Mean-Variance Hedging under Additional Market Information

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Abstract

In this paper we analyze the mean-variance hedging approach in an incomplete market under the assumption of additional market information, which is represented by a given, finite set of observed prices of non-attainable contingent claims. Due to no-arbitrage arguments, our set of investment opportunities increases and the set of possible equivalent martingale measures shrinks. Therefore, we obtain a modified mean-variance hedging problem, which takes into account the observed additional market information. Solving this we obtain an explicit description of the optimal hedging strategy and an admissible, constrained variance-optimal signed martingale measure, that generates both the approximation price and the observed option prices.

Key Words: option pricing, mean variance hedging, incomplete markets, variance-optimal martingale measure.

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1 Introduction

In an incomplete market, the determination of a unique price and of a replicating hedging strategy by means of no-arbitrage arguments is no longer possible even if the market model is arbitrage-free. A criterion for determining a ”good” hedging strategy and a ”fair” price is the mean-variance hedging approach which was first proposed by Föllmer and Sondermann (1986). It focuses on the minimization of the expected quadratic tracking error between a given contingent claim and the value process of a self-financing strategy at the terminal date.

Gouriéroux, Laurent and Pham (1998) (and independently Rheinländer and Schweizer (1997)) solve the general mean-variance hedging problem when the risky assets price process is a continuous semimartingale. Their key tool is the so-called hedging numéraire, which is used both as a deflator and to extend the primitive assets family. This idea enables them to transform the original problem into an equivalent and simpler one, which can easily be solved by means of the Galtchouk-Kunita-Watanabe theorem.

But this general mean variance hedging approach does not take into account additional information on market prices. In this paper we assume the existence of such additional market information, which is represented by a prescribed, finite set of observed prices of different contingent claims. These specific contingent claims have to be non-attainable or non-replicable by dynamic portfolio strategies in order to deliver new, relevant information on the underlying price system of the market. Due to no-arbitrage arguments, the set of all possible linear price systems or equivalent martingale measures shrinks and we have to consider a modified mean-variance hedging problem, which allows for buying or selling these specific contingent claims at the observed prices. Solving this by means of the techniques developed by Gouriéroux et al. (1998), we obtain an explicit description of the optimal hedging strategy and a constrained variance-optimal signed martingale measure, which generates both the approximation price and the observed option prices.

The paper is organized as follows. Section 2 introduces the model and derives the techniques to find a price and a hedging strategy for an attainable contingent claim. Two approaches of the option pricing theory are considered: the hedging approach and the martingale approach. It is shown that this option pricing theory is insufficient in the incomplete case when there are non-attainable contingent claims. In section 3, we assume the existence of additional information represented by a given, finite set of observed contingent claim prices. In order to satisfy the no-arbitrage condition of our financial market under this modified framework, we discuss the impact of this new information and trading possibilities on the traditional techniques of section 2. Section 4 describes in detail our modified mean-variance hedging approach, which has to be modified with respect to the assumption of the additional information and new trading possibilities of section 3. We present a solution following the idea of Gouriéroux et al. (1998). In Section 5, we discuss some examples to illustrate the relevance of the additional market information. The final section 6 is devoted to a convergence analysis.

2 Option Pricing Theory

We consider a financial market operating in continuous time and described by a probability space $(\Omega, \mathbb{F}, P)$, a time horizon T and a filtration $\mathbb{F} = \{\mathcal{F}_t,\, 0 \leq t \leq T\}$ satisfying the usual conditions, where $\mathcal{F}_t$ represents the information available at time t. A continuous semimartingale $S = (S_t)_{0 \leq t \leq T}$ describes the price evolution of a risky asset in the financial market containing also some riskless asset $B = (B_t)_{0 \leq t \leq T}$, with $B_t \equiv 1 \,\forall\, t \in [0, T]$.

A central problem in finance in such a framework is the pricing and hedging of a T-contingent claim H, which is a $\mathcal{F}_T$-measurable, square-integrable random variable H describing the net payoff at time T of some financial instrument, i.e. $H \in L^2(\Omega,\, \mathcal{F}_T,\, P)$. A famous example of a T-contingent claim is the European call option on the risky asset S with expiration date T and strike price K. The net payoff of such a European call option at time T is given by $H(\omega) = \max(S_T(\omega) - K, 0)$.

2.1 Hedging Approach

The hedging approach tries to solve the problem of pricing and hedging a given T-contingent claim H by dynamically replicating H with a dynamic portfolio strategy of the form $(\theta, \eta) = (\theta_t, \eta_t)_{0 \leq t \leq T}$ where θ is a predictable process and η is adapted. In such a strategy, θ_t describes the number of units of the risky asset at time t and η_t describes the amount invested in the riskless asset at time t.

At any time t, the value of the portfolio (θ_t, η_t) is then given by:

$$V_t = \theta_t S_t + \eta_t\,.$$

A strategy is called *self-financing* if its value process $V = (V_t)_{t \in [0,T]}$ can be written as the sum of a constant and a stochastic integral with respect to S:

$$V_t = x + \int_0^t \theta_s\, dS_s\,, \tag{1}$$

where $x = V_0$ denotes the initial cost to start the strategy.

>From this definition we see that a self-financing strategy (θ, η) is completely determined by the initial cost x and θ and can be identified with the pair (x, θ). A more mathematical formulation will be given in the next section.

The right-hand side in equation (1) represents the total earnings or capital gains which you realize on your holdings up to time t. All changes in the value of the portfolio are due to capital gains; withdrawal or infusion of cash are not allowed. After time 0, such a strategy is self-supporting: any fluctuations in S can be neutralized by rebalancing θ and η in such a way, that no further gains or losses are incurred.

A T-contingent claim H is said to be *attainable* iff there exists a self-financing strategy (x^H, θ^H) whose terminal value $V_T^{x^H, \theta^H}$ equals H almost surely:

$$H = x^H + G_T\big(\theta^H\big) \qquad P \text{ - a.s.,} \tag{2}$$

with $G_T(\theta) := \int_0^\top \theta_s \, dS_s$. H can be perfectly replicated.

If the financial market is arbitrage-free, i.e. it does not allow for arbitrage opportunities, the price of H at time 0 must be equal to x^H and (x^H, θ^H) is a hedging strategy, which replicates the contingent claim H. We speak of a complete market if all contingent claims are attainable.

This approach is the basic idea of the seminal paper of Black and Scholes (1973). Their well-known Black-Scholes model is a complete model. In such a framework the pricing and hedging of contingent claims can be done in a preference-independent fashion. But this completeness property is destroyed by modifying the original underlying stochastic source of the model and the model becomes incomplete, which means that there are non-attainable contingent claims.

For a non-attainable T-contingent claim H, it is by definition impossible to find a self-financing strategy with terminal value $V_T = H$ and representation (2).

This shows that the problem of pricing and hedging a non-attainable T-contingent claim H cannot be solved by means of the hedging approach. The next approach, the martingale approach, delivers linear price systems in form of equivalent martingale measures, which are consistent with the hedging approach in case of attainable contingent claims and compute "fair" prices in case of non-attainable contingent claims.

2.2 Martingale Approach

A second, more mathematical approach has been introduced by Harrison and Kreps (1979) and Harrison and Pliska (1981). Their basic idea is to use so-called equivalent martingale measures and the techniques of the martingale theory for a solution of the pricing and hedging problem:

Definition 1 (equivalent martingale measure):
The probability measure Q on $(\Omega, \mathcal{F}_T)$ is an *equivalent martingale measure of* P if $Q \sim P$, $\frac{dQ}{dP} \in L^2(\Omega, \mathcal{F}_T, P)$ and if the (discounted) price process S is a Q-martingale.

Let $\mathcal{M}(P)_e := \{Q \sim P : \frac{dQ}{dP} \in L^2(P),\, S \text{ is a } Q\text{-martingale } \}$ denote the set of all equivalent martingale measures of P.

The following assumption makes use of the result of the well-known "first fundamental theorem" and implies that the market is arbitrage-free:

Assumption 1:
There exists at least one equivalent martingale measure:

$$\mathcal{M}(P)_e \neq \emptyset\,.$$

We need to give a more rigorous mathematical formulation of a self-financing portfolio strategy:

Definition 2:
A strategy (x, θ) is self-financing if its value process allows a representation of the form (1) and if $x \in \mathbb{R}$ and $\theta \in \Theta$, where

$$\Theta := \Big\{\theta \text{ is a predictable process such that } G_T(\theta) \in L^2(\Omega, \mathcal{F}_T, P) \\ \text{and for each } Q \in \mathcal{M}(P)_e \text{ the process } (G_t(\theta))_{t\in[0,T]} \text{ is a } Q\text{-martingale.}\Big\}$$

$G_T(\Theta) := \big\{G_T(\theta) : \theta \in \Theta\big\}$ denotes the set of investment opportunities with initial cost 0 and $G_T(x, \Theta) := \big\{x + G_T(\theta) : x \in \mathbb{R},\, \theta \in \Theta\big\}$ denotes the set of all attainable T-contingent claims.

Remark 1:
By construction it is obvious that $G_T(\Theta) \subsetneqq L^2(\Omega, \mathcal{F}_T, P)$. The integrability conditions of the definition of a self-financing strategy ensure that $G_T(x, \Theta)$ is closed in $L^2(\Omega, \mathcal{F}_T, P)$. (see Delbaen and Schachermayer (1996a))

The well-known Galtchouk-Kunita-Watanabe projection theorem (see Ansel and Stricker (1993)) delivers a characterization of an arbitrary contingent claim H with respect to a given equivalent martingale measure Q:

Theorem 1 (Martingale Representation Theorem):
If $Q \in \mathcal{M}(P)_e$, a T-contingent claim H can be uniquely written as

$$H = \mathrm{E}^Q\big[H\big] + G_T(\psi^{Q,H}) + L_T^{Q,H} \quad \text{a.s.,} \tag{3}$$

where

(i) $(L_t^{Q,H})_{0 \le t \le T}$ is a square-integrable, strongly orthogonal martingale, i.e. $\mathrm{E}^Q\big[L_t^{Q,H} \cdot S_t\big] = 0$ for all $t \in [0, T]$ and $\mathrm{E}^Q\big[L_T^{Q,H}\big] = 0$.

(ii) $\big(\mathrm{E}^Q[H], \psi^{Q,H}\big)$ is a self-financing strategy.

Firstly, this result shows the consistency between the martingale approach and the hedging approach: If H is attainable, there exists a self-financing strategy and $L_T^{Q,H} \equiv 0$ must hold in representation (3) for all equivalent martingale measures $Q \in \mathcal{M}(P)_e$. Due to no-arbitrage arguments, $\big(\mathrm{E}^Q[H], \psi^{Q,H}\big)$ must be the unique hedging strategy of H and does not depend on the choice of $Q \in \mathcal{M}(P)_e$.

If our model is complete and all contingent claims are attainable, the equivalence of the martingale approach and the hedging approach is the statement of the next well-known theorem:

Theorem 2 (Second Fundamental Theorem):
The equivalent martingale measure is unique if and only if the market model is complete.

Secondly, in case of a non-attainable T-contingent claim H we obtain $Q\big[L_T^{Q,H} \neq 0\big] > 0$. Thus the strategy $\big(\mathrm{E}^Q[H], \psi^{Q,H}\big)$ cannot replicate H. But the martingale approach can be interpreted as an extension of the hedging approach by defining $\mathrm{E}^Q[H]$ to be the "fair" price of the contingent claim H. Hence the expectation operator of an equivalent martingale measure can be seen as a pricing function or linear price system [see Harrison and Pliska (1981), proposition 2.6]. But it should be pointed out that this "fair" price of a non-attainable contingent claim depends on the specific choice of the equivalent martingale measure $Q \in \mathcal{M}(P)_e$. Furthermore, all prices of contingent claims should be computed with the same selected equivalent martingale measure in order to avoid arbitrage opportunities.

So in case of an incomplete market there exists the selection problem to find an "optimal" equivalent martingale measure and we have to introduce an useful criterion according to which this "optimal" equivalent martingale measure (or price system) has to be chosen.

One such criterion is the mean-variance hedging approach, which was first proposed by Föllmer and Sondermann (1986) and was extended by Bouleau and Lamberton (1989), Schweizer (1994) and Schweizer (1996) (see Schweizer (2001) for an overview).

original mean-variance hedging problem

(4)

Suppose H is a T-contingent claim. Minimize $\mathrm{E}\left[\left(H - x - G_T(\theta)\right)^2\right]$ over all self-financing strategies (x, θ).

The idea of the mean-variance hedging approach is to insist on the usage of self-financing strategies and to minimize the "risk"

$$H - \big(x + G_T(\theta)\big) \tag{5}$$

between a non-attainable T-contingent claim H and the payoff of a self-financing strategy (x, θ) at the terminal date T. Here, "risk" is measured by the expected (with respect to the subjective probability measure) quadratic distance (5) at the terminal date T.

Therefore, this definition of risk does not depend on the price evolution of the self-financing strategies between time 0 and T. The quadratic terminal risk is simply the expected quadratic cost of revising the terminal portfolio in order to replicate H. But it does depend on the underlying subjective probability measure P. The question how to start with an "optimal" subjective probability measure P is still an open problem.

This original mean-variance hedging problem has been solved by Gouriéroux et al. (1998) and independently by Rheinländer and Schweizer (1997) when price processes are continuous semimartingales. The key tool of Gouriéroux et al. (1998) is the so-called *hedging numeraire*

$$V_T^* := 1 - G_T(\theta^*)\,, \tag{6}$$

which is defined to minimize $\mathrm{E}\big[(1 - G_T(\theta))^2\big]$ over all $\theta \in \Theta$:

$$\mathrm{E}\big[(1 - G_T(\theta))^2\big] \geq \mathrm{E}\big[(V_T^*)^2\big]\,.$$

Gouriéroux et al. (1998) then show that V_T^* has the following properties

$$V_T^* > 0\,, \qquad \mathrm{E}[V_T^* \cdot G_T(\Theta)] = 0\,, \qquad \mathrm{E}[(V_T^*)^2] = \mathrm{E}[V_T^*]\,. \tag{7}$$

Furthermore, they introduce a new probability measure $\widetilde{P}$ defined by

$$\frac{d\widetilde{P}}{dP} := \frac{V_T^*}{\mathrm{E}[V_T^*]}\,, \tag{8}$$

which is the *variance-optimal martingale measure*, i.e. $\widetilde{P} \in \mathcal{M}(P)_e$ and $\widetilde{P}$ minimizes $\mathrm{Var}[\frac{dQ}{dP}]$ over all $Q \in \mathcal{M}_s$. (see also Delbaen and Schachermayer (1996b) and Schweizer (1996).)

Finally, the optimal initial price $x^{P,H}$ of the optimal self-financing strategy $(x^{P,H},\ \theta^{P,H})$ can be characterized as an expected value of the T-contingent claim H under the newly introduced measure $\widetilde{P}$: $x^{P,H} = \mathrm{E}^{\widetilde{P}}\big[H\big]$. (See Gouriéroux et al. (1998) or Rheinländer and Schweizer (1997) for an explicit description of $\theta^{P,H}$.)

Since this optimization is done in a Hilbert space, this solution delivers a unique orthogonal decomposition for the T-contingent claim H under P:

$$H = \mathrm{E}^{\widetilde{P}}[H] + G_T(\theta^{P,H}) + L_T^{P,H} \tag{9}$$

with $\mathrm{E}[L_T^{P,H}] = 0$ and $G_T(\Theta) \perp L_T^{P,H}$, i.e $\mathrm{E}[G_T(\Theta) \cdot L_T^{P,H}] = 0$.

3 Option Pricing Theory under additional Market Information

We consider the financial market of the previous section, but under the assumption of additional market information, which is represented by a given, finite set of at time 0 observed T-contingent claim prices.

Assumption 2:
Given a fixed set of T-contingent claims $\big\{C_T^1, \ldots, C_T^n\big\}$ the price of the T-contingent claim $C_T^i \in L^2(P)$ at time 0 is $C_0^i \in \mathbb{R}$ for all $i \in 1 \ldots n$.

The following conditions are satisfied:

(a) The T-contingent claims $C_T := \big(C_T^1, \ldots, C_T^n\big)^\top$ are non-attainable.

(b) Let $L_T^{P,C} := \big(L_T^{P,C^1}, \ldots, L_T^{P,C^n}\big)$ be derived by the orthogonal decomposition of the T - contingent claims C_T under P like in (9) such that $L_T^{P,C^i} \perp G(x, \Theta)$. Then

$$\mathrm{E}\big[L_T^{P,C}(L_T^{P,C})^\top\big]^{-1} \text{ exists.}$$

(c) The observed T-contingent claim prices $\big\{C_0^i,\ i = 1 \ldots n\big\}$ are "admissible", i.e. there exists at least one equivalent martingale measure $Q \in \mathcal{M}(P)_e$ such that

$$E^Q\big[C_T\big] = C_0 \tag{10}$$

with $C_0 := (C_0^1, \ldots, C_0^n)^\top$.

Assumption 2 says that for each $i = 1, \ldots, n$ we exogenously observe the price C_0^i of the T-contingent claim C_T^i on the financial market. In particular, we are allowed to trade these T-contingent claims at these prices at time 0.

Item (a) implies that these observed contingent claim prices deliver new, relevant information on the underlying pricing function or price system of the market. If the C_T

were attainable we would not gain any new relevant information, because their prices would uniquely determined by no-arbitrage arguments.

Point (b) is a more mathematical assumption. The orthogonal decomposition can be derived by applying the original mean variance hedging approach. (An explanation of these notions will be given later on.) It ensures that every contingent claim C_T^i of the observed set is not redundant, but increases the information about the price system of our financial market.

The third condition (c) ensures that the observed contingent claim prices are reasonable and can be replicated by an equivalent martingale measure. Since our model has to be arbitrage-free, our computed model prices must coincide with these observed prices: Only those equivalent martingale measures are useful as pricing functions, which generate the observed contingent claim prices C_0. As a consequence, the set of equivalent martingale measures to be considered in the selection problem of the previous section shrinks to the set of *admissible equivalent martingale measures*:

Definition 3:
An equivalent martingale measure $Q \in \mathcal{M}(P)_e$ with property (10) is called *admissible*. The set of all admissible equivalent martingale measures is denoted by

$$\widetilde{\mathcal{M}}(P)_e^n := \Big\{ Q \in \mathcal{M}(P)_e \,:\, E^Q\big[C_T^i\big] = C_0^i \;\forall\, i = 1 \dots n \Big\}$$

Figure 1: *Observing the prices* $\{C_0^1, \dots, C_0^n\}$ *restricts the set of possible equivalent martingale measures.*

The definition of the admissible equivalent martingale measure and assumption 2 imply that

$$\widetilde{\mathcal{M}}(P)_e^n \neq \emptyset \quad \text{and} \quad \widetilde{\mathcal{M}}(P)_e^n \subsetneq \mathcal{M}(P)_e\,.$$

An admissible equivalent martingale measure is consistent with the observed contingent claim prices, hence it does not violate the no arbitrage condition and can be used as a pricing operator.

Assuming this kind of additional market information implies new investment opportunities: There is in addition to the self-financing strategy the possibility to buy (or to sell) δ^i units of the contingent claim C_T^i for the price $\delta^i C_0^i$ at time 0. Hence one has to take into account this additional trading possibilities in the construction of the possible portfolio strategies. Therefore we introduce *mixed portfolio strategies*:

Definition 4:
If (x,θ) is a self-financing strategy and $\delta := (\delta^1,\ldots,\delta^n)^\top \in \mathbb{R}^n$ then the value of the *mixed portfolio strategy* (x,θ,δ) at time T is given by

$$V_T^{x,\theta,\delta} := x + G_T(\theta) + \delta^\top(C_T - C_0 \cdot B_T) = x + G_T(\theta) + \delta^\top(C_T - C_0)$$

A mixed portfolio strategy can be interpreted as a composition of a dynamic strategy and a static strategy. Strategies, which trade the T-contingent claims C_T dynamically, cannot be allowed, since the price evolution of the T-contingent claims C_T between time 0 and T is unknown. Any specification of these price processes between time 0 and T would restrict the set of admissible martingale measures in a subjective way and cannot be justified by observations on our financial market.

The set of attainable T contingent claims must therefore be augmented:

Definition 5:
The set of T - contingent claims, which are attainable by using mixed portfolio strategies, is given by

$$A_T := \left\{ x + g + \delta^\top(C_T - C_0) \;:\; \text{for all } x \in \mathbb{R},\, g \in G_T(\Theta),\, \delta \in \mathbb{R}^n \right\}.$$

Remark 1 implies that $A_T \subseteq L^2(P)$ and that A_T is closed in $L^2(P)$.

$A_T(0) := \{g + \delta^\top(C_T - C_0) \;:\; \text{for all } g \in G_T(\Theta),\, \delta \in \mathbb{R}^n\}$ denotes the set of T - contingent claims, which are attainable by using mixed portfolio strategies with initial cost 0.

The following theorem generalizes theorem 1. It presents an orthogonal decomposition of a T-contingent claim H with respect to an admissible equivalent martingale measure into a part, that can be replicated by mixed portfolio strategies and belongs to A_T, and into a non-replicable, orthogonal part.

Theorem 3 (modified martingale representation theorem):
Suppose $Q \in \widetilde{\mathcal{M}}(P)_e^n$. Let $L_T^{Q,C} := \left(L_T^{Q,C^1},\ldots,L_T^{Q,C^n}\right)$ where L_T^{Q,C^i} is derived by applying the martingale representation (3) to C_T^i, for all $i = 1,\ldots,n$, and assume that

$\mathrm{E}^Q\big[L_T^{Q,C}(L_T^{Q,C})^\top\big]^{-1}$ exists. A T-contingent claim H can be uniquely written as

$$H = \mathrm{E}^Q\big[H\big] + G_T(\widetilde{\psi}^{Q,H}) + \delta^{Q,H\top}\big(C_T - C_0\big) + N^{Q,H} \qquad Q \text{ a.s.,} \tag{11}$$

where

(i) $N^{Q,H} \in L^2(\Omega, \mathcal{F}_T, Q)$, $\mathrm{E}^Q\big[N^{Q,H}\big] = 0$ and $\mathrm{E}^Q\big[N^{Q,H} \cdot a\big] = 0$ for all $a \in A_T$, i.e. $N^{Q,H} \in {A_T}^\perp$.

(ii) $\Big(\mathrm{E}^Q\big[H\big], \widetilde{\psi^{Q,H}}, \delta^{Q,H}\Big)$ is a mixed portfolio strategy, i.e. $\mathrm{E}^Q\big[H\big] + G_T(\widetilde{\psi}^{Q,H}) + \delta^{Q,H\top}\big(C_T - C_0\big) \in A_T$.

Proof. According to theorem 1 (martingale representation theorem) the T-contingent claim H can be written as

$$H = \mathrm{E}^Q\big[H\big] + G_T(\psi^{Q,H}) + L_T^{Q,H}\,,$$

with $\mathrm{E}^Q[L_T^{Q,H}] = 0$ and $\mathrm{E}^Q[G_T(\Theta)L_T^{Q,H}] = 0$.

In the same way the T-contingent claims C_T admit the representation

$$C_T^i = \mathrm{E}^Q[C_T^i] + G_T(\theta^{Q,C^i}) + L_T^{Q,C^i} \qquad \text{for all } i = 1, \ldots, n\,, \tag{$*$}$$

with $\mathrm{E}^Q[L_T^{Q,C^i}] = 0$ and $\mathrm{E}^Q[G_T(\Theta)L_T^{Q,C^i}] = 0$ for all $i = 1, \ldots, n$.

Therefore for $\delta \in \mathbb{R}^n$

$$\begin{aligned} H &= H - \delta^\top(C_T - C_0) + \delta^\top(C_T - C_0) \\ &= \mathrm{E}^Q\big[H\big] - \delta^\top(\mathrm{E}^Q[C_T] - C_0) + G_T(\psi^{Q,H} - \delta^\top\theta^{Q,C}) + (L_T^{Q,H} - \delta^\top L_T^{Q,C}) \\ &\quad + \delta^\top(C_T - C_0) \end{aligned} \tag{12}$$

Since $Q \in \widetilde{\mathcal{M}}(P)_e^n$ is an admissible equivalent martingale measure the expression $\delta^\top(\mathrm{E}^Q[C_T] - C_0)$ is equal to 0, and

$$= \mathrm{E}^Q\big[H\big] + G_T(\psi^{Q,H} - \delta^\top\theta^{Q,C}) + \delta^\top(C_T - C_0) + (L_T^{Q,H} - \delta^\top L_T^{Q,C})\,, \tag{13}$$

where $\big(\mathrm{E}^Q\big[H\big],\, \psi^{Q,H} - \delta^\top\theta^{Q,C}\big)$ is a self-financing strategy because of the linearity of stochastic integrals.

Now the parameter δ has to be chosen such that the following expression is satisfied for all $\big(x + g + \lambda^\top(C_T - C_0)\big) \in A_T$:

$$0 \overset{!}{=} \mathrm{E}^Q\Big[\big(x + g + \lambda^\top(C_T - C_0)\big) \cdot \big(L_T^{Q,H} - \delta^\top L_T^{Q,C}\big)\Big]$$

It follows from equation ($*$) and from the definition of $L_T^{Q,H}$ and $L_T^{Q,C}$ that

$$= \lambda^\top\, \mathrm{E}^Q\Big[L_T^{Q,C} \cdot \big(L_T^{Q,H} - \delta^\top L_T^{Q,C}\big)\Big]$$

This expression is equal to 0 if δ is chosen such that

$$(14)\qquad \begin{aligned}\delta^{Q,H} &:= \mathrm{E}^Q\big[L_T^{Q,C}(L_T^{Q,C})^\top\big]^{-1}\,\mathrm{E}^Q\big[L_T^{Q,H}L_T^{Q,C}\big]\\ &= \mathrm{Cov}^Q\big[L_T^{Q,C}\,,\,L_T^{Q,C}\big]^{-1}\,\mathrm{Cov}^Q\big[L_T^{Q,H}\,,\,L_T^{Q,C}\big]\end{aligned}$$

Setting

$$\widetilde{\psi^{Q,H}} := \psi^{Q,H} - \delta^{Q,H^\top}\theta^{Q,C} \quad \text{and} \quad N^{Q,H} := L_T^{Q,H} - \delta^{Q,H^\top}L_T^{Q,C}$$

yields that $\big(\mathrm{E}^Q\big[H\big],\,\widetilde{\psi^{Q,H}},\,\delta^{Q,H}\big)$ is a mixed portfolio strategy and that $N^{Q,H} \in L^2(\mathcal{F}_T, Q)$ with $\mathrm{E}^Q\big[N^{Q,H}\big] = 0$ and $N^{Q,H} \in {A_T}^\perp$.

□

This theorem shows (similar to the general approach) the consistency between the martingale approach and the hedging approach in our modified framework: If $H \in A_T$ we obtain $N^{Q,H} \equiv 0$ and a unique replicating mixed portfolio strategy for all $Q \in \widetilde{\mathcal{M}}(P)_e^n$. So again, the expectation operator of an admissible equivalent martingale measure can be interpreted as a pricing function.

Since the variance can be interpreted as a measure of risk, we obtain from the modified martingale representation that the risk of an arbitrary contingent claim H can be decomposed into a hedgeable part and an intrinsic, non-hedgeable part. The intrinsic risk is the specific risk of a non-attainable T-contingent claim, which cannot be eliminated using mixed portfolio strategies. Let $Q \in \widetilde{\mathcal{M}}(P)_e^n$, then

$$\begin{aligned}\mathrm{Var}^Q\big[H\big] &= \mathrm{Var}^Q\big[\mathrm{E}^Q\big[H\big] + G_T(\widetilde{\psi}^{Q,H}) + \delta^{Q,H}(C_T - C_0) + N^{Q,H}\big]\\ &= \mathrm{Var}^Q\Big[\mathrm{E}^Q\big[H\big] + G_T(\widetilde{\psi}^{Q,H}) + \delta^{Q,H^\top}(C_T - C_0)\Big] + \mathrm{Var}^Q\big[N^{Q,H}\big]\\ &\quad + \mathrm{Cov}^Q\Big[(\mathrm{E}^Q\big[H\big] + G_T(\widetilde{\psi}^{Q,H}) + \delta^{Q,H^\top}(C_T - C_0)), N^{Q,H}\Big]\,.\end{aligned}$$

The last term is equal to zero because of $N^{Q,H} \in {A_T}^\perp$ and $\big(\mathrm{E}^Q\big[H\big] + G_T(\widetilde{\psi}^{Q,H}) + \delta^{Q,H^\top}(C_T - C_0)\big) \in A_T$. Hence

$$= \underbrace{\mathrm{Var}^Q[\mathrm{E}^Q\big[H\big] + G_T(\widetilde{\psi}^{Q,H}) + \delta^{Q,H^\top}(C_T - C_0)]}_{\text{hedgeable risk}} + \underbrace{\mathrm{Var}^Q[N^{Q,H}]}_{\text{intrinsic risk}}\,.$$

Note that the intrinsic risk of an arbitrary contingent claim H in our modified framework is smaller than in the general case without the assumption of additional observed contingent claim prices:

$$\mathrm{Var}^Q\big[N^{Q,H}\big] \le \mathrm{Var}^Q\big[L_T^{Q,H}\big]\,.$$

If Q is an equivalent martingale measure but not admissible, similar results as in theorem 3 can be formulated

Remark 2:
In case of $Q \in \mathcal{M}(P)_e \backslash \widetilde{\mathcal{M}}(P)^n_e$, the equivalent martingale measure Q is not admissible , i.e. it does not satisfy the condition $\mathrm{E}^Q[C_T] = C_0$. But by setting $\widetilde{\psi}^{Q,H} := \psi^{Q,H} - \delta^{Q,H^\top}\theta^{Q,C}$, $\delta^{Q,H} := \mathrm{E}^Q\big[L_T^{Q,C}(L_T^{Q,C})^\top\big]^{-1}\,\mathrm{E}^Q\big[L_T^{Q,H}L_T^{Q,C}\big]$ and $N^{Q,H} := L_T^{Q,H} - \delta^{Q,H^\top}L_T^{Q,C}$ it follows from equation (12) and its following conclusions that a T-contingent claim H can be written as

$$H = \mathrm{E}^Q\big[H\big] - \delta^{Q,H^\top}\big(\mathrm{E}^Q[C_T] - C_0\big) + G_T(\widetilde{\psi}^{Q,H}) + \delta^{Q,H^\top}\big(C_T - C_0\big) + N^{Q,H}\,, \tag{15}$$

where $N^{Q,H} \in L^2(\Omega, \mathcal{F}_T, Q)$, $\mathrm{E}^Q[N^{Q,H}] = 0$ and $N^{Q,H} \in A_T^\perp$.

This observation leads us to the idea of constructing a new, appropriate measure that admits a representation like in theorem 3. The following notion has to be defined for that reason

Definition 6 (signed admissible martingale measure):
A *signed admissible martingale measure of* P is a signed measure Q on $(\Omega, \mathcal{F}_T)$ with $Q[\Omega] = 1$, $Q \ll P$, $\frac{dQ}{dP} \in L^2(\Omega, \mathcal{F}_T, P)$ and

$$\mathrm{E}^Q\big[a\big] = \mathrm{E}\Big[\frac{dQ}{dP}\,a\Big] = 0 \qquad \text{for all } a \in A_T(0). \tag{16}$$

$\widetilde{\mathcal{M}}(P)^n_s$ denotes the convex set of all signed admissible martingale measures of P.

Note that condition (16) already implies

$$\mathrm{E}^Q\big[C_T^i - C_0^i\big] = \mathrm{E}\Big[\frac{dQ}{dP}\big(C_T^i - C_0^i\big)\Big] = 0 \quad \forall \quad i = 1, \dots, n$$

for a signed admissible martingale measure $Q \in \widetilde{\mathcal{M}}(P)^n_s$.

Lemma 1:
If $Q \in \mathcal{M}(P)_e \backslash \widetilde{\mathcal{M}}(P)^n_e$ and if $\mathrm{E}^Q[L_T^C L_T^C]^{-1}$ exists, a signed admissible martingale measure W can be constructed by

$$\frac{dW}{dQ} = 1 - \mathrm{E}^Q[C_T - C_0]^\top\,\mathrm{E}^Q[L_T^{Q,C}L_T^{Q,C^\top}]^{-1}L_T^{Q,C}\,, \tag{17}$$

which satisfies

$$\mathrm{E}^W\big[H\big] = \mathrm{E}^Q\big[H\big] - \delta^{Q,H^\top}\big(\mathrm{E}^Q[C_T] - C_0\big)$$

for all T-contingent claims H.

Proof. Since $Q \in \mathcal{M}(P)_e \backslash \widetilde{\mathcal{M}}(P)_e^n$ is an equivalent martingale measure we can make use of the martingale representation theorem 1 with respect to C_T. Defining the measure W by

$$\frac{dW}{dQ} = 1 - \mathrm{E}^Q[C_T - C_0]^\top \, \mathrm{E}^Q[L_T^{Q,C} L_T^{Q,C^\top}]^{-1} L_T^{Q,C} \,,$$

it follows that W is a signed measure on $(\Omega, \mathcal{F}_T)$ and $\frac{dW}{dQ} \in L^2(\Omega, \mathcal{F}_T, Q)$.

If H is a T-contingent claim the definition of W gives us

$$\begin{aligned}\mathrm{E}^W\big[H\big] &= \mathrm{E}^Q\Big[H\big(1 - \mathrm{E}^Q[C_T - C_0]^\top \, \mathrm{E}^Q[L_T^{Q,C} L_T^{Q,C^\top}]^{-1} L_T^{Q,C}\big)\Big] \\ &= \mathrm{E}^Q[H] - \mathrm{E}^Q[C_T - C_0]^\top \, \mathrm{E}^Q[L_T^{Q,C} L_T^{Q,C^\top}]^{-1} \, \mathrm{E}^Q[L_T^{Q,C} L_T^{Q,H}] \,.\end{aligned}$$

Setting $H = C_T^i$ yields that $E^W[C_T^i] = C_0^i$ for all $i = 1, \ldots, n$.

In case of $H = g \in G_T(\Theta)$ we have

$$\mathrm{E}^W[g] = \mathrm{E}^Q[g] - \mathrm{E}^Q[C_T - C_0]^\top \, \mathrm{E}^Q\big[L_T^{Q,C} L_T^{Q,C^\top}\big]^{-1} \, \mathrm{E}^Q\big[L_T^{Q,C} L_T^{Q,g}\big] \,.$$

The martingale property of Q and $L_T^{Q,g} \equiv 0$ yields

$$= 0 \,.$$

Hence the constructed signed measure W is admissible and possesses the martingale property. □

4 Mean-Variance Hedging under additional Market Information

But the general mean-variance hedging approach does not consider the kind of additional market information introduced in the last section. It concentrates only on the approximate replication of a contingent claim by means of self-financing strategies. Additional trading and hedge possibilities like observed, non-attainable contingent claims are neglected.

Therefore we assume just as in assumption 2 of the last section the existence of additional market information, which is represented by a given, finite set of observed contingent claim prices.

According to the results of the last section, we are looking for a mixed portfolio strategy (x, θ, δ) which minimizes the expected quadratic error of replication between the T-contingent claim H and the value process of the mixed portfolio strategy (x, θ, δ) at the terminal date T. So we obtain the following

modified mean-variance hedging problem

(18)
$$\boxed{\begin{array}{l}\text{Suppose } H \text{ is a } T\text{-contingent claim. Minimize} \\ \qquad \mathrm{E}\left[\left(H - x - G_T(\theta) - \delta^\top (C_T - C_0)\right)^2\right] \\ \text{over all mixed portfolio strategies } (x, \theta, \delta).\end{array}}$$

This approach proposes to price options by L^2-approximation: we want to determine an initial capital x, a dynamic trading strategy θ and a static hedging strategy δ such that the achieved terminal wealth $x + G_T(\theta) + \delta^\top (C_T - C_0)$ approximates the T-contingent claim H with respect to the distance in $L^2(P)$.

Another interesting interpretation for the modified version of the mean-variance hedging problem (18) is that for each $i = 1, \ldots, n$ the T-contingent claim $(C_T^i - C_0^i)$ can be considered as a risk-swap between the risky T-contingent claim C_T^i and the riskless T-contingent claim $C_0^i \cdot B_T$. The price of this swap at time 0 is 0. Therefore this swap can be used in our modified mean-variance hedging approach to reduce the remaining risk of the general mean-variance hedging approach.

Remark 3:
We can rewrite problem (18) with regard to the notation introduced in the last section:

(19)
$$\boxed{\begin{array}{l}\text{Minimize} \\ \qquad \mathrm{E}\left[(H - a)^2\right] \\ \text{over all } a \in A_T.\end{array}}$$

The existence of a solution of this optimization problem is ensured by the $L^2(P)$-closedness of A_T.

An optimal strategy of the modified mean-variance hedging problem (18) is called *modified minimal variance hedging strategy* of the T-contingent claim H under P. The following property supports this name:

If $(x^*, \theta^*, \delta^*)$ is a solution of the problem (18), then (θ^*, δ^*) also solves the optimization problem:

$$\boxed{\begin{array}{l}\text{Minimize} \\ \qquad \mathrm{Var}\left[H - G_T(\theta) - \delta^\top (C_T - C_0)\right] \\ \text{over all } (\theta, \delta)\,.\end{array}}$$

Proof. For all $\theta \in \Theta$, $\delta \in \mathbb{R}^n$ we have:

$$\begin{aligned}
&\mathrm{Var}\Big[H - G_T(\theta) - \delta^\top(C_T - C_0)\Big] \\
&\qquad \stackrel{\text{def}}{=} \mathrm{E}\Bigg[\Big(H - \underbrace{\mathrm{E}\Big[H - G_T(\theta) - \delta^\top(C_T - C_0)\Big]}_{=:x \in \mathbb{R}} - G_T(\theta) - \delta^\top(C_T - C_0)\Big)^2\Bigg]
\end{aligned}$$

Since $(x^*, \theta^*, \delta^*)$ is a solution of the optimization problem (18), it minimizes the last expression.

$$\begin{aligned}
&\geq \mathrm{E}\Bigg[\Big(H - x^* - G_T(\theta^*) - {\delta^*}^\top(C_T - C_0)\Big)^2\Bigg] \\
&\geq \mathrm{Var}\Big[H - x^* - G_T(\theta^*) - {\delta^*}^\top(C_T - C_0)\Big] \\
&= \mathrm{Var}\Big[H - G_T(\theta^*) - {\delta^*}^\top(C_T - C_0)\Big] ,
\end{aligned}$$

by definition of the variance.

□

4.1 Solution of the Modified Mean-Variance Hedging Problem

In order to solve the modified mean-variance hedging problem (18) it turns out to be didactically reasonable to distinguish between three cases:

- The subjective probability measure P is already an admissible equivalent martingale measure, i.e. $P \in \widetilde{\mathcal{M}}(P)_e^n$.
- P is an equivalent martingale measure, but it is not admissible, i.e. $P \in \mathcal{M}(P)_e \backslash \widetilde{\mathcal{M}}(P)_e^n$.
- P is not an equivalent martingale measure, i.e. $P \notin \mathcal{M}(P)_e$.

4.1.1 Case 1: $P \in \widetilde{\mathcal{M}}(P)_e^n$

Recall from the modified martingale representation (11) that the T-contingent claim H can be written as

$$H = a^{P,H} + N^{P,H} \quad P \text{ a.s.}$$

with $a^{P,H} = \mathrm{E}^P\big[H\big] + G_T(\widetilde{\psi}^{P,H}) + {\delta^{P,H}}^\top(C_T - C_0) \in A_T$ and $N^{P,H} \in {A_T}^\perp$.

For each $a \in A_T$ we have

$$\begin{aligned}
\mathrm{E}\big[(H - a)^2\big] &= \mathrm{E}\big[(a^{P,H} + N_T^{P,H} - a)^2\big] \\
&= \mathrm{E}\big[(a^{P,H} - a)^2\big] + \mathrm{E}\big[(N^{P,H})^2\big] + 2\,\mathrm{E}\big[(a^{P,H} - a)N^{P,H}\big]
\end{aligned}$$

Because of $(a^{P,H} - a) \in A_T$ and $N^{P,H} \in {A_T}^{\perp}$ the last term is equal to 0:

$$= \mathrm{E}\big[(a^{P,H} - a)^2\big] + \mathrm{E}\big[(N^{P,H})^2\big]$$

Choosing $a = a^{P,H}$ minimizes this expression and delivers

$$= \mathrm{E}\big[(N^{P,H})^2\big] \,.$$

Therefore, we have shown that the optimal strategy $(\mathrm{E}^P\big[H\big], \widetilde{\psi}^{P,H}, \delta^{P,H})$ of the modified mean-variance hedging problem can be derived by means of the modified martingale representation (11) when the subjective probability measure P is already an admissible equivalent martingale measure.

Note that the price of the optimal strategy at time 0 is given by the P-expected value $\mathrm{E}^P\big[H\big]$.

4.1.2 Case 2: $P \in \mathcal{M}(P)_e \backslash \widetilde{\mathcal{M}}(P)_e^n$

If $P \in \mathcal{M}(P)_e \backslash \widetilde{\mathcal{M}}(P)_e^n$ we cannot use the modified martingale representation theorem, but we can use the results of remark 2, especially equation (15):

For each $a \in A_T$ we obtain

$$\mathrm{E}[(H-a)^2] \overset{(15)}{=} \mathrm{E}\Big[\big(\underbrace{\mathrm{E}[H] - {\delta^{P,H}}^\top(\mathrm{E}[C_T] - C_0) + G_T(\widetilde{\psi}^{P,H}) + {\delta^{P,H}}^\top(C_T - C_0)}_{=:\widetilde{a}} + N^{P,H} - a\big)^2\Big]$$

$$= \mathrm{E}\Big[\big(\widetilde{a} - a\big)^2\Big] + \mathrm{E}\Big[\big(N^{P,H}\big)^2\Big]$$

This expression is minimized by setting $a = a^{P,H} = \widetilde{a}$.

$$= \mathrm{E}\Big[\big(N^{P,H}\big)^2\Big]$$

Hence the optimal strategy is given by

$$\Big(\mathrm{E}[H] - {\delta^{P,H}}^\top(\mathrm{E}[C_T] - C_0),\, \widetilde{\psi}^{P,H},\, \delta^{P,H}\Big)\,.$$

But now, the price of the strategy is the P - expected value of H minus a correction term

$$\mathrm{E}\big[H\big] - {\delta^{P,H}}^\top\big(\mathrm{E}[C_T] - C_0\big) \qquad (\neq \mathrm{E}[H])\,.$$

Since P is an equivalent martingale measure (albeit not admissible), P is already the variance optimal martingale measure (of the original approach). Assumption 2 (b) implies that the conditions of lemma 1 are fulfilled. Applying this result, formula (17) defines a new signed measure W, which is admissible and satisfies

$$\mathrm{E}^W\big[H\big] = \mathrm{E}\big[H\big] - {\delta^{P,H}}^\top\big(\mathrm{E}[C_T] - C_0\big).$$

It will be shown later on that this newly constructed measure is the so-called *constrained variance-optimal martingale measure*.

4.1.3 Case 3: $P \notin \mathcal{M}(P)_e$

We now turn to the general situation where S is a continuous semimartingale under P. We have seen that the solution of the original approach (4) delivers the unique orthogonal decomposition for the T-contingent claim H under P

$$H = \mathrm{E}^{\widetilde{P}}[H] + G_T(\theta^{P,H}) + L_T^{P,H} \tag{20}$$

with $\mathrm{E}[L_T^{P,H}] = 0$ and $G_T(\Theta) \perp L_T^{P,H}$, i.e $\mathrm{E}[G_T(\Theta) \cdot L_T^{P,H}] = 0$.

The original approach can also be applied to the T-contingent claims C_T and delivers the orthogonal decomposition

$$C_T = \mathrm{E}^{\widetilde{P}}[C_T] + G_T(\theta^{P,C}) + L_T^{P,C} \tag{21}$$

with $G_T(\Theta) \perp L_T^{P,C}$ and $\mathrm{E}[L_T^{P,C}] = 0$.

The modified mean-variance hedging problem can be solved using these orthogonal representations. Its solution can be characterized by means of the hedging numeraire $V_T^* = 1 - G_T(\theta^*)$ (see definition (6)) and the variance-optimal martingale measure $\widetilde{P}$ (see definitions (8)):

Lemma 2:
The solution of the modified mean-variance hedging problem (18) is given by the optimal mixed portfolio strategy $(\widetilde{x}, \widetilde{\theta}, \widetilde{\delta})$ with

$$\begin{aligned}
\widetilde{x} =& \mathrm{E}^{\widetilde{P}}\big[H\big] - \mathrm{E}\big[L_T^{P,H} L_T^{P,C}\big]^\top \mathrm{E}\big[L_T^{P,C} (L_T^{P,C})^\top\big]^{-1} \mathrm{E}^{\widetilde{P}}\Big[C_T - C_0\Big] \\
=& \mathrm{E}^W\big[H\big] \\
\widetilde{\delta} =& \mathrm{E}[L_T^{P,C} (L_T^{P,C})^\top]^{-1} \mathrm{E}\big[L_T^{P,H} L_T^{P,C}\big] \\
\widetilde{\theta} =& \theta^{P,H} - \widetilde{\delta}^\top \theta^{P,C}
\end{aligned}$$

where

$$\frac{dW}{dP} := \frac{d\widetilde{P}}{dP} - \mathrm{E}^{\widetilde{P}}\big[C_T - C_0\big]^\top \mathrm{E}\big[L_T^{P,C} (L_T^{P,C})^\top\big]^{-1} L_T^{P,C} \tag{22}$$

defines an admissible signed martingale measure on $(\Omega, \mathcal{F}_T)$, i.e. $W \in \widetilde{\mathcal{M}}(P)_s^n$.

Proof. Firstly, it follows similarly as in the proof of Lemma 1 that the signed measure W is indeed an admissible signed martingale measure.

Secondly, for all $x \in \mathbb{R}$, $\theta \in \Theta$ and $\delta \in \mathbb{R}^n$ we have

$$\mathrm{E}\Big[\Big(H - x - G_T(\theta) - \delta^\top (C_T - C_0)\Big)^2\Big]$$

Using the above-mentioned orthogonal representations (20) and (21) for H and C, this expression is equal to

$$= \mathrm{E}\Big[\Big(\mathrm{E}^{\widetilde{P}}[H] - \delta^\top \mathrm{E}^{\widetilde{P}}[C_T - C_0] - x + G_T\big(\theta^{P,H} - \delta^\top \theta^{P,C} - \theta\big) + L_T^{P,H} - \delta^\top L_T^{P,C}\Big)^2\Big]$$

The orthogonality of $G_T(\Theta) \perp L_T^{P,H}$ and $G_T(\Theta) \perp L_T^{P,C}$ yields

$$= \mathrm{E}\Big[\Big(\mathrm{E}^{\widetilde{P}}[H] - \delta^\top \mathrm{E}^{\widetilde{P}}[C_T - C_0] - x + G_T\big(\theta^{P,H} - \delta^\top \theta^{P,C} - \theta\big)\Big)^2\Big] + \mathrm{E}\Big[\Big(L_T^{P,H} - \delta^\top L_T^{P,C}\Big)^2\Big]$$

Setting $x = \widetilde{x} := \mathrm{E}^{\widetilde{P}}[H] - \delta^\top \mathrm{E}^{\widetilde{P}}[C_T - C_0]$ and $\theta = \widetilde{\theta} := \theta^{P,H} - \delta^\top \theta^{P,C}$ for fixed δ minimises this expression for all x, θ.

$$\geq \mathrm{E}\Big[\Big(L_T^{P,H} - \delta^\top L_T^{P,C}\Big)^2\Big]$$

Finally, the minimum is attained if we choose $\delta = \widetilde{\delta} = \mathrm{E}[L_T^{P,C}(L_T^{P,C})^\top]^{-1} \mathrm{E}\big[L_T^{P,H} L_T^{P,C}\big]$:

$$\geq \mathrm{E}\big[(L_T^{P,H})^2\big] - \mathrm{E}\big[L_T^{P,H} L_T^{P,C}\big]^\top \mathrm{E}\big[L_T^{P,C}(L_T^{P,C})^\top\big]^{-1} \mathrm{E}\big[L_T^{P,H} L_T^{P,C}\big]\,.$$

□

This shows that the optimal price $\widetilde{x}$ for H can be described by an expected value under the measure W. This newly constructed measure admits another characterization. In order to derive this we introduce the *modified hedging numeraire*

(23) $$V_T^B := 1 - G_T(\theta^b) - {\delta^b}^\top (C_T - C_0)$$

with

$$\delta^b := \mathrm{E}[V_T^*]\, M^{-1}\, \mathrm{E}^{\widetilde{P}}[C_T - C_0]$$

$$M := \mathrm{E}\Big[\Big(V_T^* \mathrm{E}^{\widetilde{P}}[C_T - C_0] + L_T^{P,C}\Big)\Big(V_T^* \mathrm{E}^{\widetilde{P}}[C_T - C_0] + L_T^{P,C}\Big)^\top\Big]$$

and

$$\theta^b := \theta^* - \delta^\top \theta^C - \theta^* \delta^\top \mathrm{E}^{\widetilde{P}}[C_T - C_0]\,,$$

which minimizes $\mathrm{E}\Big[\Big(1 - G_T(\theta) - \delta^\top (C_T - C_0)\Big)^2\Big]$ for all $\delta \in \mathbb{R}^n$, $\theta \in \Theta$, i.e.

$$\mathrm{E}\Big[\Big(1 - G_T(\theta) - \delta^\top (C_T - C_0)\Big)^2\Big] \geq \mathrm{E}\Big[\big(V_T^B\big)^2\Big] \qquad \forall\, \delta \in \mathbb{R}^n,\, \theta \in \Theta\,.$$

Proof. For all $\theta \in \Theta$ and $\delta \in \mathbb{R}^n$ we have

$$\mathrm{E}\Big[\Big(1 - G_T(\theta) - \delta^\top (C_T - C_0)\Big)^2\Big]$$

Since the T-contingent claims C_T admit the unique orthogonal decomposition (21) under P, we can write

$$= \mathrm{E}\Big[\Big(1 - G_T(\theta) - \delta^\top\big(\mathrm{E}^{\widetilde{P}}[C_T - C_0] + G_T(\theta^C) + L_T^{P,C}\big)\Big)^2\Big]$$

Because of $V_T^* = 1 - G_T(\theta^*)$ it follows

$$= \mathrm{E}\Big[\Big(1 - G_T\big(\underbrace{\theta + \delta^\top\theta^C + \theta^*\delta^\top\,\mathrm{E}^{\widetilde{P}}[C_T - C_0]}_{=:\,\phi(\theta)}\big) - \delta^\top\big(V_T^*\,\mathrm{E}^{\widetilde{P}}[C_T - C_0] + L_T^{P,C}\big)\Big)^2\Big]$$

$$= \mathrm{E}\Big[\big(1 - G_T(\phi(\theta))\big)^2\Big] + \delta^\top\,\mathrm{E}\Big[\Big(V_T^*\,\mathrm{E}^{\widetilde{P}}[C_T - C_0] + L_T^{P,C}\Big)\Big(V_T^*\,\mathrm{E}^{\widetilde{P}}[C_T - C_0] + L_T^{P,C}\Big)^\top\Big]\delta$$
$$- 2\delta^\top\,\mathrm{E}\Big[\big(1 - G_T(\phi(\theta))\big)\big(V_T^*\,\mathrm{E}^{\widetilde{P}}[C_T - C_0] + L_T^{P,C}\big)\Big]$$

Due to $\mathrm{E}\big[V_T^* \cdot G_T(\Theta)\big] = 0$ and $\mathrm{E}\big[L_T^{P,C} \cdot G_T(\Theta)\big] = 0$ it follows

$$= \mathrm{E}\big[\big(1 - G_T(\phi(\theta))\big)^2\big] + \delta^\top\,\mathrm{E}\Big[\Big(V_T^*\,\mathrm{E}^{\widetilde{P}}[C_T - C_0] + L_T^{P,C}\Big)\Big(V_T^*\,\mathrm{E}^{\widetilde{P}}[C_T - C_0] + L_T^{P,C}\Big)^\top\Big]\delta^\top$$
$$- 2\,\mathrm{E}[V_T^*]\,\delta^\top\,\mathrm{E}^{\widetilde{P}}[C_T - C_0]$$

Since only the first term depends on θ, setting $\theta = \phi^{-1}(\theta^*) = \theta^* - \delta^\top\theta^C - \theta^*\delta^\top\,\mathrm{E}^{\widetilde{P}}[C_T - C_0]$ minimises this expression for all θ and fixed δ according to the definition of V_T^*.

$$\geq \mathrm{E}\big[(V_T^*)^2\big] + \delta^\top\,\mathrm{E}\Big[\Big(V_T^*\,\mathrm{E}^{\widetilde{P}}[C_T - C_0] + L_T^{P,C}\Big)\Big(V_T^*\,\mathrm{E}^{\widetilde{P}}[C_T - C_0] + L_T^{P,C}\Big)^\top\Big]\delta^\top$$
$$- 2\,\mathrm{E}[V_T^*]\,\delta^\top\,\mathrm{E}^{\widetilde{P}}[C_T - C_0]$$

Choosing $\delta = \delta^* := \mathrm{E}[V_T^*]\;\mathrm{E}\Big[\Big(V_T^*\,\mathrm{E}^{\widetilde{P}}[C_T - C_0] + L_T^{P,C}\Big)\Big(V_T^*\,\mathrm{E}^{\widetilde{P}}[C_T - C_0] + L_T^{P,C}\Big)^\top\Big]^{-1}\mathrm{E}^{\widetilde{P}}[C_T - C_0] = \mathrm{E}[V_T^*]\,M^{-1}\,\mathrm{E}^{\widetilde{P}}[C_T - C_0]$ minimises this expression for all δ. Using the method of modification for matrix inversion (see Stewart (1973, p. 414)) it can be shown that the existence of $\mathrm{E}[L_T(L_T)^\top]^{-1}$ implies the existence of M^{-1}.

$$\geq \mathrm{E}\big[(V_T^*)^2\big] + \delta^{*\top}\,\mathrm{E}\Big[\Big(V_T^*\,\mathrm{E}^{\widetilde{P}}[C_T - C_0] + L_T^{P,C}\Big)\Big(V_T^*\,\mathrm{E}^{\widetilde{P}}[C_T - C_0] + L_T^{P,C}\Big)^\top\Big]\delta^*$$
$$- 2\,\mathrm{E}[V_T^*]\,\delta^{*\top}\,\mathrm{E}^{\widetilde{P}}[C_T - C_0]$$
$$= \mathrm{E}\big[(V_T^*)^2\big] - \mathrm{E}[V_T^*]^2\,\mathrm{E}^{\widetilde{P}}[C_T - C_0]^\top M^{-1}\,\mathrm{E}^{\widetilde{P}}[C_T - C_0] = \mathrm{E}[(V_T^B)^2]$$

□

Additionally, this proof shows that the modified hedging numeraire can be written as

$$V_T^B = V_T^* - \mathrm{E}[V_T^*]\,\mathrm{E}^{\widetilde{P}}\big[C_T - C_0\big]^\top M^{-1}\Big(V_T^*\,\mathrm{E}^{\widetilde{P}}\big[C_T - C_0\big] + L_T\Big), \tag{24}$$

and that due to $\mathrm{E}\big[\big(V_T^*\big)^2\big] = \mathrm{E}\big[V_T^*\big]$ in the last equation of the proof

$$\mathrm{E}\big[\big(V_T^B\big)^2\big] = \mathrm{E}\big[V_T^B\big]. \tag{25}$$

Consequently, this implies

$$\mathrm{E}\big[V_T^B\big] = \mathrm{E}[V_T^*]\Big(1 - \mathrm{E}^{\widetilde{P}}\big[C_T - C_0\big]^\top M^{-1}\,\mathrm{E}^{\widetilde{P}}\big[C_T - C_0\big]\Big) > 0. \tag{26}$$

After this preliminary remarks we are able to derive the aforementioned alternative characterization for the newly constructed measure W:

Lemma 3:
The density of the signed measure W as defined by (22) can be written as

$$\frac{dW}{dP} = \frac{V_T^B}{\mathrm{E}\big[V_T^B\big]}\,. \tag{27}$$

Proof. Starting with the formula (27), we have

$$\frac{V_T^B}{\mathrm{E}\big[V_T^B\big]} \overset{(24)}{=} \frac{V_T^* - \mathrm{E}[V_T^*]\,\mathrm{E}^{\widetilde{P}}\big[C_T - C_0\big]^\top M^{-1}\Big(V_T^*\,\mathrm{E}^{\widetilde{P}}\big[C_T - C_0\big] + L_T\Big)}{\Big(1 - \mathrm{E}[V_T^*]\,\mathrm{E}^{\widetilde{P}}[C_T - C_0]^\top M^{-1}\,\mathrm{E}^{\widetilde{P}}[C_T - C_0]\Big)\,\mathrm{E}[V_T^*]}$$

$$= \frac{V_T^*}{\mathrm{E}[V_T^*]} - \frac{\mathrm{E}^{\widetilde{P}}[C_T - C_0]^\top M^{-1} L_T}{1 - \mathrm{E}[V_T^*]\,\mathrm{E}^{\widetilde{P}}[C_T - C_0]^\top M^{-1}\,\mathrm{E}^{\widetilde{P}}[C_T - C_0]}$$

Because of $M = \mathrm{E}[V_T^*]\,\mathrm{E}^{\widetilde{P}}\big[C_T - C_0\big]\,\mathrm{E}^{\widetilde{P}}\big[C_T - C_0\big]^\top + \mathrm{E}^P\big[L_T(L_T)^\top\big]$ the following equation holds: $\mathrm{Id} = \Big(M - \mathrm{E}[V_T^*]\,\mathrm{E}^{\widetilde{P}}[C_T - C_0]\,\mathrm{E}^{\widetilde{P}}[C_T - C_0]^\top\Big)\,\mathrm{E}^P[L_T(L_T)^\top]^{-1}$. Therefore, it follows

$$= \frac{d\widetilde{P}}{dP} - \frac{\mathrm{E}^{\widetilde{P}}[C_T - C_0]^\top M^{-1}\Big(M - \mathrm{E}[V_T^*]\,\mathrm{E}^{\widetilde{P}}[C_T - C_0]\,\mathrm{E}^{\widetilde{P}}[C_T - C_0]^\top\Big)\,\mathrm{E}^P[L_T(L_T)^\top]^{-1} L_T}{1 - \mathrm{E}[V_T^*]\,\mathrm{E}^{\widetilde{P}}[C_T - C_0]^\top M^{-1}\,\mathrm{E}^{\widetilde{P}}[C_T - C_0]}$$

$$= \frac{d\widetilde{P}}{dP} - \frac{\Big(1 - \mathrm{E}[V_T^*]\,\mathrm{E}^{\widetilde{P}}[C_T - C_0]^\top M^{-1}\,\mathrm{E}^{\widetilde{P}}[C_T - C_0]\Big)\,\mathrm{E}^{\widetilde{P}}[C_T - C_0]^\top\,\mathrm{E}^P[L_T(L_T)^\top]^{-1} L_T}{1 - \mathrm{E}[V_T^*]\,\mathrm{E}^{\widetilde{P}}[C_T - C_0]^\top M^{-1}\,\mathrm{E}^{\widetilde{P}}[C_T - C_0]}$$

$$= \frac{d\widetilde{P}}{dP} - \mathrm{E}^{\widetilde{P}}\big[C_T - C_0\big]^\top\,\mathrm{E}\big[L_T(L_T)^\top\big]^{-1} L_T$$

But this equal to the definition of W:

$$= \frac{dW}{dP}$$

This proofs the assertion. ∎

Now we are prepared to derive an interesting interpretation of the admissible, signed martingale measure W by means of lemma 3. W turns out to be the solution of the next optimization problem:

Minimize $$\mathrm{Var}\left[\frac{dQ}{dP}\right] = \mathrm{E}\left[\left(\frac{dQ}{dP}\right)^2\right] - 1$$ over all admissible signed martingale measures $Q \in \widetilde{\mathcal{M}}(P)_s^n$.

A solution Q^* of this dual quadratic problem is called *constrained (admissible) variance-optimal martingale measure*

Lemma 4:
The admissible signed martingale measure W as defined by (22) is the constrained variance-optimal martingale measure.

Proof. For any $a = g + \delta^\top(C_T - C_0)$ with $g \in G_T(\Theta)$ and $\delta \in \mathbb{R}^n$ and for each $Q \in \widetilde{\mathcal{M}}(P)_s^n$ we have

$$1 = \mathrm{E}^Q\big[1-a\big] = \mathrm{E}\Big[\frac{dQ}{dP}(1-a)\Big] \leq \mathrm{E}\Big[\Big(\frac{dQ}{dP}\Big)^2\Big]\,\mathrm{E}\big[(1-a)^2\big]$$

by the Cauchy-Schwarz inequality and therefore

$$\begin{aligned}
\frac{1}{\inf_{Q\in\widetilde{\mathcal{M}}(P)_s^n} \mathrm{E}[(\frac{dQ}{dP})^2]} &= \sup_{Q\in\widetilde{\mathcal{M}}(P)_s^n} \frac{1}{\mathrm{E}[(\frac{dQ}{dP})^2]} \\
&\leq \inf_{a\in\{g+\delta^\top(C_T-C_0)\,:g\in G_T(\Theta),\delta\in\mathbb{R}^n\}} \mathrm{E}\big[(1-a)^2\big] \\
&= \mathrm{E}[(V_T^B)^2] = \frac{\mathrm{E}[(V_T^B)^2]\,\mathrm{E}[V_T^B]^2}{\mathrm{E}[V_T^B]^2}
\end{aligned}$$

Due to $\mathrm{E}[V_T^B] = \mathrm{E}[(V_T^B)^2]$ it follows

$$= \frac{\mathrm{E}[(V_T^B)^2]\,\mathrm{E}[V_T^B]^2}{\mathrm{E}[(V_T^B)^2]^2} = \frac{\mathrm{E}[V_T^B]^2}{\mathrm{E}[(V_T^B)^2]} \overset{(27)}{=} \frac{1}{\mathrm{E}[(\frac{dW}{dP})^2]}\,.$$

Because of $W \in \widetilde{\mathcal{M}}(P)_s^n$, the measure W is the the constrained variance-optimal martingale measure. □

This indicates that finding the constrained variance-optimal admissible signed martingale measure is the dual problem to solving the modified hedging numeraire problem. The duality is reflected in the fact that the modified approximation price is obtained as an expectation under W.

5 Examples

In this section we analyze two examples to illustrate the impact of the assumption of additional information on different market situations.

5.1 Example 1

As first example we consider a financial market $(S^1,\, S^2,\, B)$ defined on a probability space $(\Omega, \mathcal{F}_T, P)$, where S^1 and S^2 are two risky assets and B the riskless asset. Suppose that their dynamics with respect to P are

$$\begin{aligned} dS_t^1 &= S_t^1\big(\mu\, dt + \sigma^1\, dW_t^1\big) \\ dS_t^2 &= S_t^2\big(\sigma^2\, dW_t^2\big) \qquad\qquad \text{under } P \\ B_t &\equiv 1 \end{aligned}$$

where W^1 and W^2 are two independent P Brownian motions.

This market $\big(S^1,\, S^2,\, B\big)$ is complete and the dynamics under the unique martingale measure $\widetilde{P} \in \mathcal{M}(P)_e$ are:

$$\begin{aligned} dS_t^1 &= S_t^1\big(\sigma^1\, d\widetilde{W}_t^1\big) \\ dS_t^2 &= S_t^2\big(\sigma^2\, d\widetilde{W}_t^2\big) \qquad\qquad \text{under } \widetilde{P} \\ B_t &\equiv 1 \end{aligned}$$

where $\widetilde{W}^1$ and $\widetilde{W}^2$ are two independent $\widetilde{P}$ Brownian motions thanks to the Girsanov - theorem.

But now we assume that our information is limited and the asset S^1 is not observable. Thus our dynamic investment opportunities are restricted to the basic assets $(S^2,\, B)$. This restricted market is therefore incomplete, but we assume that the "true" price system is still $\widetilde{P}$.

Starting with our subjective measure P the variance-optimal martingale measure of the original mean-variance hedging approach is P itself (not the "true" measure $\widetilde{P}$).

Consider a T-contingent claim $C_T = C_T(S_T^1)$ that depends on S_T^1 and is non-attainable with respect to the restricted market $(S^2,\, B)$. Suppose the price C_0 of this contract at time 0 can be observed and is given by $C_0 := \mathrm{E}^{\widetilde{P}}[C_T]$.

The original mean-variance hedging approach ignores this additional information and delivers $\mathrm{E}^P[C_T](\neq C_0)$ as a price of C_T. Arbitrage opportunities are possible therefore.

But our modified mean-variance hedging approach incorporates this additional information and we obtain a constrained admissible variance-optimal measure $P^* \neq P$, which is in this sense closer to the "true" martingale measure $\widetilde{P}$ than P and preserves the no-arbitrage requirement.

5.2 Example 2

The second example is a simple stochastic volatility model and is based on an example introduced by Harrison and Pliska (1981) and analyzed in detail by Müller (1985) and Föllmer and Schweizer (1991).

We consider a financial market (S, B) defined on a probability space $(\Omega, \mathcal{F}_T, P)$ with a random variable $\eta \in \{+, -\}$, where S is a risky asset and B the riskless asset. Suppose that their dynamics are given by

$$\begin{aligned} dS_t &= S_t\sigma(+)\, dW_t \qquad \text{on } \{\eta = +\} \\ dS_t &= S_t\sigma(-)\, dW_t \qquad \text{on } \{\eta = -\} \\ B_t &\equiv 1\,, \end{aligned}$$

where $(W_t)_{t\in[0,T]}$ denotes a Brownian motion, $\sigma(-) \neq \sigma(+)$ and $\sigma(-), \sigma(+) \in \mathbb{R}^+$.

If the realization of η is known at time 0, the market is complete and it follows from Black-Scholes that a T-contingent claim H (e.g. a European call option) can be written as

$$H = H_0^+ 1_{\{\eta=+\}} + H_0^- 1_{\{\eta=-\}} + \int_0^T (\psi_t^+ 1_{\{\eta=+\}} + \psi_t^- 1_{\{\eta=-\}})\, dS_t\,,$$

where $H_0^\pm$ and $\psi^\pm$ denote the usual Black-Scholes values and strategies with respect to the variance $\sigma(\pm)$ (see Föllmer and Schweizer (1991)).

Suppose now, that the realization of η is unknown at time 0, but becomes observable directly after time 0. The market is incomplete, and with $p := P[\{\eta = +\}]$ (assume $0 < p < 1$) Föllmer and Schweizer (1991) show that in this case H admits the following representation corresponding to theorem 1

$$\begin{aligned} H = \;& (p\, H_0^+ + (1-p)\, H_0^-) + G_T(1_{\{\eta=+\}}\, \psi^+ + (1 - 1_{\{\eta=+\}})\psi^-) \\ & + \underbrace{(H_0^+ - H_0^-)(1_{\{\eta=+\}} - p)}_{=L_T^{P,H}} \end{aligned} \tag{28}$$

Since P is already an equivalent martingale measure, the variance optimal martingale measure of the general approach is P.

Assume now that the price C_0 of the (non-attainable) European call option C_T at time 0 can be observed and is given by $C_0 = q\, C_0^+ + (1-q)\, C_0^-$, with $0 < q < 1$ and $q \neq p$. Note that the Black-Scholes formula implies $C_0^+ \neq C_0^-$. The original mean variance approach does not incorporate this additional information and uses the variance-optimal measure P for pricing, although it is obvious that P is not admissible and cannot be the "true" martingale measure because under P the price of C_T would be $E^P[C_T] = p\, C_0^+ + (1-p)\, C_0^-$ $(\neq C_0)$.

But our modified approach delivers the admissible, variance-optimal martingale measure W:

$$\frac{dW}{dP} = 1 - \mathrm{E}\big[C_T - C_0\big] \frac{L_T^{P,C}}{\mathrm{E}\big[(L_T^{P,C})^2\big]}$$
$$\overset{(28)}{=} 1 - \mathrm{E}\big[C_T - q\, C_0^+ - (1-q)\, C_0^-\big] \frac{(C_0^+ - C_0^-)(1_B - p)}{\mathrm{E}\big[((C_0^+ - C_0^-)(1_B - p))^2\big]}$$
$$= 1 + (q-p)(C_0^+ - C_0^-) \frac{(C_0^+ - C_0^-)(1_B - p)}{(C_0^+ - C_0^-)^2 p(1-p)}$$
$$= \frac{1-q}{1-p} \cdot (1 - 1_B) + \frac{q}{p} \cdot 1_B$$

The measure W is an equivalent martingale measure due to positivity of its density. Furthermore, W is admissible and $\mathrm{E}^W[C_T] = q\, C_0^+ + (1-q)\, C_0^-$ because of $W[B] = q$. In fact, W must be the "true" pricing measure of the market.

6 Convergence

The idea behind this section is the intuition, that the more prices of non-attainable contingent claims are observed in the market, the more information about the "true" pricing function or the "true" equivalent martingale measure P^* is revealed.

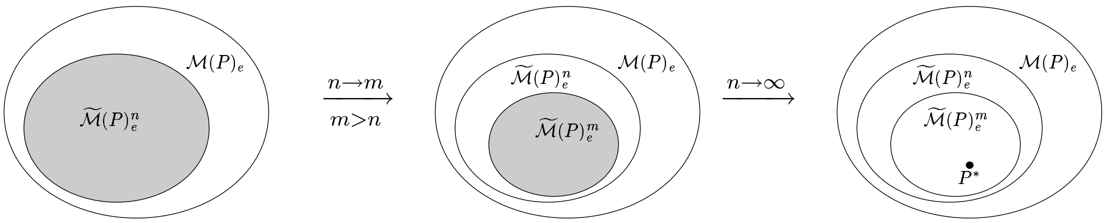

Figure 2: *convergence for* $n \to \infty$

In order to check this guess we consider a financial market consisting of a riskless asset $B \equiv 1$ and a risky asset Π. Its price process $(\Pi_t)_{t \in [0,T]}$ is defined on the probability space $(\Omega,\, \mathcal{F}_T,\, P) = (C,\, \mathcal{C},\, P)$ of continuous functions on the time interval $[0, T]$, and let $\Pi_t(\omega) := \omega(t) \in \mathbb{R}$ for all $\omega \in \Omega$ and $t \in [0, T]$. This market is complete and let $P^* \neq P$ be the unique equivalent martingale measure.

Suppose now the risky asset Π cannot be observed. Similar as in example 1, our financial market is restricted to the degenerated market (B) and set of investment opportunities shrinks to

$$G_T(\Theta) = \emptyset\,.$$

The market is now incomplete, but we assume that the underlying "true" martingale measure or price system is still given by P^*.

The σ-algebra $\sigma(\Pi_T^{-1})$ is generated by the sequence $(\Pi^{-1}(A_i))_{i\in\mathbb{N}}$ where the A_i are half-open intervals of $\mathbb{R}$. Set $\mathcal{G}_n := \{\sigma((\Pi_T^{-1}(A_i))_{i=1,\dots,n})\}$ for a fixed n. Thanks to the chosen structure of the $\{A_i, i = 1,\dots,n\}$ there exists a partition of Ω into a finite number of measurable sets $B_{n,1},\dots,B_{n,m_n}$ such that every element of $\mathcal{G}_n$ is the union of some of these sets.

Suppose we observe at time 0 the prices $\{C_0^1,\dots,C_o^n\}$ of the T-contingent claims $\left\{1_{\{\Pi_T\in A_1\}},\dots,1_{\{\Pi_T\in A_n\}}\right\}$. These are given by $C_0 = (P^*\Pi_T^{-1}[A_i])_{i=1,\dots,n}$.

The constrained variance-optimal martingale measure is then defined by

$$\frac{dW_n}{dP} = 1 - \mathrm{E}[C_T - C_0]\,\mathrm{E}[L_T^C(L_T^C)^\top]^{-1}L_T^C$$

Due to $L_T^{C^i} = 1_{\{\Pi_T\in A_i\}} - P\Pi_T^{-1}[A_i]$ this density is $\mathcal{G}_n$- measurable. Since the new measure W_n is by construction uniquely defined for each A_i, $i = 1,\dots,n$ and because $\{A_i, i = 1,\dots,n\}$ generates $\mathcal{G}_n$, this last expression can be simplified thanks to the theory of probability measures and has to be given by

$$= \sum_{i=1}^{n} 1_{\{\Pi_T\in A_i\}}\frac{\mu\Pi_T^{-1}[A_i]}{P\Pi_T^{-1}[A_i]} \quad (>0)$$

(Note that W_n is indeed an equivalent probability measure because of the positivity of its Radon-Nikodym density)

According to Meyer (1966, p.153) the last expression is an uniformly integrable $(\mathcal{G}_n)_{n\in\mathbb{N}}$-martingale and because of the martingale convergence theorem it converges to a limit in the L^1 norm when $n\to\infty$. This limit is evidently a Radon-Nikodym density of the restriction of P^* to $\sigma(\Pi_T) = \mathcal{G}_\infty$, with respect to the restriction of P to $\sigma(\Pi_T)$. This yields

$$\Pi_T^{-1}W^n \xrightarrow{\mathrm{w}} \Pi_T^{-1}P^*\,. \tag{29}$$

Therefore, for a fixed time T the one-dimensional marginal distribution converges towards the one-dimensional marginal distribution of the "true" pricing measure P^*.

7 Conclusion

In this paper, we consider the mean-variance hedging approach under the assumption of additional market information represented by a given, finite set of observed prices of non-attainable contingent claims. Taking into account these additional trading and hedge possibilities we obtain a modified mean-variance hedging problem. We present a solution of this optimization problem by applying the techniques developed by Gouriéroux et al. (1998) and obtain an explicit description for the optimal mixed portfolio strategy and derive a constraint variance optimal, admissible, signed martingale measure.

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