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MULTIPLICATIVE BACKGROUND RISK *

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Abstract

We consider random wealth of the multiplicative form $\tilde{x}\tilde{y}$, where $\tilde{x}$ and $\tilde{y}$ are statistically independent random variables. We assume that $\tilde{x}$ is endogenous to the economic agent, but that $\tilde{y}$ is an exogenous and uninsurable background risk. Our main focus is on how the randomness of $\tilde{y}$ affects risk-taking behavior for decisions on the choice of $\tilde{x}$. We characterize conditions on preferences that lead to more cautious behavior. We also develop the concept of the affiliated utility function, which we define as the composition of the underlying utility function and the exponential function. This allows us to adapt several results for additive background risk to the multiplicative case.

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1. Introduction

Consider a risk-averse economic agent whose preferences can be represented within an expected-utility framework via the continuously differentiable utility function u. The agent must decide upon choice parameters for a random variable representing final wealth, $\tilde{x}$. For example, $\tilde{x}$ might represent wealth from an individual's portfolio of financial assets, or $\tilde{x}$ might represent random corporate profits based on management decisions within the firm.

A fair amount of attention in recent years has examined how decisions on $\tilde{x}$ might be affected by the addition of an additive risk $\tilde{\varepsilon}$, where $\tilde{\varepsilon}$ and $\tilde{x}$ are statistically independent. Thus, final wealth or profits can be written as $\tilde{x}+\tilde{\varepsilon}$. We assume that $\tilde{\varepsilon}$ is not directly insurable. For example, $\tilde{\varepsilon}$ might represent future wage income subject to human-capital risks; or $\tilde{\varepsilon}$ might represent an exogenous pension portfolio provided by one's employer. Although it is interesting to examine the interdependence between $\tilde{x}$ and $\tilde{\varepsilon}$, the case of independence is of special interest and provides for many interesting observations. In order to focus on the risk effects, rather than wealth effects, it is often assumed that $E\tilde{\varepsilon}=0$, where E denotes the expectation operator. In such a case, $\tilde{\varepsilon}$ is often called a "background risk." Since any non-zero mean for $\tilde{\varepsilon}$ can be added to the $\tilde{x}$ term, this assumption does not reduce the applicability of the model. Our purpose in the present paper is to examine the effects of introducing a "multiplicative background risk" into the individual's final wealth distribution.

The modern literature on additive background risk stems from the papers of Kihlstrom, et al. (1981), Ross (1981) and Nachman (1982). Doherty and Schlesinger

(1983) incorporated their analysis into models of decision making under uncertainty, which underwent somewhat of a renaissance in the 1990's thanks to new theoretical tools provided by Pratt and Zeckhauser (1987), Kimball (1990) and Gollier and Pratt (1996).

One canonical hypothesis concerning additive background risk is that the riskiness of $\tilde{\varepsilon}$ leads to a more cautious behavior towards decisions on $\tilde{x}$, such as in Guiso, et al. (1996). However, this conclusion need not always be the case, unless particular restrictions on preferences are met. Eeckhoudt and Kimball (1992) first examined this direction of research. Rather than review the large body literature for the case of additive background risks, we refer the reader to the excellent comprehensive presentation of this material in Gollier (2001).

Surprisingly, very little attention has been given to the case where the background risk is multiplicative. Our goal in this paper is to provide a theoretical foundation for models with a multiplicative background risk. Under what conditions on preferences will the addition of a multiplicative background risk compel the agent to behave more cautiously in making decisions about the endogenous wealth variable $\tilde{x}$?

To this end, let $\tilde{y}$ be a random variable on a nonnegative support that is statistically independent of $\tilde{x}$. We consider final wealth to be given by the product $\tilde{x}\tilde{y}$. The random variable $\tilde{y}$ is considered to be exogenous to the individual and is not directly insurable. Numerous examples of such multiplicative risks include the following:

1. Let $\tilde{x}$ be the pre-tax profits of a firm and let $\tilde{y}$ represent the firm's retention net of taxes, where tax rates are random due to tax-legislation uncertainty.

2. Let $\tilde{x}$ be the random wealth in an individual's financial portfolio in period one, and let $\tilde{y}$ denote a mandatory annuity account that uses proceeds from $\tilde{x}$ in period two.

3. Let $\tilde{x}$ denote nominal wealth or profit and let $\tilde{y}$ denote an end-of-period price deflator.

4. Let $\tilde{x}$ denote profit in a foreign currency and let $\tilde{y}$ denote the end-of-period exchange rate.

5. Let $\tilde{x}$ denote the random per-unit profit for a farm commodity, based on some hedging strategy, and let $\tilde{y}$ denote an exogenous random quantity of output.

In order to isolate the risk effects of $\tilde{y}$, we will assume that $E\tilde{y}=1$. For the case where $\tilde{y}$ has a mean that differs from one, we can incorporate this mean into $\tilde{x}$ via a deterministic scaling effect.[1] Since $\tilde{x}\tilde{y} = \tilde{x} + \tilde{x}(\tilde{y}-1)$, the assumption that $E\tilde{y}=1$, together with the independence of $\tilde{x}$ and $\tilde{y}$, guarantees that $\tilde{x}\tilde{y}$ is riskier than $\tilde{x}$ alone in the sense of Rothschild and Stiglitz (1970). We will refer to $\tilde{y}$, defined in this manner, as a "multiplicative background risk."

In the next section, we introduce the basic framework. We next examine conditions on preferences that lead to more cautious behavior towards $\tilde{x}$ in the presence of a multiplicative background risk $\tilde{y}$. In section 4, we introduce the concept of the affiliated utility function and examine some of its basic properties. Section 5 uses the affiliated utility function to apply several extant results from the literature on additive background risk in the case of multiplicative risks. Section 6 examines comparative risk aversion; in particular we determine conditions that preserve the relation "more risk averse" in the presence of a multiplicative background risk. Section 7 provides some concluding thoughts.

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[1] Thus, for instance, in our first example above we can let $\tilde{x}$ represent post-tax profits based on the expected tax rates and let $\tilde{y}$ represent a deviation from these tax rates. Or, in the second example let $\tilde{x}$ denote wealth including expected annuity returns and let $\tilde{y}$ denote a multiplicative excess-return adjustment.

2. The Basic Model

Consider a risk-averse economic agent with utility function u. We wish to determine how the addition of a multiplicative background risk $\tilde{y}$ affects decision making on $\tilde{x}$. Let F and G denote the (cumulative) distribution functions associated with the random variables $\tilde{x}$ and $\tilde{y}$ respectively. Since $\tilde{x}$ and $\tilde{y}$ are independent, we can write expected utility as an iterated integral

(1) $$Eu(\tilde{x}\tilde{y}) = \int_0^\infty\int_0^\infty u(xy)dG(y)dF(x) \equiv E_F[E_G u(\tilde{x}\tilde{y})].$$

Define the derived utility function, see Nachman (1982)[2], as the interior integral given in equation (1). That is,

(2) $$v_G(x) \equiv \int_0^\infty u(xy)dG(y) = E_G u(x\tilde{y})$$

Trivially, $v_G(x)$ is increasing and concave since u is. Thus, equation (1) can be written as $Eu(\tilde{x}\tilde{y}) = E_F v_G(\tilde{x})$. Decisions on $\tilde{x}$ made in the presence of the multiplicative risk $\tilde{y}$ under utility u are isomorphic to decisions made on $\tilde{x}$ in isolation under the risk-averse utility $v_G(x)$. Let $\Gamma(\tilde{x})$ denote the set of random variables $\tilde{y}$ such that $\tilde{y}$ is statistically independent from $\tilde{x}$ and $E\tilde{y}$=1. Our focus here is in determining conditions on the utility function u such that the derived utility function, $v_G(x)$, is more risk averse than u for all $\tilde{y} \in \Gamma(\tilde{x})$. In other words, we wish to determine conditions on u that will guarantee that

[2] Actually, Nachman considers a more general relationship between $\tilde{x}$ and $\tilde{y}$. We adapt his measure to the case of multiplicative risks. The derived utility function for the additive case is described earlier by Kihlstrom, et al. (1981).

(3) $\frac{-v''_G(x)}{v'_G(x)} \equiv \frac{-E_G[u''(x\tilde{y})\tilde{y}^2]}{E_G[u'(x\tilde{y})\tilde{y}]} \geq \frac{-u''(x)}{u'(x)} \quad \forall x$.[3]

To avoid excessive notation, we will dispense with the subscripts and simply write $v(x)$ and $Eu(x\tilde{y})$, where we assume $\tilde{y}$ is an arbitrary member of $\Gamma(\tilde{x})$. We will let $r_v(x)$ and $r_u(x)$ denote the measure of absolute risk aversion for v and u respectively, i.e. the left-hand-side and right-hand-side of inequality (3) respectively.

Since we are involved with a multiplicative risk, it is often convenient to consider the corresponding measures of relative risk aversion, $R_v(x) \equiv xr_v(x)$ and $R_u(x) \equiv xr_u(x)$. For arbitrary x, straightforward manipulation of (3) shows that

(4) $R_v(x) = E_G[R_u(x\tilde{y})\frac{u'(x\tilde{y})\tilde{y}}{E_G[u'(x\tilde{y})\tilde{y}]}] \equiv \int_0^\infty R_u(xy)d\eta_x(y)$

where $\eta_x(\mathrm{y}) \equiv \int_0^{\mathrm{y}} \frac{u'(xt)tdG(t)}{E_G[u'(x\tilde{y})\tilde{y}]}$.

Note that $\eta_x(y)$ is itself a well-defined probability distribution. We define $\hat{E}_x$ to denote the expectation operator based on the probability distribution $\eta_x(y)$, which is a type of risk-adjusted probability measure. Thus, we see that relative risk aversion for v is a type of weighted average of relative risk aversion for u, namely $R_v(x) = \hat{E}[R_u(x\tilde{y})]$.

3. Risk Aversion Properties

From equation (4), it follows trivially that v inherits constant relative aversion (CRRA), whenever u exhibits CRRA. More explicitly, if $R_u(x) = \gamma \quad \forall x$, then

[3] In order to keep the mathematics simple, we will take "more risk averse" to be in the weak sense of Pratt (1964).

$R_v(x) = \gamma \ \forall x$ as well. Since it then also follows that $r_u(x) = r_v(x) \ \forall x$, we see that *u* and *v* are equivalent utility representations under CRRA. This is not surprising, since any optimal choice of an endogenous $\tilde{x}$ will be optimal for $\tilde{x}y \ \ \forall y > 0$ under CRRA preferences.

We next wish to examine conditions under which (3) holds $\forall \tilde{y} \in \Gamma(\tilde{x})$, i.e., we want to know when *v* is more risk averse than *u*. We may consider conditions for which this holds locally, with $r_v(x) \geq r_u(x)$, by examining the equivalent condition $R_v(x) \geq R_u(x)$. Our approach is to consider this last inequality for a particular value of *x*, by applying η_x as in equation (4). If the value of *x* chosen is arbitrary, so that $R_v(x) \geq R_u(x) \ \forall x$, then we are done.

Suppose that $R_u(x)$ is (not necessarily strictly) convex. Since $\eta_x(y)$ is a probability distribution, it follows from Jensen's inequality and equation (4) that

$$(5) \qquad R_v(x) \equiv \hat{E}R_u(x\tilde{y}) \geq R_u(x\hat{E}\tilde{y}),$$

where

$$(6) \qquad \hat{E}\tilde{y} = \int_0^\infty y\, d\eta_x(y) = \int_0^\infty y \frac{u'(xy)y}{E[u'(x\tilde{y})\tilde{y}]} dG(y).$$

Next, note that

$$(7) \qquad \frac{\partial^2 u(xy)}{\partial x \partial y} = \frac{\partial}{\partial y}[u'(xy)y] = u'(xy)[1 - R_u(xy)].$$

The sign of (7) tells us whether increases in the level of *y* will increase or decrease the marginal utility of *x*. The derivative in (7) will be everywhere positive [negative] if $R_u(xy) < [>]\, 1 \ \forall y$ in the support of *G*. This implies that increases in *y* reduce the marginal utility of x whenever $R_u > 1$, and increases in y increase the marginal utility of x whenever $R_u < 1$.

Since $E\left\{\frac{u'(x\tilde{y})\tilde{y}}{E[u'(x\tilde{y})\tilde{y}]}\right\} = 1$, we obtain the following result from (6) and (7).

<u>Lemma 1</u>: $\hat{E}\tilde{y} \gtreqless E\tilde{y} = 1$ if $R_u(xy) \lesseqgtr 1 \quad \forall y \in Supp(G)$.

We are now ready to prove the following result:

<u>Proposition 1</u>: *Suppose that $R_u(x)$ is convex and that one of the following conditions holds $\forall (x, y) \in Supp(F) \times Supp(G)$:*

(i) $R_u(xy) > 1$ and R_u is decreasing,

or (ii) $R_u(xy) < 1$ and R_u is increasing.

Then v is more risk averse than u.

<u>Proof</u>: Since $R_u(x)$ is convex, it follows from equation (4) that $R_v(x) \geq R_u(x\hat{E}\tilde{y})$ by Jensen's inequality. If $R_u > 1$, then $\hat{E}\tilde{y} < 1$ from Lemma 1. Hence, $R_u(x\hat{E}\tilde{y}) \geq R_u(x)$ under the assumption of decreasing relative risk aversion (DRRA). If $R_u < 1$, then it follows from Lemma 1 that $\hat{E}\tilde{y} > 1$. Hence, $R_u(x\hat{E}\tilde{y}) \geq R_u(x)$ under the assumption of increasing relative risk aversion (IRRA). Thus we have $R_v(x) \geq R_u(x)$ whenever condition (*i*) or (*ii*) holds. ■

Interestingly, if we have CRRA preferences, *u* and *v* are equivalent regardless of whether or not relative risk aversion exceeds 1. If relative risk aversion is increasing in wealth, as originally postulated by Arrow (1971) and most recently empirically supported by Guiso and Paiella (2001), then *v* will be more risk averse than *u* whenever R_u is convex and less than 1. If R_u is everywhere greater than 1 and exhibits increasing relative risk aversion, we cannot use Proposition 1 to verify that v is more risk averse than u. Indeed, if we have R_u>1 and if R_u is (not necessarily strictly) concave, it is easy to show that v is then less risk averse than u. Indeed, the following two cases are easy to show.

<u>Proposition 2</u>: *Suppose that $R_u(x)$ is concave and that one of the following conditions holds $\forall(x,y)\in Supp(F)\times Supp(G)$:*

(i) $R_u(xy)>1$ and R_u is increasing,

or (ii) $R_u(xy)<1$ and R_u is decreasing.

Then v is less risk averse than u.

<u>Proof</u>: The proof is similar to Proposition 1 and left to the reader. ■

Of course, whether risk aversion exhibits constant-, increasing-, or decreasing relative risk aversion, or none of these, is an empirical question. Certainly constant relative risk aversion is very common in equilibrium asset-pricing models. But empirical support also exists for both increasing relative risk aversion (e.g. Guiso and Paiella (2001)) and for decreasing relative risk aversion (e.g. Kessler and Wolf (1991)).

To illustrate Proposition 1 and 2, consider the following examples:

Example 1: Let $u(x)=-e^{-kx}$ where $k>0$. This is the case of constant absolute risk aversion (CARA). In this case $R_u'(x)=k$ and $R_u''(x)=0$. Thus, R_u is increasing and is both convex and concave. If we consider $\tilde{x}$ and $\tilde{y}$ such that $xy<1/k$ $\forall(x,y)\in$ $\text{Supp}(F)\times\text{Supp}(G)$, then $R_u(xy)<1$ and v is more risk averse than u by Proposition 1. However, if $xy>1/k$ $\forall(x,y)\in$ $\text{Supp}(F)\times\text{Supp}(G)$, then $R_u(xy)>1$ and v is less risk averse than u by Proposition 2.

Example 2: Let $u(x) = x - kx^2$ where $k > 0$. We restrict $x < \frac{1}{2k}$ so that marginal utility is positive. This is the case of quadratic utility. It is straightforward to show that $R_u(x) = 2kx(1-2kx)^{-1}$ and that R_u is both strictly increasing and convex. Moreover, $R_u(xy) < 1$ if $xy < \frac{1}{4k}$ $\forall (x, y) \in$ Supp(F) $\times$ Supp(G), so that v is more risk averse than u by Proposition 1. On the other hand, if $\frac{1}{4k} < xy < \frac{1}{2k}$ $\forall (x, y) \in$ Supp$(F) \times$Supp(G), then $R_u(xy) > 1$, but we cannot apply Proposition 1 (since R_u is increasing) or Proposition 2 (since R_u is convex).

Both utility functions above belong to the so-called HARA class of utility, as does CRRA utility.[4] Since we already showed that u and v are equivalent under CRRA, we see that no general results seem to apply to the HARA class of utility.

4. Affiliated Utility Functions

In this section, we obtain additional results by considering $\ln(xy) = \ln x + \ln y$. This allows us to adapt several results from the case of additive background risks. In order to accomplish this, we define the affiliated utility function, $\hat{u}$, such that $u(x) = \hat{u}(\ln x)$, where we restrict $x > 0$. Equivalently, we can substitute $\theta = \ln x$ to write $\hat{u}(\theta) \equiv u(e^{\theta})$ $\forall \theta \in \mathbb{R}$. In other words, $\hat{u}$ is the composite of u with the exponential function. Although $\hat{u}$ is increasing, it need not be concave. Since $u(xy) = \hat{u}(\ln x + \ln y)$, we will examine the additive risks $\ln \tilde{x} + \ln \tilde{y}$ in this section.

Let $\hat{r}(\theta)$ denote absolute risk aversion for $\hat{u}(\theta)$, i.e. $\hat{r}(\theta) = -\hat{u}''(\theta)/\hat{u}'(\theta)$. Straightforward calculations show that

$$(8) \qquad R_u(x) = 1 - \frac{\hat{u}''(\ln x)}{\hat{u}'(\ln x)} = 1 + \hat{r}_u(\ln x).$$

[4] Utility belongs to the HARA class if $[r(x)]^{-1}$ is linear in x.

Note that $R_u(x) < 1$ implies that $\hat{r}_u(\ln x) < 0$. Thus, if $R_u(x) < 1 \quad \forall x < 0$, then $\hat{u}$ exhibits risk-loving behavior and is convex. This is not surprising given the construction of the affiliated utility function.

If u is more concave than the natural logarithm function, $\hat{u}$ will be concave. That is, $\hat{u}$ will be everywhere risk averse if and only if u is everywhere more risk averse than log utility. If $u(x) = \ln x$, then $\hat{u}$ is risk neutral. Note that $\hat{u}$ does not represent utility of wealth, however. To refer to $\hat{u}$ as "risk averse, risk loving or risk neutral" is only a technical convenience, since in all cases, we are assuming that true preferences u are risk averse. Still, by examining the nature of $\hat{r}$, we will be able to adapt several existing results on additive background risk to the multiplicative case.

A few examples can help to illustrate the relationship between utility functions and the corresponding affiliated utility functions:

(*i*) If $u(x) = x$, so that preferences are risk neutral, then $\hat{u}(\theta) = e^{\theta}$, which is risk loving with constant absolute risk aversion.

(*ii*) If $u(x) = \frac{1}{1-\gamma} x^{1-\gamma}$, $\gamma > 0$, $\gamma \neq 1$, so that preferences exhibit constant relative risk aversion, then $\hat{u}(\theta) = \frac{1}{1-\gamma} e^{(1-\gamma)\theta}$. Note that affiliated utility functions exhibit constant absolute risk aversion of degree γ-1, which is risk averse only if γ>1.

(*iii*) If $u(x) = x - bx^2$, $b > 0$, $x < \frac{1}{2b}$, so that utility is quadratic, then $\hat{u}(\theta) = e^{\theta} - be^{2\theta}$.

(iv) The above examples are all special cases of HARA utility. Let $u(x) = \xi(\eta + \frac{x}{\gamma})^{1-\gamma}$, $\eta + \frac{x}{\gamma} > 0$, $\frac{\xi(1-\gamma)}{\gamma} > 0$. Then $\hat{u}(x) = \xi\left(\eta + \frac{e^{\theta}}{\gamma}\right)^{1-\gamma}$.

From the definition of $v(x)$ in (2), in a manner analogous to equation (8) we can derive

$$(9) \qquad R_v(x) = 1 - \frac{E_G\,\hat{u}''(\ln x + \ln \tilde{y})}{E_G\,\hat{u}'(\ln x + \ln \tilde{y})} \equiv 1 + \hat{r}_v(\ln x).$$

From (8) and (9), we easily obtain the following result.

Lemma 2: *(i)* $R_v(x) \geq R_u(x)$ if and only if $\hat{r}_v(\ln x) \geq \hat{r}_u(\ln x)$,

and *(ii)* $R_t(x)$ *is decreasing if and only if* $\hat{r}_t(\ln x)$ *is decreasing*, *t = u,v.*

Equivalent to (i) above, $r_v(x) \geq r_u(x)$ if and only if $\hat{r}_v(\ln x) \geq \hat{r}_u(\ln x)$. For the case where $R_u < 1$, so that $\hat{u}$ is risk loving, we can still interpret $\hat{r}_v > \hat{r}_u$ as meaning "$\hat{v}$ is more risk averse than $\hat{u}$," but in the sense of being less risk loving.

Consider now the set of $\tilde{y} \in \Gamma(\tilde{x})$, so that $E\tilde{y} = 1$. For any $\tilde{y} \in \Gamma(\tilde{x})$, $E(\ln \tilde{y}) \leq 0$, with equality only holding in the degenerate case, where $\tilde{y} = 1$ a.s. If we restrict utility such that $R_u > 1$, so that $\hat{u}$ is risk averse, then we know from Gollier and Pratt (1996) that v is more risk averse than u for an arbitrary $\tilde{y} \in \Gamma(\tilde{x})$ if and only if $\hat{u}(x)$ is risk vulnerable.[5] Since risk vulnerability is not an easy trait to verify, Gollier and Pratt offer us several useful sufficient conditions for risk vulnerability that are easy to check. In particular, we can apply their results in equation (9) to obtain the following.

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[5] More directly, we would use $\hat{u}$ to examine the behavior of $\hat{v}(\ln x) \equiv E_G\hat{u}(\ln x + \ln \tilde{y})$ for any nondegenerate $\tilde{y}$ with $E \ln \tilde{y} \leq 0$, rather than with $E \ln \tilde{y} < 0$. However, the distinction is nil if utility is differentiable.

Proposition 3: Suppose that $R_u(x) > 1 \;\; \forall x$. *Then v is more risk averse than u if either*

(*i*) $\hat{r}_u$ *is decreasing and convex,*

or (*ii*) $\hat{u}$ *exhibits standard risk aversion (see Kimball, 1993, and below).*

5. Properties of Affiliated Utility

In this section, we examine conditions on the utility function $u(x)$ that must hold if its affiliated utility function $\hat{u}(\theta)$ satisfies the properties given in Proposition 3(*i*) or 3(*ii*). In particular, we first show that $R_u(x)$ is decreasing and convex, whenever $\hat{r}_u(\theta)$ is decreasing and convex. We then show how there is a close relationship between standardness of the affiliated utility function $\hat{u}$ and standard relative risk aversion of u.

From equation (8), we see that

$$(10) \qquad R_u'(x) = \frac{1}{x} \; \hat{r}_u'(\ln x)$$

and

$$(11) \qquad R_u''(x) = \frac{1}{x^2}[\hat{r}_u''(\ln x) - \hat{r}_u'(\ln x)].$$

Consequently, since $x>0$, it follows from equation (10) that $R_u(x)$ is decreasing if and only if $\hat{r}_u(\theta)$ is decreasing. Moreover, if $\hat{r}_u(\theta)$ is decreasing and convex, it follows from equation (11) that $R_u(x)$ is also convex. As a consequence, the conditions holding in Proposition 3(*i*) imply those of Proposition 1(*i*), so that Proposition 3(*i*) also might be thought of as a corollary to Proposition 1.

The property of standard risk aversion, as presented in Kimball (1993), is analyzed at length in Gollier (2001). It is especially useful since it is easily characterized by decreasing absolute risk aversion and decreasing absolute prudence, where absolute prudence is measured as $p(x) = -\frac{u'''(x)}{u''(x)}$. If $u'''(x) > 0$, preferences are said to be prudent. If the affiliated utility function is standard risk averse, we may apply Proposition 3(*ii*) to conclude that *v* is more risk averse than *u*.

We first obtain a preliminary result that will prove useful. Straightforward calculations show that

$$(12) \qquad R_u'(x) = \frac{-u''(x)u'(x) - xu'''(x)u'(x) + x[u''(x)]^2}{[u'(x)]^2} = r_u(x)[1 - P_u(x) + R_u(x)],$$

where $P_u(x) \equiv \frac{-xu'''(x)}{u''(x)}$ denotes the measure of relative prudence. Consequently, we directly obtain the following result.

<u>Lemma 3</u>: $R_u'(x) \gtreqless 0$ *if and only if* $P_u(x) \lesseqgtr 1 + R_u(x)$.

We already know that $\hat{u}(\theta)$ is risk averse whenever $R_u(x) > 1$. Lemma 4 shows a condition on the underlying preferences that is equivalent to the prudence of $\hat{u}(\theta)$.

<u>Lemma 4</u>: *The affiliated utility function* $\hat{u}(\theta)$ *exhibits prudence,* $\hat{u}'''(\theta) > 0 \ \ \forall\theta$, *if and only if* $P_u(x) > 3 - \frac{1}{R_u(x)}$.

<u>Proof</u>: Recall that $\hat{u}(\ln x) = u(x)$, so that we obtain the following by differentiating with respect to $\ln x$:

$$\hat{u}'(\ln x) = xu'(x)$$

$$\hat{u}''(\ln x) = xu'(x) + x^2u''(x)$$

$$\hat{u}'''(\ln x) = xu'(x) + 3x^2u''(x) + x^3u'''(x).$$

Thus, dividing $\hat{u}'''(\ln x)$ by $-x^2u''(x) > 0$ we obtain

$$\hat{u}'''(\ln x) > 0 \quad \Leftrightarrow \quad -\frac{xu'''(x)}{u''(x)} - 3 - \frac{u'(x)}{xu''(x)} > 0 \quad \Leftrightarrow \quad P_u(x) > 3 - \frac{1}{R_u(x)}.$$

■

From Lemmata 3 and 4, we can easily now show the following.

<u>Lemma 5</u>: *If u exhibits decreasing relative risk aversion, the affiliated utility function* $\hat{u}$ *exhibits prudence.*

<u>Proof</u>: From Lemmata 3 and 4, the conclusion follows if $1 + R_u(x) \geq 3 - [R_u(x)]^{-1}$. Since $R_u(x)$ is positive, this is equivalent to $\{[R_u(x)]^2 - 2R_u(x) + 1\} = \{R_u(x) - 1\}^2 \geq 0$, which obviously holds. ■

We can use the derivatives in the proof of Lemma 4 to calculate the measure of absolute prudence for the affiliated utility function. In particular, we obtain

(13) $$\hat{p}(\ln x) \equiv -\frac{\hat{u}'''(\ln x)}{\hat{u}''(\ln x)} = -\frac{x^2u'''(x) + 2xu''(x)}{xu''(x) + u'(x)} - 1 = \frac{P_u(x) - 2}{1 - (R_u(x))^{-1}} - 1,$$

where the last step follows from dividing both the numerator and denominator in (13) by $xu''(x)$.

We are now ready to prove that standard relative risk aversion of *u* is a necessary condition for $\hat{u}$ to be standard:

Proposition 4: *Suppose that $\hat{u}$ exhibits standard risk aversion. Then* $R_u(x) > 1$ and u *exhibits standard relative risk aversion; that is, both $P_u(x)$ and $R_u(x)$ are positive and decreasing.*

Proof: From equation (8), we know that $\hat{u}$ risk averse implies that $R_u(x) > 1$. Since $\hat{u}$ exhibits decreasing absolute risk aversion, it follows from Lemma 2 that u exhibits decreasing relative risk aversion. Thus, we must show that u also exhibits positive and decreasing relative prudence. That relative prudence is positive follows easily from Lemma 3.

Differentiating equation (13) with respect to $\ln x$ we obtain

$$\frac{d\hat{p}(\ln x)}{d\ln x} = \frac{x[1-(R_u(x))^{-1}]P_u'(x) - x[P_u(x)-2](R_u(x))^{-2}R_u'(x)}{[1-(R_u(x))^{-1}]^2}.$$

Because $R_u(x) > 1$, it follows that $[R_u(x)]^2 - R_u(x) > 0$ and, from Lemma 3, that $P_u(x) - 2 > 0$. Thus, it follows that $\frac{d\hat{p}(\ln x)}{d\ln x}$ is negative if and only if

$$\text{(14)} \qquad P_u'(x) < \frac{P_u(x)-2}{[R_u(x)]^2 - R_u(x)} R_u'(x) < 0. \quad \blacksquare$$

From the proof of Proposition 4, we see that u exhibiting standard relative risk aversion is necessary, but not quite sufficient to imply that the affiliated utility function $\hat{u}$ is standard risk averse. However, we do obtain the following result.

Corollary 1: *Let* $R_u(x) > 1$. *If u exhibits standard relative risk aversion and the inequality in (14) holds, then the affiliated utility function $\hat{u}$ is standard risk averse.*

Proof: Since $R_u(x) > 1$, it follows from equation (8) that $\hat{r}(\theta) > 0$. Standard relative risk aversion of u implies, from Lemma 2, that $\hat{u}$ exhibits decreasing absolute risk aversion. It also follows, from Lemma 5, that $\hat{u}''' > 0$. Since (14) holding implies that $\hat{u}$ also exhibits decreasing absolute prudence, the Corollary follows. ■

Example: Let u belong to the HARA class of utility functions, $u(x) = \xi(\eta + \frac{x}{\gamma})^{1-\gamma}$ and suppose that $\gamma > 1$. Now $R_u(x) = \dfrac{x}{\eta + \frac{1}{\gamma}x}$. Thus, it follows easily that u exhibits decreasing relative risk aversion if and only if $\eta < 0$. Hence, $x > \eta + \frac{1}{\gamma}x$, so that $R_u(x) > 1$. To see that u exhibits standard relative risk aversion, note that $P_u(x) = \frac{1+\gamma}{\gamma} R_u(x)$. Thus, u exhibits decreasing relative prudence if and only if u exhibits decreasing relative risk aversion. Thus, u is standard relative risk averse and $R_u(x) > 1$. We now wish to show that $\hat{u}$ is standard risk averse.

By Corollary 1, we would be done if the inequality in (14) holds. Since both $P_u'(x) < 0$ and $R_u'(x) < 0$, inequality (14) is equivalent to

$$\frac{1+\gamma}{\gamma} > \frac{P_u(x) - 2}{R_u(x)[R_u(x) - 1]} = \frac{(\frac{1+\gamma}{\gamma})R_u(x) - 2}{R_u(x)[R_u(x) - 1]}$$

$$\Leftrightarrow \qquad R_u(x)[R_u(x) - 1] > R_u(x) - 2(\tfrac{\gamma}{1+\gamma})$$

$$\Leftrightarrow \qquad [R_u(x) - 1]^2 > \frac{1-\gamma}{1+\gamma}.$$

But, this last inequality follows, since $\gamma > 1$. Since $\hat{u}$ is standard risk averse, it follows from Proposition 3(*ii*) that the derived utility function v is more risk averse than u.

We can extend the example above to the following more general result.

Corollary 2: *Let u belong to the HARA class of utility functions,* $u(x) = \xi(\eta + \frac{x}{\gamma})^{1-\gamma}$, *with the domain of u given as* $[a, \infty)$, *where* $a > (-\eta\gamma)$. *Then the following two conditions are equivalent:*

(i) *u is standard relative risk averse with* $R_u(x) > 1$.

(ii) $\hat{u}$ *is standard risk averse.*

Proof: The domain of *u* guarantees risk aversion if and only if $\gamma > 0$. As in the above example, it follows that *u* exhibits decreasing relative risk aversion if and only if $\eta < 0$. Moreover, $R_u(x) > 1$ as $x \to \infty$ requires that $\gamma > 1$. Thus $(i) \Rightarrow (ii)$ follows from identical arguments to those made in the preceding example, whereas $(ii) \Rightarrow (i)$ follows from Proposition 4. ■

6. Comparative Risk Aversion

We start here by examining some intrapersonal characteristics of risk aversion. We will later examine some interpersonal characteristics. From equation (4), we see trivially that $R_v(x)$ will be everywhere greater than [less than] 1 if $R_u(x)$ is everywhere greater than [less than] 1. This result is more than just a technicality. Since many results in the literature on choice under uncertainty specify a global condition that either $R_u(x) > 1$ or $R_u(x) < 1$, such results also will hold in the presence of a multiplicative background risk, since $R_v(x)$ also will satisfy the appropriate property.

More generally, it follows trivially from equation (4) that

Proposition 5: Given any $\tilde{y} \in \Gamma(x)$, with distribution function G, $\inf\{R_u(xy)\} \le R_v(x) \le \sup\{R_u(xy)\} \quad \forall y \in Supp(G)$.

A key result in the literature on additive background risk is that the properties of constant absolute risk aversion and decreasing absolute risk aversion for utility are carried over to the derived utility function. On the other hand, the property of increasing absolute risk aversion does not always carry over. We next develop analogous results for relative risk aversion in the case of a multiplicative background risk. We have already seen that *v* inherits constant relative risk aversion from *u*. Indeed, the level of constant risk aversion is identical. To see that the same holds true for decreasing relative risk aversion, we first require the following Theorem, which is due to Gollier and Kimball (1996). A proof of this Theorem can also be found in Gollier (2001).

Lemma 6 (Diffidence Theorem, Gollier and Kimball): *Let Λ denote the set of all random variables with support contained in the interval* $[a,b]$ *and let f and g be two real-valued functions. The following two conditions are equivalent:*

(i) *For any* $\tilde{y} \in \Lambda, \quad Ef(\tilde{y}) = 0 \quad \Rightarrow \quad Eg(\tilde{y}) \ge 0.$

(ii) $\exists m \in \mathbb{R}$ *such that* $g(y) \ge mf(y) \quad \forall y \in [a,b].$

We now are ready to show that *v* also inherits decreasing relative risk aversion from *u*.[6]

Proposition 6: *Let* $\tilde{y}$ *have a bounded support. If u exhibits nonincreasing relative risk aversion, then so does the derived utility function v.*

[6] Although aesthetically unappealing, the limitation to bounded supports is not particularly restrictive. We already limit $\tilde{y}$ to be positive, so set a=0. Now, for any ε>0, we can always find a value for b such that the probability that $\tilde{y} > b$ is less than ε.

<u>Proof</u>: It follows from Lemma 3, that we need to show that, $\forall x$,

(15) $\quad P_u(x) \geq 1 + R_u(x) \quad \Rightarrow \quad P_v(x) \geq 1 + R_v(x)$.

That is, we must show that

(16) $$\frac{-Eu'''(x\tilde{y})\tilde{y}^3 x}{Eu''(x\tilde{y})\tilde{y}^2} \geq \frac{-Eu''(x\tilde{y})\tilde{y}^2 x}{Eu'(x\tilde{y})\tilde{y}} + 1.$$

Inequality (16) is equivalent to the following:

(17) $$E[u''(x\tilde{y})\tilde{y}^2 x + (\lambda - 1)u'(x\tilde{y})\tilde{y}] = 0 \quad \Rightarrow \quad E[u'''(x\tilde{y})\tilde{y}^3 x + \lambda u''(x\tilde{y})\tilde{y}^2] \geq 0.$$

By the Diffidence Theorem, (17) will hold if we can find a real number *m*, such that

(18) $$u'''(xy)y^3 x + \lambda u''(xy)y^2 \geq m[u''(xy)y^2 x + (\lambda - 1)u'(xy)y] \quad \forall y.$$

The left-hand side of (18) can be written as

(19) $$\frac{xyu''(xy)}{u'(xy)} \quad \frac{u'(xy)y}{x}\left[\frac{xyu'''(xy)}{u''(xy)} + \lambda\right] = -R_u(xy)\frac{u'(xy)y}{x}\left[\lambda - P_u(xy)\right].$$

Since $P_u(x) \geq 1 + R_u(x)$, it follows from (18) and (19) that

(20) $$u'''(xy)y^3 x + \lambda u''(xy)y^2 \geq -R_u(xy)\frac{u'(xy)y}{x}\left[\lambda - 1 - R_u(xy)\right].$$

From (18) and (20), we would be done if we could find an *m*, such that

(21) $$\begin{aligned} -R_u(xy)\frac{u'(xy)y}{x}\left[\lambda - 1 - R_u(xy)\right] &\geq m[u''(xy)y^2 x + (\lambda - 1)u'(xy)y] \\ &= mu'(xy)y[\lambda - 1 - R_u(xy)]. \end{aligned}$$

This follows by taking $m = (1-\lambda)/x$, since we then obtain (21) is equivalent to

$$(22) \qquad -R_u(xy)\left[\lambda-1-R_u(xy)\right]+(\lambda-1)[\lambda-1-R_u(xy)]=[\lambda-1-R_u(xy)]^2 \geq 0 .$$

Hence, (15) holds and *v* exhibits decreasing relative risk aversion. ■

We next turn to examining some interpersonal characteristics of comparative risk aversion. Kihlstrom, et al. (1981) and Ross (1981) examined these for the case of an additive background risk.[7] Their results are special cases of more general results found in Nachman (1982). Nachman is one of the few who considers the case of multiplicative background risks as a special case of his general results, albeit briefly. The basic question we address is the following: If agent 1 is more risk averse than agent 2, will this property be preserved in the presence of a multiplicative background risk? That is, if u_1 is more risk averse than u_2, when will it follow that v_1 is also more risk averse than v_2? One result that is quite easy to obtain is the following:

Corollary 3: *Let u^a and u^b be risk-averse utility functions such that u^a is more risk averse than u^b, i.e. $R_u^a(x) \geq R_u^b(x) \quad \forall x$. If $\exists \lambda \in \mathbb{R}$ such that $\forall x \quad R_u^a(x) \geq \lambda \geq R_u^b(x)$, then v^a is more risk averse than v^b.*

Proof: Follows directly from Proposition 5 and equation (4). ■

The proof of Corollary 3 also follows directly from the following result, which is due to Nachman (1982). We include it here for completeness.

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[7] Actually, Ross considers the background risk to be mean-independent, which is not as restrictive as the assumption of independence.

<u>Proposition (Nachman)</u>: Let u^a and u^b be risk-averse utility functions such that u^a is more risk averse than u^b, i.e. $R_u^a(x) \geq R_u^b(x) \ \forall x$. If there exists a function u^c such that $R_u^a(x) \geq R_u^c(x) \geq R_u^b(x) \ \forall x$ and $R_u^c(x)$ is nonincreasing, then v^a is more risk averse than v^b.

It follows easily from Nachman's result that v^a will be more risk averse than v^b if either of the utility functions, u^a or u^b, exhibits nonincreasing relative risk aversion. This result is a direct counterpart to the result by Kihlstrom, et al. in the case of additive background risk.

7. Concluding Remarks

The notion that markets are complete is a mathematical nicety that does not hold true in practice. Many types of political, human-capital and social risks, as well as some financial risks, are not represented by direct contracts. Obviously, many of these risks can be hedged indirectly - - so-called "cross hedging." However, even when such "background risks" are independent of other risks and cannot be "hedged" per se, they may have an impact upon risk-taking strategies that are within the control of the economic agent. Much has been done over the past twenty years in examining the effects of additive background risks. But surprisingly little has been done to systematically study economic decision making in the presence of a multiplicative background risk.

This paper is a first step towards developing a comprehensive theory of background risk in this direction. As the few examples in our introduction show, models with such multiplicative background risks are not hard to find within the literature. An understanding of the basic concepts presented here hopefully might help us understand a multitude of results for which standard theories (in the absence of any background risk) yield predictions that seem at odds with everyday observations of reality.

Since risk aversion captures all the essential information about preferences within an expected-utility framework, our focus here has been on comparing risk aversion with and without the background risk. As we learn more about these inherent properties, we hopefully will be able to find better models to use in the realm of positive theories.

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