma(T)$ is affected by a variation of time to maturity and volatility. Furthermore, we regard the aging-effect and discuss the impact of further parameters on $\gamma(T)$.

4.1 Impact of Time to Maturity

When a variation of the time to maturity T is regarded, the time Δt between each two consecutive trading dates is held constant. This implies that $T/\Delta t$ is an integer.

With an increase of time to maturity T for a fixed Δt, both option values divided by the liquid zero bond price, $\overline{S}(t,T)/P(t,T)$ and $\underline{S}(t,\alpha;t_1,...,t_N)/P(t,T)$, rise. However, the first ratio benefits from an increase of T more than the second one. This is due to the fact that an increase of T is associated with more opportunities when holding a lookback option with continuously monitoring of the underlying rather than a lookback option with discretely monitoring. Therefore, the ratio between the values of a liquid and an illiquid zero bond, given by $e^{\gamma(T)\cdot T} = \overline{S}(t,T)/\underline{S}(t,\gamma(T);t_1,...,t_N)$, increases with T.

In the next step, we regard the liquidity spread $\gamma(T)$ rather than the relation $e^{\gamma(T)\cdot T}$. At first glance, it is not clear whether the increase of $e^{\gamma(T)\cdot T}$ with T arises from a potential increase of $\gamma(T)$ or only from the increase of T. It turns out that the liquidity spread $\gamma(T)$ first rises with T and then declines, while $\gamma(T)\cdot T$ always increases with T. This behavior is illustrated by figure 1. Thus, for high values of T, the decrease of $\gamma(T)$ is less severe than the increase of T. In particular, $\gamma(T)$ approaches zero when T tends to zero or infinity. This property can be shown as follows. According to equation (10), $\gamma(T)$ equals $\frac{1}{T}\ln\left(\overline{S}(0,T)/\underline{S}(0,\gamma(T);t_1,...,t_N)\right)$. Since $\underline{S}(t,\alpha;t_1,...,t_N)$ always exceeds $P(t,T)$, the inequality $\gamma(T)\leq\frac{1}{T}\ln\left(\overline{S}(0,T)/P(0,T)\right)$ holds. The derived upper bound for the liquidity spread converges to zero when T tends to zero or infinity. As the the spread is always positive, it also approaches zero for $T\to 0$ or $T\to\infty$.

Summarizing up, $\gamma(T)$ first increases with T and then tends to zero.

4.2 Impact of Volatility

There are two ways to look at the volatility of zero bonds. First, the higher σ, the higher the uncertainty of the short rate r_t, and consequently, the higher the uncertainty of the liquid zero bond price $P(t,T)$. As seen in section 3, both option values, $\overline{S}(t,T)$ and $\underline{S}(t,\alpha;t_1,...,t_N)$, benefit from an increase of σ. But in case of $\overline{S}(t,T)$, the option value benefits from the increase of σ over the total time period $[t,T]$, while in case of $\underline{S}(t,\alpha;t_1,...,t_N)$ the increase is only in effect at the discrete trading dates $t_1,...,t_N$. Thus, $\overline{S}(t,T)$ rises more sharply with σ than $\underline{S}(t,\alpha;t_1,...,t_N)$ which leads to an increase of the liquidity spread $\gamma(T)$.

Figure 2: Liquidity Spread as a Function of σ

The figure shows the liquidity spread measured in bp as a function of σ. The distance between each pair of adjacent trading dates is a constant equal to Δt. The used parameter values are: $\kappa = 0.16643$, $T = 5$, $\Delta t = 0.1$.

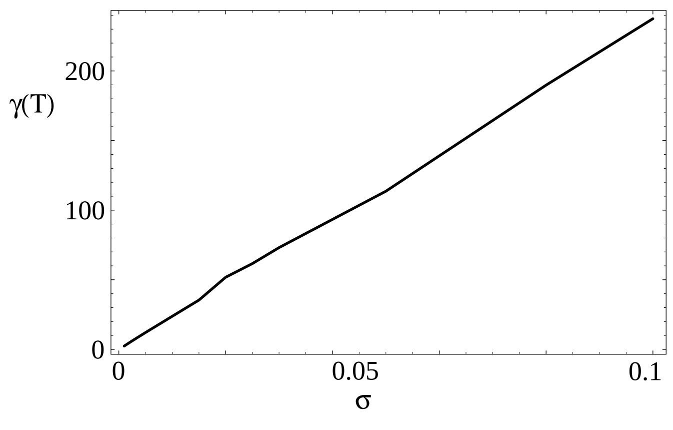


Figure 2 illustrates that the liquidity spread increases with σ. In this example, an increase of σ by 0.01 is related to an increase of $\gamma(5)$ of about 23 bp.

The second way to impact the volatility is to vary κ, the speed of reversion. The higher κ, the stronger is the mean-reversion effect that drives the short rate r_t towards $\theta(t)$; this decreases the volatility of r_t as well as the volatility of the zero bond price. Following the same argumentation as for σ, we also find that the liquidity spread increases when the volatility increases by a lower κ.

One can argue that volatility stems from the arrival of new information. According to this view, the higher the volatility, the higher the density of new information and/or the greater the importance of new information. In case of interest-rate certainty, i.e. $\sigma = 0$, the complete information about the succeeding short rates r_t are fully reflected by the current zero bond prices $P(t,T)$. Therefore, the ability to find the optimal date to sell the zero bond is worthless and $\gamma(T) = 0$.

Since the arrival of new information can be seen as an incentive to trade, the willingness of investors to trade becomes more severe with the density and the importance of new information; this results in a higher liquidity spread. In other words, when the volatility rises due to a higher density of the arrival of new and important information, the incentive to trade increases and, therefore, the liquidity spread is higher. This intuitive view concerning the effect of the volatility on the liquidity premium is reflected by the liquidity spread arising from our model.

4.3 Impact of the Aging-Effect

As our model is not restricted to the case of equidistant trading dates considered in the previous two subsections, we can also deal with arbitrarily distributed trading dates $t_1, ..., t_N$. It is found empirically that many bonds are traded more often at the beginning of their lifetime than at the end. Typically, bond trading volume decreases with age since more and more bonds are absorbed by buy-and-hold investors. In the following, we examine how this bond aging-effect impacts the liquidity spread. We compare the liquidity spread of equidistantly distributed trading dates as in subsection 4.1 with a different distribution of the same number of trading dates exhibiting the aging-effect property. As $\xi(t)$ is a strictly concave function of t, a distribution of the trading dates t_i, such that the differences $\xi(t_{i+1}) - \xi(t_i)$ are constant for all $i = 1, ..., N-1$, obviously exhibits this aging-property. Note that this particular distribution ensures that every difference between the discounted zero bond values for two consecutive trading dates $P(t_{i+1}, T) \cdot e^{-\int_0^{t_{i+1}} r_s ds} - P(t_i, T) \cdot e^{-\int_0^{t_i} r_s ds}$ has the same variance. According to figure 1, we get a liquidity spread equal to 27.2 bp for $T = 5$ and a constant $\Delta t = 0.1$. Using the same parameter values except for the fact that the trading dates t_i are distributed — as described above — according to the conditions $t_1 = 0$, $\xi(t_{i+1}) - \xi(t_i) = const$ for all $i = 1, ..., N-1 = 50$, and $t_N = T = 5$, we get a much lower liquidity spread of 15.4 bp.

Hence, this particular distribution exhibiting the aging-effect is more favorable to the value of the illiquid zero bond than the equidistant distribution of the same number of trading dates. This example suggests that the observable aging-effect distribution might result in lower liquidity spreads and higher bond values respectively.

The reason for this lower liquidity spread in case of the aging-effect-distribution is that the volatility of the zero bond $P(t,T)$ decreases when time to maturity shortens. According to Ito's lemma the instantaneous variance of the zero bond return is given by $B(t,T)^2 \cdot \sigma^2 = \left(1 - e^{-\kappa \cdot (T-t)}\right)^2 \cdot \kappa^{-2} \cdot \sigma^2$. Following subsection 4.2, the illiquid zero bond price benefits from the opportunity to trade as a response to new information. Taking the view that a higher volatility causes stronger trading incentives, we can see that the illiquid zero bond price might be higher, when most of the trading dates are at the beginning of the lifetime during a period with a high volatility and a high information density. Conversely, there is already an idea of a sort of aging-effect embedded in the property of our term structure model that the zero bond volatility is higher at the beginning of its lifetime, since high volatility is usually accompanied with high trading activity.

4.4 Impact of Other Parameters

Recall that the current interest rate level r_t and the mean-reversion $\theta(t)$ neither impact $\overline{S}(t,T)/P(t,T)$ nor $\underline{S}(t,\alpha;t_1,...,t_N)/P(t,T)$ as shown in section 3. Therefore, a change of r_t and $\theta(t)$ influences the price of the liquid and the illiquid zero bond but the liquidity spread $\gamma(T)$ is independent of both variables.

5 Design of the Empirical Study

An empirical analysis of our liquidity spread model requires the analysis of bonds which differ only in their liquidity. We describe the data set in section 5.1. Furthermore we estimate the model parameters in section 5.2.

5.1 The Data

A perfect test for a model of liquidity spreads should be performed with a data set containing bonds that are homogenous in all respects but liquidity. To achieve this goal we test our model with data of two segments of the German fixed income market.

The first segment is formed by German government securities. We take the price of government securities as the reference value of perfect liquid securities. Prices are calculated from the parameters of the Svensson (1994) term structure model. The used parameters are estimated and publicly provided by the Deutsche Bundesbank on a daily basis using all government securities.

For the illiquid sector we have to take bonds with lower liquidity that bear (nearly) the same risk. Therefore, we use the prices of €-denominated German Jumbo Pfandbriefe. Pfandbriefe are straight bonds that are backed by mortgages or loans to the public sector. They are subject to strict regulation statutes concerning the quality of the cover assets, the risk of the issuer's activities, and issuing limits to ensure the safety of Pfandbrief creditors. Furthermore, we include only Jumbo Pfandbriefe that are Aaa-rated by Moody's[13] and we exclude – in order to get a complete time series for every bond – all bonds that are issued or mature within the sample period. Because of the special safety requirements and our additional sample selection, the analyzed bonds bear only a small portion of credit risk that is not supposed to be a relevant determinant of their spread over government securities. We nevertheless control in our empirical analysis for effects of residual credit risk and the different capital requirements for the two security classes.[14]

Our sample period reaches from January 2000 to December 2001. We use weekly bond prices of a total of 37 Pfandbrief bonds issued by nine different institutions. We use the prices of the last official trade at the Frankfurt Stock Exchange on each Wednesday. The data is provided by Datastream. All bonds have been issued between June 1995 and October 1999. Their initial maturity is on average 7.63 years and varies between three and 15 years. The coupon ranges from 3% to 6% (4.62% on average).

5.2 Estimation of Model Parameters

Our model of liquidity spreads requires two groups of parameters. The first group of parameters describes the form and the dynamics of the term structure of interest rates. These parameters are the same for all bonds. The second group contains bond specific parameters that describe the illiquidity of a specific Jumbo Pfandbrief.

5.2.1 Estimation of the Term Structure Dynamics

Within the extended Vasicek term structure model both the shape and the evolution of the term structure are governed by the same parameter set. Therefore, we use an estimation methodology that uses simultaneously information about the time series and the cross-section of bond prices. The approach consists of a maximum likelihood estimation applied to a state space model with a latent factor. The methodology which is quite reminisce about the Kalman filter has been introduced by Chen and Scott (1993) and Pearson and Sun (1994).

The data sample for our econometric procedure consists of daily zero bond prices for maturities between three and ten years. The prices are derived from time series of Svensson

[13]Actually, the rating of Standard & Poors for all Jumbo Pfandbriefe is AAA given that they are rated at all.

[14]Rules by the Basle Committee for Banking Supervision oblige banks to maintain a capital reserve of 0,8 % of the book value for Pfandbriefe while German government securities are not subject to any capital requirements.

(1994) parameters for German government bonds. In order to reduce the standard error of our estimates, we use a higher data frequency (daily) for the parameter estimation than in the econometric study in section 6.

The basis for our econometric model is formed by the dynamics equation of the short rate under the empirical measure. We assume that the market price of risk behaves in a way such that the long-term mean drift θ^{real} under the empirical measure is constant. Thus, the short rate process is given by a standard Vasicek (1977) process

$$dr_t = \kappa \left(\theta^{real} - r_t\right) dt + \sigma dz_t' \tag{11}$$

with z_t' denoting a standard Wiener process under the empirical measure. A discretization of equation (11) shows that the short rate r_t is conditionally normally distributed given the initial value at $t - \Delta t$ with the following mean and variance:

$$r_t | r_{t-\Delta t}\,; \kappa, \sigma, \theta^{real} \sim N\left(e^{-\kappa\Delta t} r_{t-\Delta t} + \theta^{real}\left(1 - e^{-\kappa\Delta t}\right), \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa\Delta t}\right)\right)$$

The corresponding transition density is denoted by $\varphi(r_t | r_{t-\Delta t}\,; \kappa, \sigma, \theta^{real})$.

Our econometric model for each bond price with maturity $T = 3, 4, 5, ..., 10$ (respectively the observation equation) takes the following form:

$$\ln P(t,T) = A(t,T) - B(t,T) \cdot r_t + \tilde{\epsilon}_t^T \tag{12}$$

The formulation in terms of the logarithm of the price yields a linear model. The observation errors $\tilde{\epsilon}_t^T$ are assumed to be normally distributed and mutually independent. The state variable r_t itself is unobservable. We estimate its trajectory together with the parameters κ, σ, θ, θ^{real} and the error variances from the data. Therefore, we assume that the price of the bond with a maturity of $\bar{T} = 5$ years is measured without error. This yields an analytical expression for the latent r_t by rearranging equation (12). In principle, we could also apply filtering methods in the spirit of the Kalman filter to estimate the latent short rate. As noted by Duffie and Singleton (1997) the approach taken here has among other things the advantage that a subset of bonds is fitted without error. This is a useful property especially for pricing purposes. Assuming further a constant standard deviation σ_B of the measurement errors for the other bonds, we can derive the density $f(t)$ of the observed log-prices in t conditional on the value of the state variable $r_{t-\Delta t}$ at the previous date.

$$f(t) = \varphi(r_t | r_{t-\Delta t}\,; \kappa, \sigma, \theta^{real}) \frac{1}{B(t,\bar{T})} \prod_{T=3}^{10} \frac{1}{\sqrt{2\pi\sigma_B^2}} \exp\left(-\frac{\left(\epsilon_t^T\right)^2}{2\sigma_B^2}\right)$$

The parameters are estimated by maximizing the log-likelihood function for the complete sample period from 2000 to 2001:

$$\max_{\kappa,\ \sigma,\ \theta,\ \theta^{real},\ \sigma_B} \sum_{t=2000/01/04}^{2001/12/28} \log f(t)$$

The estimation of the parameters is repeated several times using different starting values. Further estimations using data only from the year 2000 or only from the year 2001 yield nearly the same results which speaks in favor of the assumed Vasicek term structure model. The results are reported in table 1.

Table 1: Estimates of the Term Structure Parameters for 2000 – 2001

κ	σ	θ	θ^{real}	σ_B
0.16643	0.01097	0.06390	0.01748	0.00907

5.2.2 Determination of the Illiquidity Parameters

The illiquidity parameters specify the possible trading dates during the lifetime of each Jumbo Pfandbrief. Under the assumption of equidistant trading dates the distance Δt between two consecutive trading dates suffices for a complete characterization. Otherwise, further parameters are needed.

It is hard to observe or estimate Δt directly. In principle, Δt can be derived by analyzing trading volume data, turnover frequency or other related measures. Unfortunately, this is not possible because of the market structure. Detailed transaction data is only available for trades on the stock exchange. Since most Pfandbrief bonds are traded over-the-counter (OTC) by banks or institutional investors, the stock exchange volume data is heavily biased.[15]

Therefore, we determine the bond specific illiquidity parameters implicitly by inverting the pricing relation. The exact procedure is described in the context of the empirical analysis in section 6.

6 Empirical Results

In our theoretical model, we focus on liquidity spreads of zero bonds. In order to perform an empirical analysis of liquidity spreads in the German Jumbo Pfandbrief market, we have to determine appropriate spreads for coupon bonds. In principle, our valuation method for zero bonds could directly be applied to coupon bonds. However, the intermediate coupon payments would increase the complexity of the necessary calculations.[16] To

[15] Though most of the trading volume is on the OTC market, the prices on the stock exchange are likely to reflect fair equilibrium prices. For each bond, at least three banks act as market makers. They are obliged to submit bid and offer quotes without exceeding a specified spread. The maximal spread ranges from four cents up to 20 cents depending on the maturity.

[16] In principle, it is straightforward to apply numerical methods to determine the required values of continuously and discretely monitored lookback options for the case of coupon bonds. However, it is no longer possible to obtain a closed-form solution for the continuously monitored lookback option. This is

keep things simple and applicable, we treat coupon bonds as if they were zero bonds that mature at the time of the duration of the bond. This procedure ensures that we consider a zero bond that shows the same sensitivity towards a parallel shift of the interest rate curve as the corresponding coupon bond. Since the coupon payments in our sample are rather low relative to the face value, the duration is not far below the time to maturity and the coupon payments are not supposed to play an essential role.

Thus, the theoretically predicted liquidity spread of a Jumbo Pfandbrief j is equal to $\gamma(D_j)$, where D_j denotes the Macaulay duration of bond j, $j = 1, ..., J = 37$. Accordingly, the empirically observed liquidity spread $\Gamma_j(t)$ of a Jumbo Pfandbrief j is given by:

$$\Gamma_j(t) = -\frac{1}{D_j} \ln \left(\frac{P_j(t) + \text{accrued interest}}{PV(j)} \right)$$

$P_j(t)$ denotes the observed price of Jumbo j. The term $PV(j)$ is equal to the present value of the outstanding payments of bond j discounted with the current term structure for government bonds. I.e., $PV(j)$ is the value of a bond which is liquid and otherwise identical to j.

Before we analyze the empirical quality of our model, we first regard some statistical properties of the observed liquidity spreads in the Jumbo Pfandbrief market. Next, we test some qualitative results of our model. Finally, we calibrate our model to the data and analyze its quantitative predictions.

6.1 Statistical Properties of the Liquidity Spread

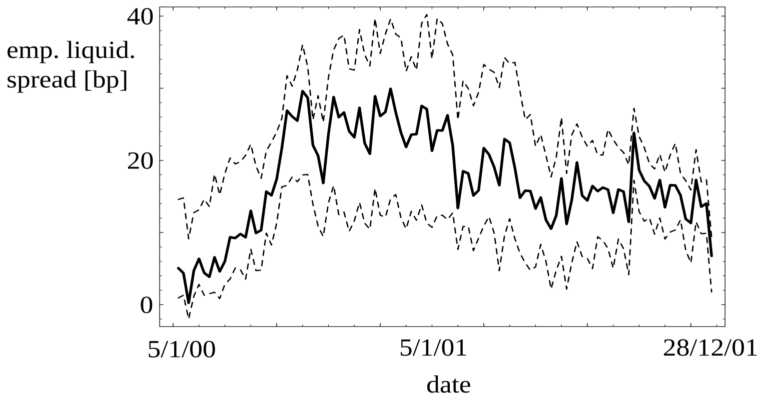
The analyzed data sample contains in total 3848 observations of the liquidity spread. Most of the liquidity spreads are in the range of 0 up to 30 basis points. The average spread is 17.1 bp and the standard deviation amounts to 11.6 bp. It turns out that 4.7% of the spreads are negative. A closer look on the data reveals that mainly bonds with a very short time to maturity show a negative spread, even though there is no theoretical justification for this behavior. Therefore, we think that these effects are not driven by liquidity. A possible reason for the deviation consists of the estimation errors for the fitted parameters of the term structure since there are only few short term government bonds available. Other reasons include nonsynchronous prices, demand shocks or simply noise in the data.

Not surprisingly, the liquidity spreads of the different bonds show a similar evolution through time. Figure 3 graphs the median, the 20%-quantile and the 80%-quantile of the spreads over time.

because the price of a coupon bond as a sum of zero bond prices with each being lognormally distributed within the Vasicek framework is in general not lognormally distributed. A numerical method for valuing a continuous lookback option is rather costly, since there is a large number of discrete time steps required to approximate the distribution of the relevant maximum with a sufficient precision.

Figure 3: Evolution of the Empirical Liquidity Spreads

The figure graphs the empirical liquidity spreads measured in bp over the observation period from 5/1/00 until 28/12/01. The solid line indicates the median, while the dashed lines stand for the 20 % and the 80 % quantile respectively.



As figure 3 shows, the cross-sectional median of the liquidity spreads evolves between 0.3 and 29.9 bp. In addition, most of the spreads are close to the (time-varying) average spread. This is also indicated by the average inter-spread correlation of 0.43.

6.2 Qualitative Behavior of Liquidity Spreads

As shown in section 4, the theoretical liquidity spread is influenced by parameters of the term structure dynamics as well as by bond-specific parameters.

Since we determine the illiquidity parameter Δt implicitly we consequently cannot measure its influence on spreads. Therefore, we restrict our analysis on the influence of the duration of a bond (resp. time to maturity in our theoretical model) and the influence of the short rate r. First, we focus on the impact of time to maturity T. For each observation date, we perform a cross-sectional regression of liquidity spreads (measured in bp). Besides the duration D_i, we include several other variables that are likely to influence the spread: the time since issuance age_i measured in days, the amount issued $issue\_size_i$ (measured in million €) and the coupon size $coupon_i$ (in per cent) of each Jumbo bond i. Formally, the estimation equation is given by:

$$\Gamma_i = \beta_0 + \beta_1 \cdot D_i + \beta_2 \cdot \mathrm{age}_i + \beta_3 \cdot \mathrm{issue\_ size}_i + \beta_4 \cdot \mathrm{coupon}_i + \epsilon_i \qquad (13)$$

The amount issued and the age of a bond are commonly used proxies for liquidity. Also, the coupon size might have an impact due to tax effects. The results of the regressions are reported in table 2.

In our theoretical model, the liquidity spread is a humped-shaped function of T. Table 2

Table 2: Results of the Cross-Sectional Regressions

Parameter	β_0	β_1	β_2	β_3	β_4
Average parameter value (all 104 regressions)	-7.5687	3.7318	-0.0004	-0.0002	1.8268
No. of regressions, where parameter is significant*	58	92	4	8	18
Average parameter value when significant*	-11.5775	4.2015	-0.0036	-0.0012	2.9850

* significant at the 5% level

indicates that empirically, duration resp. time to maturity has in most cases a significant positive influence on the liquidity spread.[17] The 12 insignificant estimates are all located in the last 16 weeks of 2001 which may indicate a change of this dependence. Note that the liquidity proxies age and issue size have almost no significant impact on the spread. Furthermore, the results indicate that there is no notable effect of coupon size.

Second, we explore the influence of the level of the short rate r on the liquidity premium. Our theoretical model predicts no impact of the short rate. In order to check this claim, we run regressions of the time series of liquidity spread changes for each Jumbo Pfandbrief. We proxy the unobserved short rate by the interest rate for $t = \frac{1}{10}$ calculated with the current Svensson (1994) parameters. In order to control for the influence of the time to maturity T as observed in the cross-sectional analysis, we include the duration and its first differences as exogenous variables. The regression equation for each bond reads as

$$\Delta\Gamma_t = \beta_0 + \beta_1 \cdot \Delta r_t + \beta_2 \cdot D_t + \beta_3 \cdot \Delta D_t + \epsilon_t, \tag{14}$$

where Δx_t denotes the change of variable x from week t to week $t+1$. As one can see in equation (14), we allow for a more general dependence of the spread on the maturity T; we consider both a proportional and an integrated relation. The variances of the estimates are adjusted for first-order autocorrelation of the residuals. In order to improve the readability of the results, we measure the short rate as well as the liquidity spread in basis points. Table 3 summarizes our results.

Table 3 shows that there is — in accordance to our theoretical model — barely no support for a dependence of the liquidity spread on the short term rate. Furthermore, the relation between liquidity spreads and duration is weaker in the time series than in the cross section. A possible explanation is the existence of a systematic time-varying influence on all spreads due to credit risk. We pursue this issue further in the next subsection.

[17] All significant estimates of β_1 are positive.

Table 3: Results of the Time Series Regressions

Parameter	β_0	β_1	β_2	β_3
Average parameter value (all 37 regressions)	-1.7965	-0.0038	0.3283	-1.6736
No. of regressions, where parameter is significant*	8	5	10	1
Average parameter value when significant*	-7.4468	-0.0459	1.4852	-41.3482

* significant at the 5% level

6.3 Results of a Practical Implementation

In our theoretical model, perfect market timing ability is the motive for trading and determines the price of the option to trade, i.e. liquidity. As we have argued in section 2, this specification is likely to overestimate the trading wishes of real investors. Consequently, this will result in too high liquidity spreads.

In the practical implementation of our model, we account for this possible misspecification. The theoretically obtained liquidity spread $\gamma(T)$ measures the opportunity costs of an investor with perfect foresight. If the investor has less information, her desire to trade will be lower.[18] Consequently, the corresponding opportunity costs of non-trading together with the liquidity spread will decrease. We model this decrease by introducing a correction factor c^* such that the liquidity spread of a zero bond maturing at T is given by

$$a + c^* \cdot \gamma(T). \tag{15}$$

The information correction factor c^* can be interpreted as a measure of the ratio between the realizable gains from trading with the actual information of investors and the potential gains with perfect information. This ratio is not a bond specific property but a property of the investors in the fixed income market. Therefore, we assume c^* to be constant for all bonds. Furthermore, we add another bond independent parameter a to our empirical model. This parameter allows us to control for other systematic effects on the spreads that are not driven by liquidity. In particular, these effects include a credit risk premium for Aaa-rated bonds, different capital requirements, and errors in the estimation of the term structure.

Moreover, an implementation of our model requires the knowledge of the distribution of the trading dates for each illiquid bond. For our empirical analysis, we assume that the trading dates of every illiquid bond are distributed equidistantly over time. As already mentioned in section 5, estimates of the distance between two consecutive trading dates

[18]Since the role of information in our model is to create trading wishes, it is irrelevant whether the investor actually has superior information or just believes to have such.

Δt are hard to determine directly. Therefore, we estimate Δt implicitly out of time series of bond prices.

Although implicit parameters are formally obtained by inverting the pricing relation, they bear an advantage over simply regarding the spreads. Implicit parameters make spreads obtained for different bonds at different dates comparable since the effects of other parameters that are relevant for pricing are ruled out. Thus, implicit parameters allow to get an intuition about the 'pure liquidity' of a bond.

The precise parameter values are obtained by a nonlinear least square regression of the observed spreads $\Gamma_i(t)$ over the theoretical spreads from equation (15). Formally, we calculate:

$$\min_{a,\ c^*,\ \Delta t_1,\ldots,\Delta t_J} \sum_{t=1}^{\#obs} \sum_{j=1}^{J} \left(\Gamma_j(t) - (a + c^* \cdot \gamma_{j,t}(T)\right)^2$$

The minimum has to be determined numerically, since there is no analytical formula for $\gamma(T)$. The results for the full sample of bonds $J = 37$ over the total observation period 2000 to 2001 ($\#obs = 104$) are reported in table 4.

Table 4: Parameter Values from the Implicit Estimation

Parameter	Value
a [bp]	4.756
c^*	0.598
Average Δt [days]	31.66
Median Δt [days]	13.89
Standard deviation Δt [days]	37.62
Minimal Δt [days]	0.008
Maximal Δt [days]	110.07

Table 4 shows, that approximately half of the potential gains from perfect information can be realized by the actual investors in the Pfandbrief market. This confirms our conjecture that the trading motives are exaggerated by the assumption of perfect foresight. Around 50 % of the bonds are traded at least twice a month. Of course, Δt cannot be interpreted literally. Our intuition about Δt is rather such that e.g. a Δt of 10 trading days means that a position of reasonable size can be built up or liquidated within 10 days without a substantial impact on the price.

Implicit parameters like a, c^* and Δt_i can be reasonably employed if they show a stable or at least a predictable behavior. In this case it is possible to build up an intuition for the values of the parameters.[19] Therefore, we perform several stability test of the obtained parameters within the next two sections.

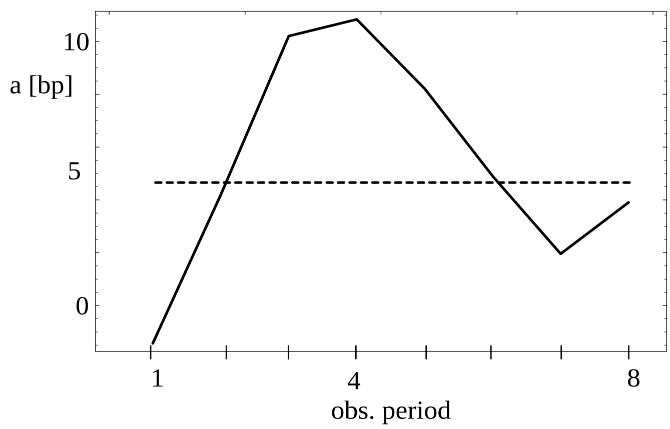
[19] In particular, we think of the case of the implicit volatility in option pricing. Though implicit volatilities show e.g. a smile effect, practitioners are able to develop a reasonable intuition for it.

6.3.1 Parameter Stability over Time

First, we analyze the behavior of the implicit parameters over time. We divide the observation period of two years into 8 subperiods with the length of three months. Then we redo the nonlinear estimation procedure for each period. Figure 4 compares the estimates for parameter a in the subperiods with those for the whole period. It turns out that the

Figure 4: Estimation Results for a

The figure shows the estimates of a for the eight observation subperiods. The dashed line indicates the estimates obtained when using the whole sample period from 2000 to 2001.



estimates of a lie between -1.4 and 10.8 bp. The magnitude of a suggests that the corresponding changes are mainly caused by a time-varying credit risk premium for Aaa-rated bonds. This component may also be the reason for the observed weaker influence of the time to maturity T in the time series regressions of the liquidity spreads.

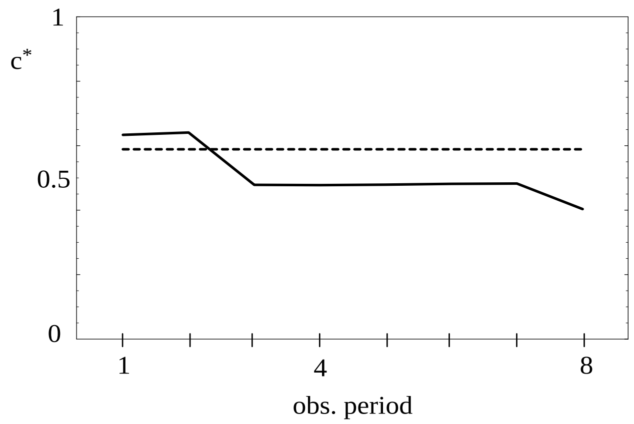
The estimations of the information correction factor c^* yield quite stable results. As figure 5 shows, the estimates do only vary little over time and are close to the values obtained in the analysis of the whole sample period. This provides evidence for the fact that the correction factor c^* which values the foresight opportunity is rather time-invariant.

An analysis of the stability of the Δt_i supports our model to a certain amount. The results for bonds with a long time to maturity are rather encouraging. E.g., figure 6 graphs the Δt in each subperiod obtained for a 5.5 % - bond of the Depfa bank which matures in January 2013. This is the bond with the longest time to maturity within the sample. Its values of Δt lie between 80 and 114 trading days and are rather constant over time.

The corresponding results for all bonds are summarized in figure 7, where the differences between the estimated Δt of each subperiod and the Δt of the whole sample period are graphed. Though there are in each subperiod a significant proportion of spreads that do not change much, there is also a considerable quantity of bonds with very pronounced changes in Δt. Additionally, the probability of a positive difference rises over time, which

Figure 5: Estimation Results for c^*

The figure shows the estimates of c^* for the eight observation subperiods. The dashed line indicates the estimates obtained when using the whole sample period from 2000 to 2001.



might indicate a systematic effect.

6.3.2 Parameter Stability over Subsamples

Second, we conduct a similar examination of the parameters when we change the analyzed set of bonds. Therefore, we divide up our data into two samples of 18 bonds (one bond was skipped). To get comparable samples, we arranged the bonds by maturity date. We then formed one sample out of the bonds with an even number and the other sample out of the bonds with an odd number. Again, we perform implicit parameter estimations.

The results indicate that the parameters a and c^* are stable over the two subsamples. The parameter values are reported in table 5.

Table 5: Stability of a and c^* over Subsamples

Parameter	Full sample (37 bonds)	Subsample 1 (18 bonds)	Subsample 2 (18 bonds)
a [bp]	4.756	5.088	5.366
c^*	0.598	0.597	0.488

As in the previous section, the estimates for the Δt_i change more pronounced than those of a and c^*. As table 6 shows, the Δt-estimate for only about one third of all bonds change less than one week. The other parameter values vary more than one week which is a substantial change since the median of the Δt is about 2.5 weeks.

Figure 6: Estimates of Δt of the 5.5 % Depfa Bank Bond maturing in 2013

The estimated Δt for the eight subperiods are plotted for the 5.5 % Depfa bank Jumbo Pfandbrief maturing in 2013. The dashed line indicates the corresponding value obtained with the whole data set.

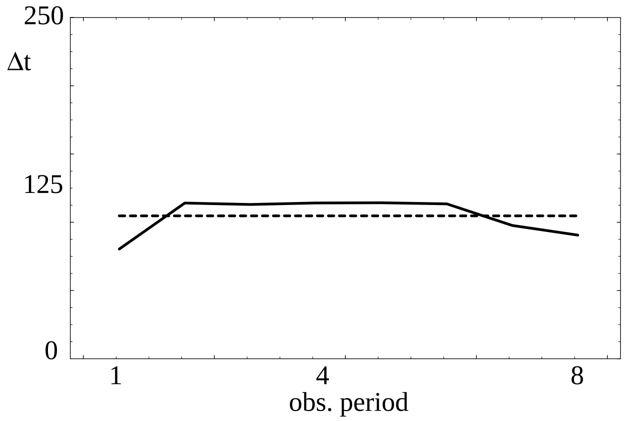


Figure 7: Differences of the Estimates of Δt

For each subperiod and bond, figure 7 graphs the difference between the estimated values of Δt obtained from a subperiod and the whole data set respectively.

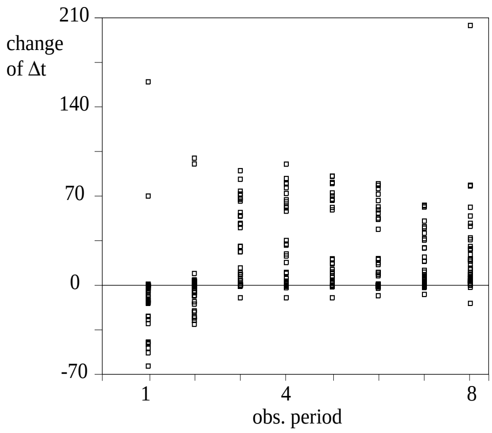


Table 6: Stability of Δt over Subsamples

The table shows the magnitude of the variations of Δt when the data set used for estimation is switched from the full sample to a subsample containing half of the bonds.

Magnitude of Change	< 1 day	< 1 week	< 4 weeks	> 4 weeks
Number of bonds	4	11	11	10

6.3.3 An Assessment

The stability analysis shows that the information correction factor c^* is rather stable. We interpret this property as positive evidence for the applicability of our model. The observed variations of a suggest the existence of a systematic time-varying factor in the liquidity spreads that probably stems from credit risk.

Obviously, the parameter Δt substantially varies for some bonds. This indicates that our assumption of an equidistant distribution of the trading dates over time cannot be sustained for all bonds. Since our model is flexible enough to incorporate arbitrary distributions of trading dates, it is in principle possible to cope with the observed time-varying Δt within our framework. There is, however, a limitation because of the implicit estimation used in our approach. The introduction of further parameters increases the dimensionality and makes it more difficult or — depending on the number of new parameters — even infeasible to identify the parameters.

7 Conclusion

Liquidity is a major determinant of bond prices. In this paper, we derive an easy-to-apply option-theoretical approach to calculating price discounts for illiquid bonds. We take the view that liquid assets offer a valuable option to trade. This interpretation of liquidity together with the fiction of an investor with perfect foresight allow us to represent the benefits of liquidity in the spirit of Longstaff (1995) as the payoff of certain financial options, namely lookback options.

We extend the work of Longstaff in two directions. First, since we want to focus on illiquid bonds rather than on equity, we introduce interest rate uncertainty. Second, we allow for different degrees of liquidity. In contrast to Longstaff who considers only one single non-trading period, our model is flexible enough to deal with multiple non-trading periods which can be arbitrarily distributed over time. It turns out that the liquidity spread in our case is determined by the ratio of the price of a lookback option written on a zero bond with continuously monitoring of the underlying and the price of a corresponding discrete lookback option.

A theoretical analysis shows that the behavior of the liquidity spreads towards the variation of other economic variables has plausible properties. Liquidity spreads are humped-shaped functions of maturity. As one expects, they increase with bond volatility. Furthermore, the liquidity spreads are not only influenced by the number of trading dates but also by their distribution over time. Another essential feature of the model is the invariance of the liquidity spreads to the level of the short rate.

To assess the empirical quality of our model, we consider data of German Jumbo Pfandbrief bonds. We are able to reveal several parallels between the theoretical predictions of our model and the behavior of the regarded bond data. Since reliable volume data is not available for these bonds, the main obstacle to applying our model is to determine the precise dates, when it is possible to trade an illiquid bond. We solve this problem through an implicit procedure. An analysis of the stability of the implied parameters give rise to some evidence for the empirical relevance of our model. Interestingly, our analysis shows that the actual liquidity spreads are about half as high as they would be if investors have perfect market timing ability — or if investors think that they have this ability. In addition, they are interfered with a further time-varying factor which probably stems from credit risk.

This paper provides several opportunities for ongoing research.

First, it is interesting to perform more direct tests of our model. If data about the trading volume of illiquid bonds is available and allows to draw conclusions about the trading dates, it is possible to directly compare theoretical and observed spreads.

Second, we demonstrate how to use a mapping $\xi : t \to t$ to transform the motion of zero bond prices plus earned interest into a geometric Brownian motion with constant coefficients. This approach allows to determine easily the prices of various other exotic options on zero bonds within an extended Vasicek framework.

Third, our model makes several simplifying assumptions to derive an easy-to-apply method to compute liquidity spreads. Therefore, a comparison of the outcomes of our simple approach with those of a fully-fledged equilibrium model will lead to further insights in the nature of liquidity spreads.

A Derivation of Equation (7)

Applying Ito's lemma, we can write for the process of $M_t := P(t,T) \cdot e^{-\int_0^t r_s ds}$:

$$\begin{aligned}
dM_t &= \frac{\partial M_t}{\partial r_t} dr_t + \frac{\partial M_t}{\partial t} dt + \frac{1}{2}\frac{\partial^2 M_t}{\partial r_t^2} dr_t^2 \\
&= -B(t,T) M_t \cdot dr_t + \left(\frac{\partial A(t,T)}{\partial t} - r_t \cdot \frac{\partial B(t,T)}{\partial t} - r_t \right) \cdot M_t \cdot dt \\
&+ \frac{1}{2} B(t,T)^2 M_t \sigma^2 \cdot dt \\
&= 0 \cdot dt - B(t,T) \cdot \sigma \cdot M_t \cdot dz_t
\end{aligned}$$

Accordingly, the process of $d\ln M_t$ reads:

$$d\ln M_t = -\frac{1}{2}B(t,T)^2\sigma^2\cdot dt - B(t,T)\cdot\sigma\cdot dz_t \tag{16}$$

This leads us to a representation of $\ln M_\tau$ as follows:

$$\ln M_\tau = \ln M_0 - \int_0^\tau \frac{1}{2}B(s,T)^2\sigma^2\cdot ds - \int_0^\tau B(s,T)\sigma\cdot dz_s$$

At this point, it is convenient to make the following definition:

$$\begin{aligned}\xi(\tau) &:= \int_0^\tau B(s,T)^2\sigma^2\cdot ds \\ &= \sigma^2\frac{4e^{-T\kappa}\left(1-e^{\kappa\tau}\right)-e^{-2T\kappa}\left(1-e^{2\kappa\tau}\right)+2\tau\kappa}{2\kappa^3}\end{aligned}$$

This allows us to represent $\ln M_\tau$ as follows:

$$\ln M_\tau = \ln M_0 - \frac{1}{2}\xi(\tau) - \int_0^\tau B(s,T)\sigma\cdot dz_s$$

As the variance of $\ln M_\tau$ is equal to

$$\int_0^\tau (B(s,T)\sigma)^2\cdot ds = \xi(\tau),$$

we can rewrite $\ln M_\tau$ by using a second standard Wiener-process z'_t at time $t=\xi(\tau)$:

$$\ln M_\tau = \ln M_0 - \frac{1}{2}\xi(\tau) + z'_{\xi(\tau)} \tag{17}$$

As $\xi(t)$ strictly increases from a value of 0 at $t=0$ to $\xi(\tau)$ at $t=\tau$, each value of $\ln M_0 - \frac{1}{2}\xi(t) + z_{\xi(t)}$ for $t\in[0,\tau]$ is uniquely related to a value of $\ln M_0 - \frac{1}{2}t + z_t$, where $t\in[0,\xi(\tau)]$. Hence, we can interpret $\ln M_\tau$ as the result from an arithmetic Brownian motion at time $\xi(\tau)$ with drift $-\frac{1}{2}$ and unit variance. This allows us to deal with the less complex process $\ln M_0 - \frac{1}{2}t + z_t$ from $t=0$ to $t=\xi(\tau)$ in order to examine the path of $\ln M_t$ from 0 to τ. Following Harrison (1985), we can determine the density function of $y_T := \ln\left(\max_{0\le s\le\xi(T)}\{M_s\}\right) - \ln(M_0)$ as follows:

$$f(y_T) = \frac{1}{\sqrt{\xi(T)}}\Phi'\left(\frac{y_T+\frac{1}{2}\xi(T)}{\sqrt{\xi(T)}}\right) + e^{-y_T}\cdot\Phi\left(-\frac{y_T-\frac{1}{2}\xi(T)}{\sqrt{\xi(T)}}\right) + e^{-y_T}\cdot\Phi'\left(\frac{y_T-\frac{1}{2}\xi(T)}{\sqrt{\xi(T)}}\right)$$

Therewith, we can evaluate the integral $\int_0^\infty e^{y_T+\ln P(0,T)}\cdot f(y_T)\,dy_T$ and we obtain:

$$\begin{aligned}&\int_0^\infty e^{y_T+\ln P(0,T)}\cdot f(y_T)\,dy_T \\ =\;& P(0,T)\cdot\left(2\cdot\Phi\left(\frac{1}{2}\sqrt{\xi(T)}\right) + e^{-\frac{\xi(T)}{8}}\sqrt{\frac{\xi(T)}{2\pi}} + \frac{\xi(T)}{2}\Phi\left(\frac{1}{2}\sqrt{\xi(T)}\right)\right)\end{aligned}$$

B Derivation of Equation (9)

In accordance with equation (17), we can represent $\ln M'_\tau$, defined by the increment $d \ln M'_t := d \ln \left(P(t,T) \cdot e^{-\alpha \cdot (T-t) - \int_0^t r_s ds} \right)$, as follows:

$$\ln M'_\tau = \ln M'_0 + \alpha \cdot t - \frac{1}{2} \xi(\tau) + z'_{\xi(\tau)}$$

This definition enables us to write for the expectation in equation (8):

$$\begin{aligned} &\mathbb{E}_0 \left(\max_{i=1,\ldots,N} \left\{ P(t_i,T) \cdot e^{-\alpha \cdot (T-t_i) - \int_0^{t_i} r_s ds} \right\} \right) \\ &= \mathbb{E}_0 \left(\max_{i=1,\ldots,N} \left\{ M'_{t_i} \right\} \right) \end{aligned}$$

The expectation of the maximum value which is monitored discretely results in

$$\begin{aligned} &\mathbb{E}_0 \left(\max_{i=1,\ldots,N} \left\{ M'_{t_i} \right\} \right) \\ &= \sum_{i=1}^N \mathbb{E}_0 \left(M'_{t_i} \cdot 1_{\left\{ M'_{t_i} > M'_{t_j},\ i \neq j,\ j=1,\ldots,N \right\}} \right), \end{aligned} \tag{18}$$

where $1_{\{.\}}$ denotes the indicator function. We can rewrite the expectation as

$$\begin{aligned} &\mathbb{E}_0 \left(M'_{t_i} \cdot 1_{\left\{ M'_{t_i} > M'_{t_j},\ i \neq j,\ j=1,\ldots,N \right\}} \right) \\ &= M'_0 \cdot e^{\alpha \cdot t_i} \cdot \widetilde{\mathbb{E}}_0 \left(1_{\left\{ M'_{t_i} > M'_{t_j},\ 1 \leq j < i,\ j=1,\ldots,N \right\}} \right) \cdot \mathbb{E}_{t_i} \left(1_{\left\{ M'_{t_i} > M'_{t_j},\ i < j \leq N,\ j=1,\ldots,N \right\}} \right), \end{aligned}$$

where $\widetilde{\mathbb{E}}_0(\cdot)$ stands for the expectation under an equivalent probability measure $\widetilde{\mathbb{P}}$. According to this measure $\widetilde{\mathbb{P}}$, $\ln M'_\tau$ can be represented by $\ln M'_0 + \alpha \cdot t + \frac{1}{2}\xi(\tau) + \widetilde{z}_{\xi(\tau)}$ if $\widetilde{z}_t$ is a standard Wiener-process under $\widetilde{\mathbb{P}}$. Heynen and Kat (1995) also make use of this equation in order to price discrete lookback options. The intuition for the idea behind this change of measure can be found in Johnson (1987). The two expectations $\mathbb{E}_0 \left(1_{\left\{ M'_{t_i} > M'_{t_j} 1 \leq j < i, j=1,\ldots,N \right\}} \right)$ and $\mathbb{E}_{t_i} \left(1_{\left\{ M'_{t_i} > M'_{t_j} i < j \leq N, j=1,\ldots,N \right\}} \right)$ can be represented by the values of multivariate cumulative standard normal distribution functions. Plugging them in equation (18) yields formula (9).

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