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Valuation with Risk-Neutral Probabili-ties: Attempts to Quantify Q

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Content

Abstract

An important and complex question in corporate finance is how to value uncertain cash-flow streams. Common practice is to discount expected cash flows with a constant risk-adjusted discount rate. The risk-adjusted discount rate approach is the basis for discounted cash flow approaches applied in practice for capital budgeting purposes or for the valuation of companies. The capital asset pricing model is usually used to determine the discount rate. This model has, however, theoretical and empirical shortcomings, as, for example, the expected rate of return of the market portfolio and thereby the expected market risk premium is not observable. An alternative approach is applied in the field of option pricing theory: The risk neutral valuation approach does not require an assumption on the risk premium. Instead, the analyst needs to quantify the risk-neutral probability. The aim of this paper is to show the relation between the two approaches and to find estimates for the risk-neutral probability by using logical arguments and empirical data of the 30 German DAX companies. We illustrate the risk-neutral valuation approach on the basis of an example that we developed from publicly available valuation documentation of a recent merger in Germany.

Keywords: Risk-neutral Probability, Risk Premium, Cost of Capital, Discounted Cash Flow, Certainty Equivalent Approach

JEL-class.: G 32, G12

1 Introduction

An important and complex question in corporate finance is how to value uncertain future cash-flow streams. Common practice is to discount expected cash flows with a constant risk-adjusted discount rate. The concept of risk-adjusted discount rates (RADR) is the basis for common discounted cash flow (DCF) approaches applied in practice for capital budgeting purposes or for the valuation of companies. The capital asset pricing model (CAPM) is usually used to determine the RADR. This model has well-known theoretical and empirical shortcomings, as, for example, the expected rate of return of the market portfolio and thereby the expected market risk premium is not observable. Instead, historical estimates are used to specify key parameters for the CAPM. This, however, is inconsistent, because future cash flows are valued with RADR based on historical data. An alternative approach is applied in the field of option pricing theory: The risk neutral valuation (RNV) approach does not require an explicit assumption on the risk premium. Instead, the analyst needs to quantify the risk-neutral probability. Risk-neutral probabilities are used to transform uncertain cash flows into their certainty equivalents. These certainty equivalents can be discounted at the risk-free interest rate.

The aim of this paper is to show the relation between the two approaches and to find estimates for the risk-neutral probability by using logical arguments and empirical data of the 30 German DAX companies. We illustrate the risk-neutral valuation approach on the basis of an example that we developed from publicly available valuation documentation of a recent merger in Germany. Chapter 2 summarises the RADR approach and points out key issues regarding the application of the CAPM. The RNV approach is presented in chapter 3. We will use a binomial model as simplified structure for the cash-flow stream and develop a valuation formula, which is consistent with but not dependent on the CAPM. After the two valuation approaches have been presented we discuss their relationships in chapter 4.

Model-based betas and risk premiums are derived from the RNV approach for different assumptions on the risk-neutral probabilities. We will derive the logical limits for risk-adjusted probabilities. In addition we use empirical data to narrow the range. Based on this analysis we suggest a heuristic approach to quantify risk-neutral probabilities. This approach is applied to a case based on real data (chapter 5). Chapter 6 closes with a brief summary.

2 Risk-adjusted discount rates

2.1 Interpretation of common practice

The common approach to value uncertain cash-flow streams e.g. for capital budgeting purposes is to discount unconditional expected cash flows, $E_P[\tilde{C}_{ti}]$, with a risk-adjusted discount rate called the cost of capital of the asset i, r_i, which produces the cash-flow stream:[1]

(1) $$V_{0i} = \sum_{t=1}^{T} E_P[\tilde{C}_{ti}]\,(1+r_i)^{-t}$$

The subscript P of the expectation operator indicates that the subjective probability p is used to formulate expected cash flows.[2] The risk aversion of investors is taken into account by using the cost of capital as a discount rate, which typically exceed the risk-free interest rate by a risk premium.[3] This approach is an application of the concept of risk-adjusted discount rates (RADR).

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1 Our interpretation of common practice is based on standard text books in finance such as Brealey, R.A./ Myers, S.C. (1996) and management literature, e.g., Copeland, T.E./ Koller, T./ Murrin, J. (2000).

2 Expectations without reference to a specific filtration indicate unconditional expectations.

3 For simplicity purposes we assume only positive risk premiums and therefore risk-averse

2.2 Constant discount rate and constant risk premium

We assume that the risk-free rate r_f is certain and not time dependent, although this is only for simplicity purposes. More important is the assumption that is found in the majority of practical applications, i.e., the assumption that the risk premium π_i is constant through time:

(2) $$r_i = r_f + \pi_i$$

It shall be pointed out that the use of a constant risk premium is warranted only under specific conditions. Sufficient conditions for the use of constant risk premiums have been described as “simplified discounting rules”. If, for example, the uncertain cash-flow stream follows a martingale with constant growth, the application of a constant risk premium to value this stream is sensible.[4]

In practice, it is often not possible to verify whether the conditions for simplified discounting rules are met, given that the cash-flow streams taken from business plans do not unveil the assumptions on the stochastic properties of that stream. Thus, the application of constant risk premiums is more a heuristic approach than anything else.

2.3 Using the CAPM to determine the risk premium

The Capital Asset Pricing Model (CAPM) is a positive theory of expected rates of return, which often serves as a basis to estimate the risk premium. In order to do so the analyst needs to quantify his or her expectation regarding the market risk premium, i.e., the difference between the expected rate of return of the market portfolio, $E_P[\tilde{r}_m]$, relative to the risk-free inter-

investor.

est rate. In addition, the beta is needed that relates the rate of return of the asset i to the rate of return of the market portfolio:[5]

(3) $$\pi_i = \beta_i \ E_P[\tilde{r}_m - r_f]$$

$$\beta_i = \frac{\mathrm{cov}[\tilde{r}_i, \tilde{r}_m]}{\mathrm{var}[\tilde{r}_m]}$$

The application of the CAPM is conceptually easy, however, empirical issues are evident: What is a reliable estimate for the expected market rate of return? Where does the beta come from, if the rates of returns of the asset i are not observable?

The heuristic "solution" is to use estimators based on historical data and comparable assets that are traded in a market. For example, the historic market risk premium often is estimated in the range between 3 % and 7 %[6] per annum. These estimates are based on time series of historic rates of return of a diversified stock market index versus interest rates of government bonds. Betas are taken from traded assets, which are assumed to be comparable in their risk profile. Table 1 shows the cost of capital for the DAX 30 companies, using a risk premium of 5%,[7] a risk-free rate of 5 % and betas from Bloomberg. Raw betas are only based on historical data, in contrast to adjusted betas which are weighted averages of the raw beta (2/3) and the market beta (1/3). Based on this information, the average cost of capital is 10 %.

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4 See Richter, F. (2001) and (2002).

5 See e.g., Weston, J.F./ Copeland, T.E. (1988) p. 195 - 198, or the original article by Sharpe, W. (1964) p. 425 - 442.

6 Claus, J./ Thomas, J. (2001), Fama, E./ French, K.R. (2000) or Cornell, B. (1999).

7 Stehle, R (1999) estimates, Morawietz, M., (1994), Bimberg, L. (1991), Uhlir, H./ Steiner, P. (1991), and other studies.

Company	**raw Beta**	**CoC on raw Beta**	**adjusted Beta**	**CoC on adj. Beta**
Adidas	0.62	8.1%	0.75	8.7%
Allianz	0.91	9.5%	0.94	9.7%
BASF	0.82	9.1%	0.88	9.4%
Bayer	0.76	8.8%	0.84	9.2%
HypoVereinsbank	0.68	8.4%	0.78	8.9%
BMW	0.87	9.3%	0.91	9.6%
Commerzbank	0.87	9.3%	0.91	9.6%
DaimlerChrysler	0.91	9.5%	0.94	9.7%
Deutsche Bank	1.14	10.7%	1.09	10.5%
Degussa	0.70	8.5%	0.80	9.0%
Lufthansa	0.99	10.0%	1.00	10.0%
Deutsche Post	n.a.	n.a.	n.a.	n.a.
Telekom	1.23	11.2%	1.15	10.8%
Eon	0.31	6.5%	0.53	7.7%
Epcos	1.51	12.5%	1.34	11.7%
Fresenius	0.50	7.5%	0.66	8.3%
Henkel	0.16	5.8%	0.44	7.2%
Infineon	n.a.	n.a.	n.a.	n.a.
Linde	0.52	7.6%	0.68	8.4%
MAN	0.62	8.1%	0.75	8.7%
Metro	0.59	7.9%	0.72	8.6%
MLP	1.00	10.0%	1.00	10.0%
Münchner Rück	0.79	9.0%	0.86	9.3%
Preussag	1.02	10.1%	1.01	10.1%
RWE	0.44	7.2%	0.63	8.1%
SAP	1.62	13.1%	1.42	12.1%
Schering	0.37	6.8%	0.58	7.9%
Siemens	1.50	12.5%	1.33	11.7%
ThyssenKrupp	0.79	8.9%	0.86	9.3%
Volkswagen	0.81	9.0%	0.87	9.4%
Market Portfolio	**1.00**	10.0%	**1.00**	10.0%

Table 1 Betas and Cost of Capital for DAX members

3 Certainty equivalents, risk-adjusted probabilities, and risk-adjusted growth rates

Even if the empirical issues of the RADR in combination with CAPM are resolved to the satisfaction of the analyst, one issue remains: It is not consistent to value future cash-flow streams with cost of capital based on historic risk premiums. The expected future risk premium should be employed instead. Therefore, we search for an approach that does not need an exogenous assumption on the market risk premium. In addition, the approach should allow for a time-varying risk premium.

3.1 Outline of the risk-neutral valuation approach

The certainty equivalent approach works as follows: Instead of discounting unconditional cash-flow expectations with a risk-adjusted discount rate, certainty equivalent cash flows are discounted at the risk-free interest rate:[8]

$$(4) \qquad V_{0i} = \sum_{t=1}^{T} E_Q[\tilde{C}_{ti} \mid F_0]\,(1+r_f)^{-t}$$

The risk aversion of investors is taken into account by reducing the expected cash flow under P. This is done on the basis of an equivalent probability measure Q, which creates conditional certainty equivalents as indicated by the filtration F. A risk-averse investor relates expected cash flows to the certainty equivalent as follows: $E_Q[\tilde{C}_{ti} \mid F_0] < E_P[\tilde{C}_{ti}]$. In the option pricing literature, this approach is called “risk-neutral valuation” (RNV). In a first step the expected cash flows are reduced via the risk-neutralised probabilities. In a second step, this modified cash-flow expectation is valued as if investors would be risk neutral, i.e., by discounting with the risk-free rate.

8 See e.g. Duffie, D. (1988) for more background on this approach.

3.2 Simplification of the time-state space: Binomial model

The application of the RADR approach requires simplifications, e.g., when determining the risk premium. The same holds true for the RNV approach, however, the heuristic simplification follows a different route. A simplified stochastic model for the uncertain cash-flow stream is assumed, which makes the time-state space tractable. The application of the binomial model is an example for this. Within this model future cash flows can either move up by the factor u_{ti} or shrink by d_{ti} as shown in exhibit 1, i.e., $\tilde{C}_{t+1,i} \in \{\tilde{C}_{t,i} u_{ti}; \tilde{C}_{t,i} d_{ti}\}$, with $0 < d_{ti} \le u_{ti}$. Both probability measures P and Q can be applied to the cash-flow process, being it to determine the subjective (unconditional) expectations or certainty equivalents.

The binomial model is a heuristic because the true stochastic model of the future cash-flow stream often is not known. Assuming that this is an acceptable simplification, the next question is how to estimate the parameters of the model. These parameters can be taken from the business plan for the asset to be valued. The analyst shall be able to articulate the expectations of the period-by-period growth rates $g_{ti} = E_P[\tilde{C}_{t+1,i}] / E_P[\tilde{C}_{t,i}] - 1$. Fixing p and assuming a recombining binomial model the up and down factors are determined:[9]

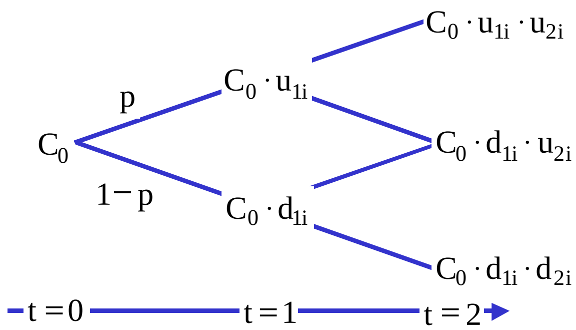
(5) $$u_{ti} = \frac{1+g_{ti}}{2p} + \sqrt{\left(\frac{1+g_{ti}}{2p}\right)^2 - \frac{1-p}{p}}$$

$$d_{ti} = \frac{1}{u_{ti}}$$

[9] Formulas for non-recombining trees are shown in the appendix.

With that we have a binomial model that is consistent with the expected growth rates of the business plan: $1 + g_{ti} = pu_{ti} + (1 - p)d_{ti}$.

Binomial Model: In each time-step only two possible states



Calculation of (conditional) expected cash flows:

$E_P[\tilde{C}_1 | F_0] = p \cdot C_0 \cdot u_{1i} + (1-p) \cdot C_0 \cdot d_{1i}$

$E_P[\tilde{C}_2 | F_1] = p \cdot C_0 \cdot u_{1i} \cdot u_{2i} + (1-p) \cdot C_0 \cdot d_{1i} \cdot u_{2i}$ ***or:***

$E_P[\tilde{C}_2 | F_1] = p \cdot C_0 \cdot d_{1i} \cdot u_{2i} + (1-p) \cdot C_0 \cdot d_{1i} \cdot d_{2i}$

Exhibit 1 Binomial Model

Tables 2 and 3 contain the growth factors derived from the I/B/E/S database for the DAX 30 companies (p = ½). We use two sets of data, based on the growth rates of expected dividends per share, DPS, and the growth rates of expected earnings per share, EPS. These are proxies for the growth rates of expected cash flows to shareholders, both of which are not perfect, given that the first lacks share buy-backs and capital increase and the second often is not fully distributed to shareholders. However, better estimates are not available.[10]

[10] The following abbreviations are used: na = not available, neg = negative growth rate, caa = change of accidental, nc = not considered. In all of these cases the derivation of the growth factor u is not possible or not sensible with our approach. Therefore we left the corresponding companies out. Furthermore companies with negative growth rates have negative correlation with a market which has positive growth rates, and we assume positive corre-

Comapny	Growth rate 2002	up-factor	down-factor	Growth rate 2003	up-factor	down-factor	Growth rate 2004	up-factor	down-factor
Adidas	5.8%	1.404	0.712	12.9%	1.652	0.605	-3.0%	neg.	neg.
Allianz	22.4%	1.931	0.518	14.2%	1.693	0.591	n.a.	n.a.	n.a.
BASF	4.8%	1.363	0.734	4.3%	1.340	0.746	-6.9%	neg.	neg.
Bayer	8.9%	1.520	0.658	11.4%	1.606	0.623	11.9%	1.621	0.617
HypoVereinsbank	9.9%	1.555	0.643	13.7%	1.677	0.596	-15.8%	neg.	neg.
BMW	5.7%	1.400	0.714	10.2%	1.565	0.639	24.0%	1.974	0.507
Commerzbank	14.1%	1.691	0.591	7.2%	1.459	0.685	-39.4%	neg.	neg.
DaimlerChrysler	10.6%	1.579	0.633	13.9%	1.683	0.594	n.a.	n.a.	n.a.
Deutsche Bank	9.2%	1.532	0.653	14.2%	1.693	0.591	-3.4%	neg.	neg.
Degussa	6.1%	1.416	0.706	8.9%	1.520	0.658	11.5%	1.608	0.622
Lufthansa	22.4%	1.929	0.518	13.1%	1.661	0.602	10.8%	1.585	0.631
Deutsche Post	3.5%	1.300	0.769	5.1%	1.375	0.727	n.a.	n.a.	n.a.
Telekom	-1.7%	neg.	neg.	3.6%	1.305	0.766	2.9%	1.270	0.788
Eon	8.4%	1.501	0.666	5.1%	1.376	0.727	6.6%	1.435	0.697
Epcos	8.2%	1.496	0.668	2.1%	1.227	0.815	58.1%	2.806	0.356
Fresenius	12.6%	1.644	0.608	5.3%	1.384	0.723	n.a.	n.a.	n.a.
Henkel	11.7%	1.615	0.619	9.2%	1.531	0.653	15.9%	1.744	0.573
Infineon	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
Linde	6.0%	1.413	0.708	3.0%	1.276	0.784	13.3%	1.665	0.601
MAN	1.1%	1.160	0.862	11.9%	1.622	0.617	n.a.	n.a.	n.a.
Metro	2.9%	1.269	0.788	3.9%	1.320	0.757	6.4%	1.428	0.700
MLP	29.3%	2.112	0.473	50.8%	2.636	0.379	27.4%	2.064	0.484
Münchner Rück	15.7%	1.740	0.575	9.4%	1.538	0.650	29.4%	2.115	0.473
Preussag	6.5%	1.431	0.699	10.4%	1.572	0.636	6.7%	1.438	0.696
RWE	10.0%	1.560	0.641	12.7%	1.647	0.607	21.5%	1.906	0.525
SAP	33.9%	2.229	0.449	22.6%	1.934	0.517	40.4%	2.389	0.419
Schering	16.4%	1.758	0.569	10.8%	1.584	0.631	23.9%	1.972	0.507
Siemens	5.8%	1.402	0.713	15.6%	1.736	0.576	17.0%	1.778	0.562
ThyssenKrupp	-1.0%	neg.	neg.	12.7%	1.648	0.607	-2.0%	neg.	neg.
Volkswagen	5.1%	1.375	0.727	5.3%	1.382	0.723	38.2%	2.335	0.428

Table 2 DPS growth rates, and related growth factors

lation in general.

Comapny	Growth rate 2002	up-factor	down-factor	Growth rate 2003	up-factor	down-factor	Growth rate 2004	up-factor	down-factor
Adidas	12.7%	1.648	0.607	17.0%	1.777	0.563	-19.2%	neg.	neg.
Allianz	49.1%	2.597	0.385	11.8%	1.618	0.618	n.a.	n.a.	n.a.
BASF	35.1%	2.259	0.443	25.2%	2.004	0.499	10.1%	1.563	0.640
Bayer	23.8%	1.968	0.508	27.9%	2.077	0.481	9.8%	1.552	0.644
HypoVereinsbank	25.2%	2.006	0.498	35.8%	2.277	0.439	21.5%	1.904	0.525
BMW	5.4%	1.388	0.721	12.8%	1.651	0.606	0.6%	1.114	0.898
Commerzbank	74.3%	3.170	0.315	26.7%	2.044	0.489	n.a.	n.a.	n.a.
DaimlerChrysler	190.5%	5.633	0.178	58.0%	2.802	0.357	38.9%	2.354	0.425
Deutsche Bank	9.1%	1.526	0.655	15.2%	1.724	0.580	-0.3%	neg.	neg.
Degussa	0.3%	1.076	0.929	26.4%	2.037	0.491	8.5%	1.506	0.664
Lufthansa	174.3%	5.298	0.189	50.2%	2.622	0.381	71.3%	3.105	0.322
Deutsche Post	-3.4%	neg.	neg.	0.8%	1.134	0.882	n.a.	n.a.	n.a.
Telekom	-12.5%	neg.	neg.	caa	caa	caa	66.2%	2.990	0.334
Eon	20.2%	1.869	0.535	13.1%	1.660	0.603	-0.3%	neg.	neg.
Epcos	-7.8%	neg.	neg.	47.6%	2.562	0.390	62.2%	2.899	0.345
Fresenius	21.3%	1.898	0.527	15.9%	1.744	0.573	20.6%	1.880	0.532
Henkel	9.1%	1.528	0.655	9.8%	1.552	0.644	3.7%	1.309	0.764
Infineon	21.6%	1.907	0.524	caa	caa	caa	-73.8%	neg.	neg.
Linde	14.5%	1.703	0.587	15.4%	1.731	0.578	21.9%	1.917	0.522
MAN	14.3%	1.696	0.590	22.5%	1.932	0.518	0.1%	1.054	0.949
Metro	13.5%	1.671	0.598	15.2%	1.723	0.581	2.3%	1.239	0.807
MLP	33.1%	2.209	0.453	41.1%	2.406	0.416	-3.3%	neg.	neg.
Münchner Rück	58.1%	2.804	0.357	18.5%	1.821	0.549	24.4%	1.985	0.504
Preussag	8.5%	1.506	0.664	5.3%	1.382	0.724	36.7%	2.300	0.435
RWE	19.1%	1.839	0.544	15.2%	1.723	0.580	22.9%	1.945	0.514
SAP	36.3%	2.289	0.437	39.5%	2.367	0.422	27.0%	2.054	0.487
Schering	14.4%	1.700	0.588	15.1%	1.721	0.581	16.0%	1.748	0.572
Siemens	75.4%	3.196	0.313	80.2%	3.301	0.303	24.5%	1.988	0.503
ThyssenKrupp	11.7%	1.614	0.619	28.1%	2.081	0.480	32.9%	2.205	0.454
Volkswagen	-0.4%	neg.	neg.	5.8%	1.404	0.712	36.5%	2.295	0.436

Table 3 EPS growth rates, and related growth factors

3.3 Risk-adjusted growth rates

Next, we apply the risk-neutral probability q instead of p to determine the risk-adjusted growth rate:

$$(6) \qquad 1+g_{ti}^{*} = qu_{ti} + (1-q)d_{ti}$$

The risk-adjusted growth rates now can be used to derive the certainty equivalents needed to apply the RNV approach as indicated by (4):

$$(7) \qquad E_Q[\tilde{C}_{ti} \mid F_0] = C_{0i}\prod_{j=1}^{t}(1+g_{ji}^{*})$$

Notice that (1) is bound to the case of constant growth rates while (7) is not. To apply (1) the empirical issue of determining π has to be resolved. (7) requires quantification of q instead. Before we come to that the relationships between (1) and (7) shall be made more transparent.

4 Relationship between the two approaches

4.1 Model-based betas and risk premiums

From a conceptual view both approaches are equivalent if the RADR is applied with consistent risk premiums. Under the assumption of the binomial model as described in the previous section, the cost of capital and thereby the risk premium is given by:[11]

$$(8) \qquad r_{ti} = x_{ti}(1+r_f)-1 = r_f + \pi_{ti}$$

[11] See Richter, F. (2002), p. 138.

$$x_{ti} = \frac{1+g_{ti}}{1+g_{ti}^{*}}$$

$$\Rightarrow \pi_{ti} = (x_{ti} - 1)(1 + r_f)$$

To determine the risk-adjusted discount rate the risk-free rate is grossed-up by a factor of x_{ti}, which depends on the growth rate of the expected cash flow relative to the growth rate of the certainty equivalent of the cash flow. This relation is supposed to hold for any asset, a portfolio of assets and for the market as a whole, i.e. for $i = m$. As shown in the appendix this immediately provides us with an interpretation of beta in the sense of the CAPM:

(9) $$\beta_{ti} = \frac{x_{ti} - 1}{x_{tm} - 1}$$

The RADR approach and the RNV approach yield the same result if the discount rate according to (8) or (9) is used.

The next two exhibits illustrate the relation between growth and expected rate of return, and beta, respectively. The graphs are plotted for various assumptions on q. With q = p = ½ we expected rates of return equal to the risk-free rate of return. Investors who do not differentiate between p and q are risk-neutral. Therefore, there is no risk premium, although beta is positive and increasing with growth. The other extreme, q = 0, characterises maximum risk aversion, given that q is the risk-adjusted probability for upward movements. With q = 0, only downward movements are taken into account. This implies maximum betas and maximum expected rates of return.[12]

[12] We assume that the correlation between the cash flows from asset i and the cash flows from the alternative investment, i.e. from the market, is positive and perfect. Therefore our results will be maximum risk premiums.

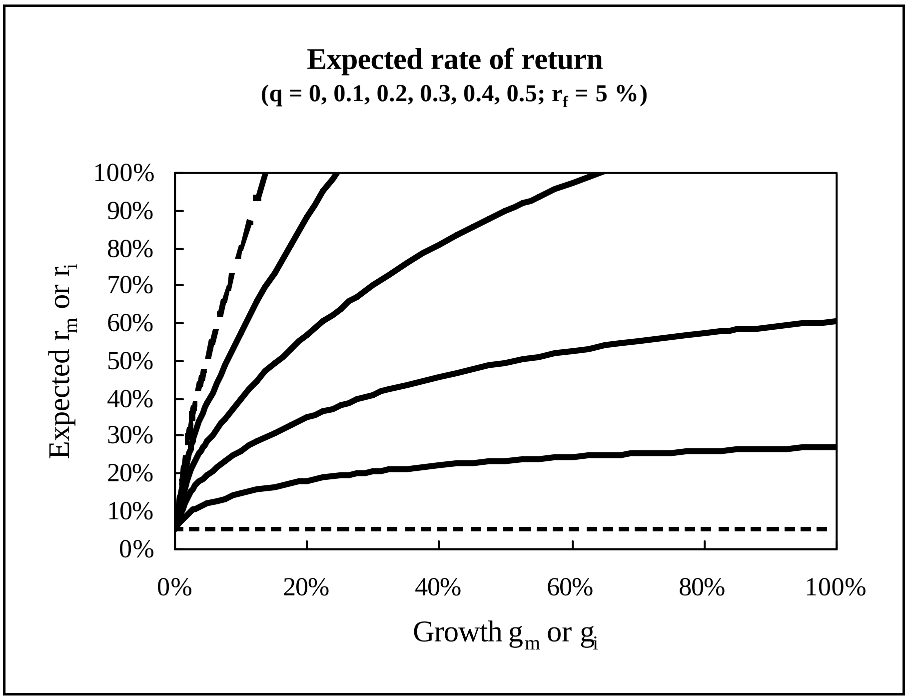


Exhibit 2 Expected Rate of Return as a function of growth and q

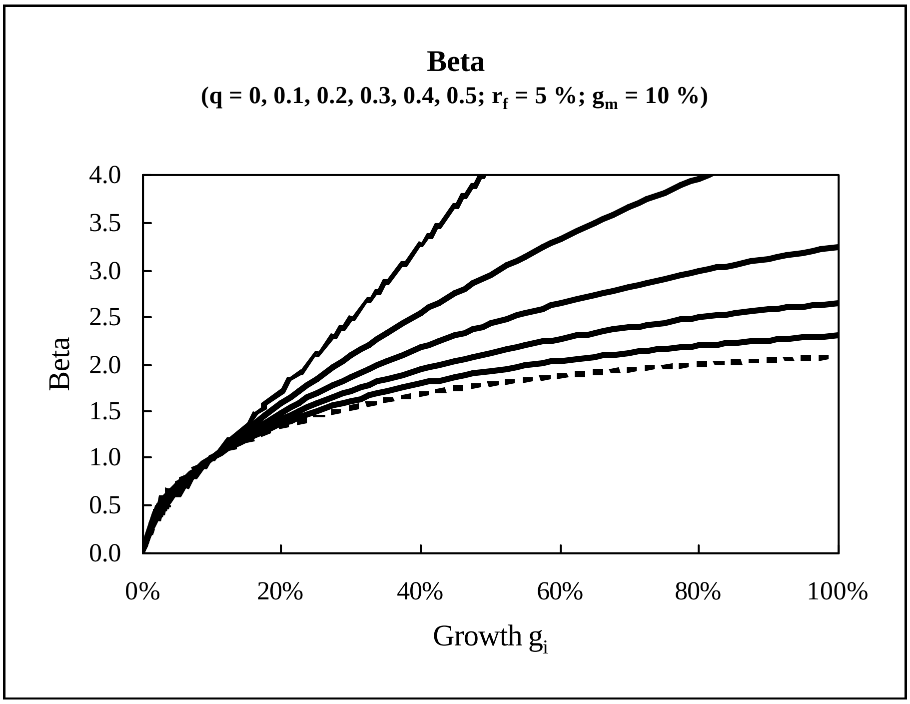


Exhibit 3 Beta as a function of growth and q

4.2 Estimating the market risk premium

We can use the model to estimate the expected market risk premium. Here is our idea: We plan to derive a forward-looking risk premium of the market portfolio from the weighted average of the risk premiums of all assets, which constitute this portfolio. We will again use analyst expectations as shown in tables 2 and 3. The expected growth rate for the market portfolio and its corresponding risk-adjusted growth rate are given by (10) and (11):

$$(10) \quad g_{tm} = \frac{\sum_{i=1}^{n} E_P[\tilde{C}_{ti}]}{\sum_{i=1}^{n} E_P[\tilde{C}_{t-1,i}]} - 1 \text{ with } E_P[\tilde{C}_{ti}] = C_{0i} \prod_{j=1}^{t} (1+g_{ti})$$

$$(11) \quad g_{tm}^{*} = \frac{\sum_{i=1}^{n} E_Q[\tilde{C}_{ti} \mid F_0]}{\sum_{i=1}^{n} E_Q[\tilde{C}_{t-1,i} \mid F_0]} - 1 \quad \text{ with } E_Q[\tilde{C}_{ti} \mid F_0] = C_{0i} \prod_{j=1}^{t} (1+g_{ji}^{*})$$

Given these two growth rates we can employ (8) to determine x_{tm} and thereby π_{tm}.

4.3 How to estimate q?

The final parameter that we need is q. We take four approaches: First, we look at the range of attainable values for q, which are given by its logical limits. Then, we use an increasing set of empirical data to narrow the bandwidth of potential values for q. The second approach is based on exogenous estimates for the expected market rates of return. Third, we include an estimate for the volatility of the market rate of return. Fourth, we use the exogenous estimate for the expected market rate of return in combination with observed betas.

4.3.1 Logical boundaries for q

Our model still lacks quantification of q. Here we are at the main purpose of this paper. Without further specification (other than the binomial model) we can define the interval of attainable values for q: q has to exceed zero, otherwise Q would not be an equivalent probability measure to P, given that we assume $p > 0$. Furthermore, q may not exceed p in order to transform the expectation under P into certainty equivalents.[13] For the purpose of our analysis the boundaries for q are as follows:

(12) $0 < q \leq p = 0.5$

With this boundaries we can determine the maximum and the minimum risk premium and thereby the minimum and the maximum value of cash-flow streams. This interval of attainable values is bound to the assumptions of the binomial model and the assumption of arbitrage-free markets only. No historic data is used at all. However, the bandwidth of attainable risk premiums is very broad and sensitive to q.

	Expected Market Rate of Return (DPS)		
q	2002	2003	2004
0	63.9%	78.0%	60.8%
0.1	47.4%	56.3%	45.4%
0.2	33.9%	39.3%	32.6%
0.3	22.6%	25.6%	21.9%
0.4	13.1%	14.4%	12.8%
0.5	5.0%	5.0%	5.0%

Table 4 Expected Market Rate of Return for different q based on DPS

[13] Again we assume positive correlation between the asset and the market.

q	Expected Market Rate of Return (EPS) 2002	2003	2004
0	202.2%	209.9%	126.9%
0.1	119.7%	122.9%	84.2%
0.2	72.6%	74.1%	55.0%
0.3	42.1%	42.8%	33.7%
0.4	20.8%	21.0%	17.6%
0.5	5.0%	5.0%	5.0%

Table 5 Expected Market Rate of return for different q based on EPS

4.3.2 Using an exogenous estimate for the expected market rate of return

Assume, the expected risk premium for the market portfolio would be given, e.g., with $\hat{\pi} \in [3\%, 7\%]$. The risk-free interest rate can be derived from observable data, say $r_f = 5\,\%$. With that we have an estimate for the expected rate of return, which in turn can be used to derive q:

$$\text{(13)} \quad \hat{r}_m = \left(\frac{1+g_{tm}}{1+g^*_{tm}}\right)(1+r_f) - 1$$

$$\Rightarrow g^*_{tm} = (1+g_{tm})\frac{1+r_f}{1+\hat{r}_m} - 1$$

$$\Rightarrow q_t = \frac{1+g^*_{tm} - d_{tm}}{u_{tm} - d_{tm}} = \frac{(1+g_{tm})\dfrac{1+r_f}{1+\hat{r}_m} - d_{tm}}{u_{tm} - d_{tm}}$$

The advantage of this approach is that we get a point estimate for the risk-neutral probability. However, we need an estimate for the expected rate of return of the market portfolio. As pointed out earlier, it is not consistent to use historic data to produce such an estimate. Notice, that if a constant risk premium is assumed, the risk-neutral probabilities become time-dependent.

Nevertheless, based on a fairly broad range of market risk premiums, the range of attainable q values is between 0.410 and 0.479.

Risk Premium	Implied value for q (DPS) 2002	2003	2004
3%	0.461	0.466	0.460
4%	0.449	0.455	0.447
5%	0.437	0.445	0.435
6%	0.425	0.434	0.422
7%	0.413	0.424	0.410

Table 6 Possible q-values based on exogenous estimates for the market risk premium (on DPS)

Risk Premium	Implied value for q (EPS) 2002	2003	2004
3%	0.479	0.479	0.474
4%	0.472	0.472	0.466
5%	0.465	0.466	0.458
6%	0.459	0.459	0.450
7%	0.452	0.453	0.442

Table 7 Possible q-values based on exogenous estimates for the market risk premium (on EPS)

4.3.3 Exogenous estimate for the market rate of return and its volatility

Assume we would have a reliable estimate for the rate of return of the market portfolio (with $r_f = 5\%$ and $\hat{\pi} \in [3\%, 7\%]$). In addition, the volatility of the market rate of return shall be given, e.g., $\hat{\sigma}_m \in \{0.25, 0.275, 0.3\}$.[14] If these parameters are supposed to be constant through time, we can derive the underlying value for q:

(14) $$E_Q[\tilde{r}_{tm} \mid F_{t-1}] = q_t (r_{t-1,m} + \hat{\sigma}_m) + (1 - q_t)(r_{t-1,m} - \hat{\sigma}_m) \overset{!}{=} r_f$$

$$\Rightarrow q_t = \frac{1 - \phi_{tm}}{2}$$

with $\phi_{tm} = \dfrac{r_{t-1,m} - r_f}{\hat{\sigma}_m}$ (also known as Sharpe-ratio)

The expected market rate of return under the risk-adjusted probability q has to equal the risk free rate. Under the assumed structure for the market rate of return q could be derived from the expected risk premium in connection with the volatility of the market rate of return. Formula (14) rests on the absence of arbitrage. It is clear that we have the implicit assumptions of a perfect correlation between the rates of return of the security i and the rates of return on the market portfolio.

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[14] See Bimberg, L. (1991).

Market Rate of Return	**q for given volatility**		
	0.25	**0.275**	**0.3**
8%	0.44	0.4455	0.4500
9%	0.42	0.4273	0.4333
10%	0.40	0.4091	0.4167
11%	0.38	0.3909	0.4000
12%	0.36	0.3727	0.3833

Table 8 Implied q for given r_m and σ_m

The advantage of the approach is again a point estimate for q. We find a range for q between 0.36 and 0.45. The disadvantages are the same as outlined in the previous section. However, it shall be pointed out that the volatility can be measured much more precisely than the market rate of return[15].

4.3.4 Using an exogenous estimate for risk premium and current betas

Finally, we use current betas in combination with the exogenous estimate for the risk premium to determine the cost of capital for the DAX companies (see table 1). We now select q such that our model-based cost of capital mimics the former estimate of cost of capital. This leads to different ranges for q each year. For these ranges of q the model-based cost of capital mimics the observable cost of capital 100 %. This indicates that current risk aversion of the market is covered by q from 0.456 to 0.472 based on medians to exclude outlier. The individual coverage is given for each year and separately for DPS and EPS data. Exhibit 4 summarises the results.

15 See Campbell et al.

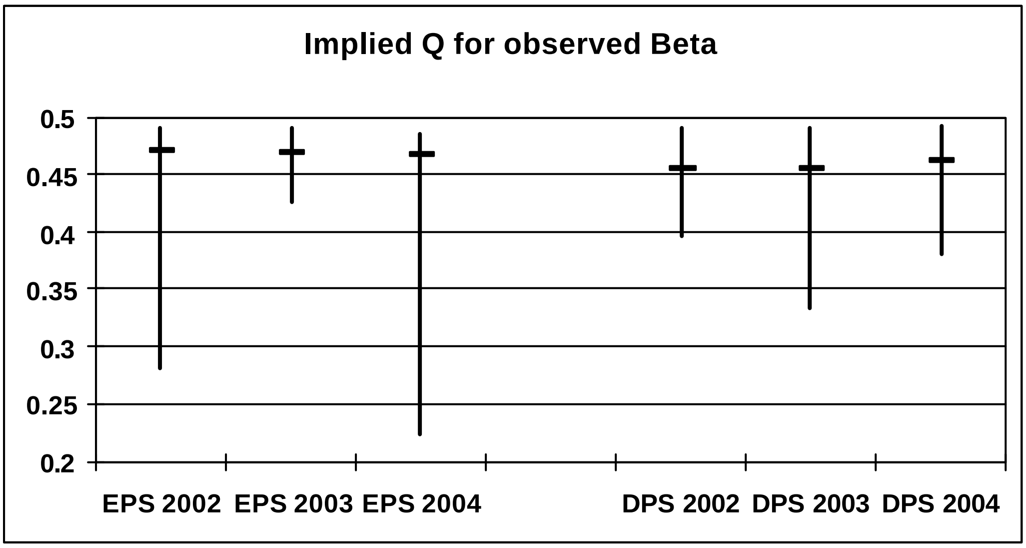

Exhibit 4 Necessary q to cover the observed CoC

4.3.5 Heuristic approach

Now we have ranges for the value of q based on its logical boundaries, and we have some guidance based on empirical data (which is, admittedly, based on historic information). If we combine all results it appears reasonable to us to assume a range for q between 0.36 and 0.48. This range lies within the logical boundaries, fits with the implied values for frequently used risk premiums, and covers the majority of observable cost of capital estimates for the DAX companies. As our aim is to estimate a company value eventually it is more appropriate to take the average of the range and use 0.42 as value for q. Therefore we suggest using this as a heuristic to value uncertain cash-flow streams. We think that this procedure is more robust than the RADR in combination with the CAPM, given that (i) we do not have to rely on its assumptions, (ii) we allow for time-varying risk premiums, (iii) we are using forward looking data to the extend possible. The application of this heuristic is illustrated in the next chapter.

5 Example for practical application

After the merger of two diversified companies in Germany the CEO decided also to merge the subsidiaries D and S in the area of chemical specialities to realise synergies and achieving a leading position in this market. As both subsidiaries have been quoted it is necessary with regard to German law to publish a merger report with detailed information about the share exchange ratios. This exchange ratio is determined on the basis of company valuation with discounted cash flow methods. For the following case study we used these publicly available data, i.e. the planned earnings before interest and tax (EBIT) for the next three years and an assumed long term growth rate of 1 %. Furthermore we considered an average corporate tax rate of 38.5 % according to German tax code. Finally we have no indication for a negative correlation between growth rate and market rate of return. Therefore we could apply our heuristic approach on the figures of company D and S as follows:

Company D	**t=0**	**1**	**2**	**3**	**4 +**
EBIT	**680.6**	**844.9**	**1,001.4**	**1,224.5**	
Corporate Tax	262.0	325.3	385.5	471.4	
Unlevered Cash Flow	418.6	519.6	615.9	753.1	
Growth Rate of Cash Flow		24.14%	18.52%	22.28%	1.00%
Probability (P)		**0.5**	**0.5**	**0.5**	**0.5**
up		1.98	1.82	1.93	1.15
down		0.51	0.55	0.52	0.87
Risk adjusted Probability (Q)		**0.42**	**0.42**	**0.42**	**0.42**
Risk adjusted Growth Rate		12.37%	8.34%	11.02%	-1.27%
Certainty Equivalent of Cash Flow		470.4	563.0	683.7	11,861.3
Discount Factor		0.95	0.91	0.86	0.86
NPV of Cash Flow	**11,795.5**	448.0	510.6	590.6	10,246.3
Implied Cost of Capital		16.00%	9.82%	8.44%	8.42%

Table 9 Application of the heuristic on company D

Company S	**t=0**	**1**	**2**	**3**	**4 +**
EBIT	**376.7**	**420.4**	**472.5**	**526.1**	
Corporate Tax	145.0	161.9	181.9	202.5	
Unlevered Cash Flow	231.7	258.5	290.6	323.6	
Growth Rate of Cash Flow		11.60%	12.39%	11.34%	1.00%
Probability (P)		**0.5**	**0.5**	**0.5**	**0.5**
up		1.61	1.64	1.60	1.15
down		0.62	0.61	0.62	0.87
Risk adjusted Probability (Q)		**0.42**	**0.42**	**0.42**	**0.42**
Risk adjusted Growth Rate		3.67%	4.18%	3.51%	-1.27%
Certainty Equivalent of Cash Flow		240.2	269.4	300.8	5,096.2
Discount Factor		0.95	0.91	0.86	0.86
NPV of Cash Flow	**5,135.2**	228.7	244.3	259.8	4,402.3
Implied Cost of Capital		13.03%	9.06%	7.58%	8.42%

Table 10 Application of the heuristic on company S

Under the RNV theorem with the heuristic of q = 0.42 the value for company D is about € 11,795.5 million and for company S € 5,135.2 million. This leads to an share exchange ratio of 1 : 2.3 in favour for company D. The results of a detailed valuation for both companies are represented as value bandwidths in exhibit 6 and 7.

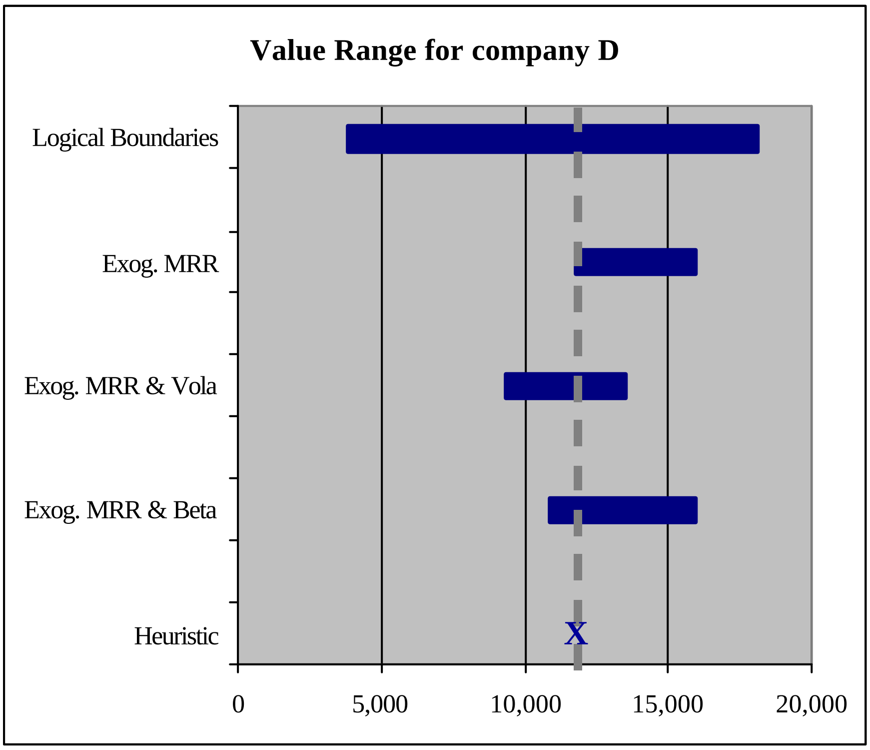

Exhibit 5 Bandwidth for company Ds value

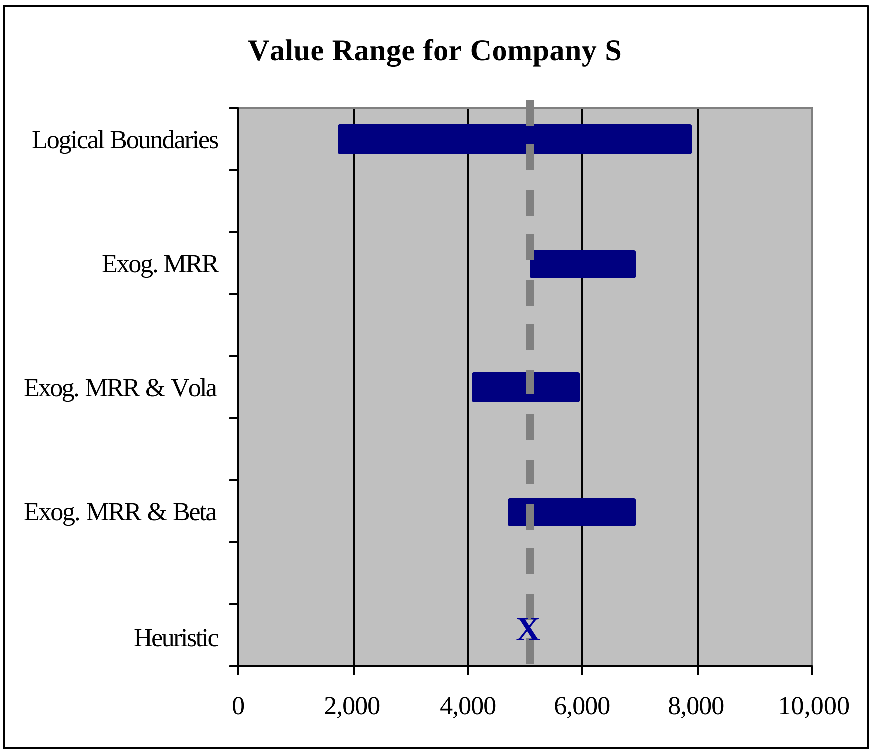

Exhibit 6 Bandwidth for company Ss value

6 Conclusion

The commonly used risk-adjusted discount rate approach rests on the CAPM, assumes constant risk premiums and relies heavily on historical data to determine the discount rate for future cash flows. The risk-neutral valuation approach overcomes these shortcomings. We think that the risk-neutral valuation approach is preferable relative to the risk-adjusted discount rate approach in combination with the CAPM, provided that we have an estimate for the risk-neutral probability (q), which was the purpose of this paper. We derived the logical boundaries for q and used empirical data as guidance to find a range for likely values for q. We think that q lies in the area of 0.36 – 0.48, although more empirical work using a broader set of companies seems sensible.

A value for q can be used for a heuristic application of the risk-neutral valuation approach. The value has to be in line with the logical boundaries of q and should be consistent with cost of capital estimates. Our indication for this value (0.42) fulfils these requirements for the DAX 30 companies.

The practical application of this heuristic is straightforward, as has been illustrated in our case study.

Appendix

(i) Formulas for non-recombining trees

Define g as the expected growth rate and σ as the standard deviation of this growth rate. In a binomial model we have the following relations:

$$1+g = pu + (1-p)d$$

$$\sigma^2 = p(u-1-g)^2 + (1-p)(d-1-g)^2$$

From the first definition we get a formula for u:

$$u = \frac{1+g-(1-p)d}{p}$$

Inserting this into the second definition gives us the expression for d:

$$\sigma^2 = p\left(\frac{1+g-(1-p)d}{p} - 1 - g\right)^2 + (1-p)(d-1-g)^2$$

$$\Leftrightarrow \frac{1}{p}\left(1+g-(1-p)d-p(1+g)\right)^2 + (1-p)(d-1-g)^2$$

$$\Leftrightarrow \frac{(1-p)^2}{p}\left(1+g-d\right)^2 + (1-p)(d-1-g)^2$$

$$\Leftrightarrow \left(\frac{(1-p)^2}{p} + (1-p)\right)\left(1+g-d\right)^2$$

$$\Rightarrow d^2 - 2(1+g)d + (1+g)^2 - \sigma^2 \frac{p}{1-p}$$

$$\Rightarrow d = 1+g-\sigma\sqrt{\frac{p}{1-p}}$$

Given g, σ and p, we clearly can derive d. In the case of p = ½ it is simply one plus growth minus one standard deviation.

Finally, we can use this result and substitute d in the formula for u:

$$u = \frac{1+g-(1-p)d}{p} = \frac{1+g-(1-p)(1+g)+(1-p)\sigma\sqrt{\frac{p}{1-p}}}{p}$$

$$\Leftrightarrow u = 1+g+\sigma\sqrt{\frac{1-p}{p}}$$

Again, in the special case of p = ½ we get one plus growth plus one standard deviation for the upward factor.

In the case of a recombining tree, the variance and the standard deviation of the growth rate are implicitly defined:

$$d = 1+g-\sigma\sqrt{\frac{p}{1-p}} = \frac{1}{1+g+\sigma\sqrt{\frac{1-p}{p}}}$$

$$\Rightarrow \sigma^2 = \left(0.5(1+g)\alpha + \sqrt{(0.5(1+g)\alpha)^2 + (1+g)^2 - 1}\right)^2$$

with $\alpha = \sqrt{\frac{p}{1-p}} - \sqrt{\frac{1-p}{p}}$

For the special case of p = ½ this expression reduces with α = 0 to:

$$\sigma^2 = (1+g)^2 - 1$$

Thus, the use of recombining trees replaces the need to specify the standard deviation of future growth rates by an implicit assumption. In recombining models, the standard deviation depends on the expected growth rate only.

(ii) Derivation of Beta

According to the CAPM we have:

$$r_{ti} = r_f + \beta_{ti}(r_{tm} - r_f)$$

$$\Rightarrow \beta_{ti} = \frac{r_{ti} - r_f}{r_{tm} - r_f}$$

We use (9) to substitute the expected rate of return for the asset i as well as for the market rate of return:

$$r_{ti} = x_{ti}(1 + r_f) - 1$$

$$r_{tm} = x_{tm}(1 + r_f) - 1$$

$$\Rightarrow \beta_{ti} = \frac{x_{ti}(1 + rf) - 1 - r_f}{x_{tm}(1 + rf) - 1 - r_f} = \frac{x_{ti}(1 + rf) - (1 + r_f)}{x_{tm}(1 + rf) - (1 + r_f)} = \frac{x_{ti} - 1}{x_{tm} - 1}$$

q.e.d.

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